ELSEVIER Physica A 243 (1997) 141–151

PHYSICA A

Two new hierarchies containing the sine-Gordon and sinh-Gordon equation, and their Lax representations[1]

Zhijun Qiao[a,b,c,*]

[a] School of Mathematical Science, Peking University, Beijing 100871, People's Republic of China
[b] Institute of Mathematics, Fudan University, Shanghai 200433, People's Republic of China
[c] Department of Mathematics, Liaoning University, Shenyang 110036, People's Republic of China

Received 3 January 1997

Abstract

An approach for obtaining a new Lax pair from a given old Lax pair is proposed, and two hierarchies of nonlinear evolution equations, which include the well-known sine-Gordon and sinh-Gordon equation, are presented in this article. By the use of the general structure of commutator representations, the Lax representations of the two new hierarchies are obtained. Particularly, the new Lax pairs of the sine-Gordon and sinh-Gordon equation are given. All of these Lax representations are of operator form.

1. Introduction

The sine-Gordon equation (SEGE)

$$u_{xt} = \sin u \tag{1}$$

and sinh-Gordon equation (SHGE)

$$u_{xt} = \operatorname{sh} u \tag{2}$$

are of considerable significance both in mathematics and in physics. Eq. (1) is the prototype of an integrable nonlinear equation in that it can be solved via the inverse scattering transformation [1]. Also, it has solitons, multisolitons, breathers quasiperiodic, π pulse and similarity solutions, and other properties of integrable nonlinear evolution equations (NLEEs) [2]. The SEGE (1) has a long history that begins in the latter part of the 19th century when this equation was found to occur in

* Correspondence address: School of Mathematical Science, Peking University, Beijing 100871, People's Republic of China. E-mail: zjqiao@lnu.edu.cn.

[1] The work has been partly supported by NNSFC and PMTPPC.

PII S0378-4371(97)00267-7

differential geometry [3], and has attracted a lot of attention. Some particular solutions of this equation were successively obtained in the past decades [4–8]. Among various techniques for getting these solutions there are the method of Backlund transformations [9–12], the method of expressing them as functions of the independent variables [2] $\xi = ax + t/a$ and $\eta = ax - t/a$ in the form $u(\xi,\eta) = 4\tan^{-1}[U(\xi)/V(\eta)]$, and the method of inverse scattering transformations [1]. All of these facts are very interesting.

It is well-known that the SEGE (1) processes the following Lax pair [13] (called the old Lax pair of Eq. (1))

$$y_x = \begin{pmatrix} \lambda & -\frac{1}{2}u_x \\ \frac{1}{2}u_x & -\lambda \end{pmatrix} y \quad \text{(spatial part)}\,, \tag{3$_1$}$$

$$y_t = \frac{1}{4\lambda}\begin{pmatrix} \cos u & \sin u \\ \sin u & -\cos u \end{pmatrix} y \quad \text{(time part)}\,. \tag{3$_2$}$$

The spatial part $(3)_1$ is exactly the special case of the Zakharov–Shabat [14] eigenvalue problem

$$y_x = \begin{pmatrix} \lambda & q \\ r & -\lambda \end{pmatrix} y\,,$$

with the potentials $q = -r = -\frac{1}{2}u_x$. So, the SEGE (1) is a particular equation of the AKNS hierarchy [15] with the specific potentials $q = -r = -\frac{1}{2}u_x$. In Ref. [16], it was pointed out that the SEGE (1) is not an evolution, there exists the recursion operator with it, and the flows $u_t = \mathscr{D}^k(u_x)$, $k = 0, 1, 2, \ldots, \mathscr{D} = D^2 + u_x^2 - u_x D^{-1} u_{xx}$, $D = \partial/\partial x$, are small symmetries of it. In Refs. [17,18], the authors gave a Lax pair of the SEGE (1) and SHGE (2) by making use of the chiral field hierarchy and the chiral field spectral problem. Afterwards, the author of Ref. [19] presented the sine-Gordon and sinh-Gordon equation via some transformation by virtue of the so-called inverse of the recursion operator $\Phi = D^2 - 4u^2 - 4u_x D^{-1} u (D^{-1}D = DD^{-1} = 1)$. In recent years, Capel, Nijhoff and their colaborators have achieved a series of developments (e.g., integrable mapping, Miura transformation, r-matrices, similarity reductions, etc.) in the field of integrable lattice systems such as discrete KdV, MKdV, KP, MKP, Calogero–Moser, Ruijsenaars–Schneider models, etc. [20,21–24]. The main methods they adopted is the versions of integrable discretizations in the time part of the integrable models, which is called the time discretization of of integrable model by them [21].

Motivated by their time-discrete versions of integrable systems, in the present articles we shall consider the time part of Lax representations for the continuous integrable equations. Concertely speaking, the paper is organized as follows. In the next section, first starting from the time part $(3)_2$ of the old Lax pair and viewing it as a new eigenvalue problem, we shall give a new hierarchy of NLEEs (called the sine-Gordon hierarchy) whose first equation is the well-known SEGE (1). Then, by virtue of the general structure of commutator representations [25,26], we construct the Lax representations of the sine-Gordon hierarchy. Particularly, a new Lax pair of the SEGE (1) is given. In Section 3, by the use of the procedure which is similar to Section 2, another new hierarchy of NLEEs (called the sinh-Gordon hierarchy) containing the SHGE (2) is found, and the Lax representations are established. Additionally, a new Lax pair of the SHGE (2) is also obtained. All of these Lax representations are of operator form. In the last section, we give some discussions and explanations of the method proposed in this article.

2. The sine-Gordon hierarchy, Lax representations and a new Lax pair of SEGE (1)

2.1. Generation of the sine-Gordon hierarchy

We view Eq. $(3)_2$ as a new eigenvalue problem and rewrite it as

$$y_x = \frac{1}{4\lambda}\begin{pmatrix} \cos u & \sin u \\ \sin u & -\cos u \end{pmatrix} y\,, \tag{4}$$

where λ is an eigenparameter, u is a scalar function, $y = (y_1, y_2)^{\mathrm{T}}$. It is easy to calculate the spectral gradient $\delta\lambda/\delta u$ of the eigenparameter λ with respect to the potential function u

$$\delta\lambda/\delta u = (4\lambda)^{-1}(y_2^2 - y_1^2)\cos u - 2y_1 y_2 \sin u)\,, \tag{5}$$

where y_1, y_2 satisfy Eq. (4).

Following the ideas of the general procedure for producing the hierarchy of NLEEs (nonlinear evolution equations) and obtaining Lax representations in Ref. [25]. Now, we are going to look for pair of the so-called Lenard's operators $K = K(u)$ and $J = J(u)$ satisfy

$$K\delta\lambda/\delta u = \lambda^{-2} J\delta\lambda/\delta u\,. \tag{6}$$

Noticing the relations

$$\begin{cases} -\frac{1}{2}(y_1^2 + y_2^2)_x = \delta\lambda/\delta u\,, \\ \frac{1}{2}(y_1^2 - y_2^2)_x = (4\lambda)^{-1}(y_1^2 + y_2^2)\cos u\,, \\ (y_1 y_2)_x = (4\lambda)^{-1}(y_1^2 + y_2^2)\sin u\,, \end{cases} \tag{7}$$

we only chose the pair of Lenard's operators K, J as

$$K = \partial^{-1}, \quad J = \tfrac{1}{4}(\partial^{-1}\cos u\partial^{-1}\cos u\partial^{-1} + \partial^{-1}\sin u\partial^{-1}\sin u\partial^{-1})\,,$$
$$\partial = \partial/\partial x, \quad \partial^{-1}\partial = \partial\partial^{-1} = 1\,, \tag{8}$$

then Eq. (6) holds. Obviously, K and J are two skew-symmetric operators.

Choose $G_{-1} = 0 \in Ker\,K$, define the Lenard's recursive sequence $\{G_j\}_{j=-1}^{-\infty}$ as follows

$$G_{j-1} = \mathscr{L} G_j, \qquad \mathscr{L} \triangleq K^{-1}J = \tfrac{1}{4}(\cos u\partial^{-1}\cos u\partial^{-1} + \sin u\partial^{-1}\sin u\partial^{-1}),$$
$$j = -1, -2, \ldots . \tag{9}$$

The vector fields $X_m \triangleq JG_m (m = -1, -2, \ldots)$ produce a new hierarchy of NLEEs:

$$u_{t_m} = X_m(u), \quad m = -1, -2, \ldots . \tag{10}$$

As $m = -1$, let $t = t_{-1}$, $\partial^{-1}0 = 0$, $\partial^{-1}\cos u\partial^{-1}0 = \alpha(t)$ and $\partial^{-1}\sin u\partial^{-1}0 = \beta(t)$, then Eq. (10) reads

$$u_t = X_{-1}(u) = JG_{-1} = \tfrac{1}{4}\partial^{-1}(\alpha(t)\cos u + \beta(t)\sin u), \tag{11}$$

where $\alpha(t)$ and $\beta(t)$ are two arbitrary smooth scalar functions. Set $\alpha(t) = 0$ and $\beta(\mathrm{t}) = 4$, then Eq. (11) exactly becomes the well-known SEGE (1). For that reason, $X_m = JG_m$ $(m = -1, -2, \ldots)$ are called the SG vector fields, and *the hierarchy* Eq. (10) *is called the sine-Gordon hierarchy of* NLEEs.

2.2. *Lax representations of the sine-Gordon hierarchy* (10)

Rewrite Eq. (4) as

$$Ly = \lambda^{-1}y, \quad L = L(u) = 4\begin{pmatrix} \cos u & \sin u \\ \sin u & -\cos u \end{pmatrix}\partial, \quad \partial = \partial/\partial x. \tag{12}$$

Apparently, we have

Proposition 1.

(i)
$$I_{2\times 2}\partial = \tfrac{1}{4}WL\,, \tag{13}$$

where

$$I_{2\times 2} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad W = \begin{pmatrix} \cos u & \sin u \\ \sin u & -\cos u \end{pmatrix}. \tag{14}$$

(ii) The Gateau derivative operator [27] $L_*(\xi)$ of the spectral operator $L = L(u)$ in the direction ξ is

$$L_*(\xi) \triangleq \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} L(u + \varepsilon\xi) = \begin{pmatrix} 0 & -\xi \\ \xi & 0 \end{pmatrix} L\,. \tag{15}$$

(iii) L_* is an injective homomorphism, i.e.

$$L_*(\xi) = 0 \Leftrightarrow \xi = 0, \qquad L_*(a\xi + b\eta) = aL_*(\xi) + bL_*(\eta), \quad \forall\, a, b \in R.$$

In order to construct the Lax representations of the sine-Gordon hierarchy (10), according to the frame structure of commutator representations [25, 26] we consider the following operator equation of $V = V(G)$:

$$[V, L] = L_*(KG)L^{-1} - L_*(JG)L, \tag{16}$$

where G is an arbitrary smooth function; $[\cdot,\cdot]$ stands for the commutator; K, J, L and L_* are expressed by Eqs. (8), (12) and (15), respectively.

Theorem 1. Let G be an arbitrary given smooth function. Let K, J, and L_* be defined by Eqs. (8) and (15), respectively. Then, for the spectral problem (12) the operator equation (10) possesses the following operator solution:

$$V = V(G) = \frac{1}{4}\begin{pmatrix} -\partial^{-1}\sin u\partial^{-1}G & -\partial^{-1}\cos u\partial^{-1}G \\ \partial^{-1}\cos u\partial^{-1}G & \partial^{-1}\sin u\partial^{-1}G \end{pmatrix} - \tfrac{1}{8}(\partial^{-1}\cos u\partial^{-1}\cos u\partial^{-1}G + \partial^{-1}\sin u\partial^{-1}\sin u\partial^{-1}G)\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}L. \tag{17}$$

Proof. Set

$$\begin{cases} V_0 = \frac{1}{4}\begin{pmatrix} -\partial^{-1}\sin u\partial^{-1}G & \partial^{-1}\cos u\partial^{-1}G \\ \partial^{-1}\cos u\partial^{-1}G & \partial^{-1}\sin u\partial^{-1}G \end{pmatrix}, \\ V_1 = -\frac{1}{8}(\partial^{-1}\cos u\partial^{-1}\cos u\partial^{-1}G + \partial^{-1}\sin u\partial^{-1}\sin u\partial^{-1}G)\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \end{cases} \tag{18}$$

Then the commutator $[V, L]$ of $V = V(G) = V_0 + V_1 L$ and L is

$$[V, L] = -4WV_{0x} + (V_0 - WV_0W - 4WV_{1x})L + (V_1 - WV_1W)L^2. \tag{19}$$

Substituting Eqs. (18) and (14) into Eq. (19), and calculating it, we find

$$\begin{cases} V_0 - WV_0W - 4WV_{1x} = 0, \\ -4WV_{0x} = L_*(KG)L^{-1}, \\ V_1 - WV_1W = -L_*(JG)L^{-1}, \end{cases} \tag{20}$$

which completes the proof Theorem 1. □

Theorem 2. Let $\{G_j\}_{j=-1}^{-\infty}$ be defined by the Lenard's recursive sequence (9). Then the sine-Gordon hierarchy (10), has the Lax representations (operator form)

$$\begin{cases} L_{t_m} = [W_m, L], \quad m = -1, -2, \ldots, \\ W_m = -\sum_{j=m}^{-1} V(G_j)L^{2(m-j)-1}, \end{cases} \tag{21}$$

where $V(G_j)$ is the expression (17) with $G = G_j$.

Proof. On one hand, $L_{t_m} = L_*(u_{t_m})$. On the other hand,

$$[W_m, L] = \left[-\sum_{j=m}^{-1} V(G_j) L^{2(m-j)-1}, L \right] = -\sum_{j=m}^{-1} [V(G_j), L] L^{2(m-j)-1}$$

$$= -\sum_{j=m}^{-1} (L_*(KG_j) L^{2(m-j-1)} - L_*^{2(m-j)})$$

$$= L_*(JG_m) - L_*(KG_{-1}) L^{2m} = L_*(X_m) .$$

Hence, $L_{t_m} = [W_m, L] \Leftrightarrow L_*(u_{t_m}) = L_*(X_m) \overset{\text{(iii) of Proposition 1}}{\Longleftrightarrow} u_{t_m} = X_m$. □

2.3. A new Lax pair of SEGE (1)

In Theorem 2 letting $m = -1$ and $t_{-1} = t$, then according to Eqs. (11) and (17) we can obtain a new Lax pair (operator form) of the SEGE (1).

Proposition 2. SEGE (1) possesses the following Lax representation:

$$L_t = [W_{-1}, L] , \tag{22}$$

where the operators L and W_{-1} are

$$L = L(u) = 4 \begin{pmatrix} \cos u & \sin u \\ \sin u & -\cos u \end{pmatrix} \partial , \tag{23$_1$}$$

$$W_{-1} = -V(G_{-1}) L^{-1} = \frac{1}{4} \begin{pmatrix} \partial^{-1} \cos u & \partial^{-1} \sin u - 2(\partial^{-1} \sin u) \\ -\partial^{-1} \sin u + 2(\partial^{-1} \sin u) & \partial^{-1} \cos u \end{pmatrix}, \tag{23$_2$}$$

here $(\partial^{-1} \sin u)$ is a function (i.e, $(\partial^{-1} \sin u) f = f(\partial^{-1} \sin u)$, for an arbitrary function f), and $\partial^{-1} \cos u$, $\partial^{-1} \sin u$ are the two operators (i.e, $\partial^{-1} \cos u f = \partial^{-1}(f \cos u)$, $\partial^{-1} \sin u f = \partial^{-1}(f \sin u)$, for an arbitrary function f).

Proof. Substituting Eqs. $(23)_1$ and $(23)_2$ into Eq. (22), we can know that Eq. (22) is equivalent to Eq. (1) through a lengthy calculations. □

Remark 1. Eqs. $(23)_1$ and $(23)_2$ can be considered to generate the non-linearized integrable systems [28] of SEGE (1) under some constraints between the potential function u and the eigenfunction vector y of Eq. (4), which is in organization.

3. The sinh-Gordon hierarchy, Lax representations and a new Lax pair of SHGE (2)

It has been known that the SHGE (2) is associated with the following Lax pair [13] (called the old Lax pair of Eq. (2))

$$y_x = \begin{pmatrix} \lambda & -\frac{1}{2}u_x \\ \frac{1}{2}u_x & -\lambda \end{pmatrix} y \quad \text{(spatial part)}\,, \tag{24$_1$}$$

$$y_t = \frac{1}{4\lambda}\begin{pmatrix} \operatorname{ch} u & -\operatorname{sh} u \\ \operatorname{sh} u & -\operatorname{ch} u \end{pmatrix} y \quad \text{(time part)}\,. \tag{24$_2$}$$

In this section, for Eq. $(24)_2$ we continue to use the procedure which is analogous to Section 2, produce the so-called sinh-Gordon hierarchy, and gives its Lax representations and the new Lax pair of the SHGE (2). Thus, we change Eq. $(24)_2$ as the following spectral problem:

$$y_x = \frac{1}{4\lambda}\begin{pmatrix} \operatorname{ch} u & -\operatorname{sh} u \\ \operatorname{sh} u & -\operatorname{ch} u \end{pmatrix} y\,. \tag{25}$$

We easily calculate the following results:

$$\delta\lambda/\delta u = (4\lambda)^{-1}(-(y_2^2 + y_1^2)\operatorname{ch} u - 2y_1 y_2 \operatorname{sh} u)\,, \tag{26}$$

$$K\delta\lambda/\delta u = \lambda^{-2} J\delta\lambda/\delta\,, \tag{27}$$

where the pair of Lenard's operators K, J are the two skew-symmetric operators

$$K = \partial^{-1}, \quad J = \tfrac{1}{4}(\partial^{-1}\operatorname{ch} u\partial^{-1}\operatorname{ch} u\partial^{-1} - \partial^{-1}\operatorname{sh} u\partial^{-1}\operatorname{sh} u\partial^{-1}),$$

$$\partial = \partial/\partial x, \quad \partial\partial^{-1} = \partial^{-1}\partial = 1\,. \tag{28}$$

The Lenards's sequence $\{G_j\}_{j=-1}^{-\infty}$ are recursively defined by

$$G_{j-1} = \mathscr{L} G_j, \qquad G_{-1} = 0 \in Ker\, K, \quad j = -1, -2, \ldots,$$

$$\mathscr{L} = K^{-1}J = \tfrac{1}{4}(\operatorname{ch} u\partial^{-1}\operatorname{ch} u\partial^{-1} - \operatorname{sh} u\partial^{-1}\operatorname{sh} u\partial^{-1})\,. \tag{29}$$

Here, we make the convention: $\partial^{-1}0 = 0$, $\partial^{-1}\operatorname{ch} u\partial^{-1}0 = \gamma(t)$, $\partial^{-1}\operatorname{sh} u\partial^{-1}0 = \delta(t)$ ($\gamma(t)$, $\delta(t)$ are two arbitrary functions). Then we can produce *the sinh-Gordon hierarchy of NLEEs* as follows:

$$u_{t_m} = X_m(u) = JG_m, \quad m = -1, -2, \ldots, \tag{30}$$

with the representative equation

$$u_{t_m} = X_{-1}(u) = \tfrac{1}{4}\partial^{-1}(\gamma(t)\operatorname{ch} u - \delta(t)\operatorname{sh} u), \quad t = t_{-1} \tag{31}$$

which can become the well-known SEGE (2) as $\gamma(t) = 0$, $\delta(t) = -4$. Eq. (25) reads

$$Ly = \lambda^{-1} y, \quad L = 4\tilde{M}\partial, \quad \tilde{M} = \begin{pmatrix} \operatorname{ch} u & -\operatorname{sh} u \\ \operatorname{sh} u & -\operatorname{ch} u \end{pmatrix}. \tag{32}$$

Hence,

$$L_*(\xi) = \begin{pmatrix} 0 & \xi \\ \xi & 0 \end{pmatrix} L, \tag{33}$$

and L_* is injective.

Theorem 3. Let G be an arbitrary given smooth function. For the spectral problem (32), we consider the operator equation of $V = V(G)$ generated by Eqs. (28) and (33)

$$[V, L] = L_*(KG)L^{-1} - L_*(JG)L. \tag{34}$$

Then Eq. (34) has the operators solution

$$V = V(G) = \frac{1}{4}\begin{pmatrix} \partial^{-1}\operatorname{sh} u\partial^{-1}G & -\partial^{-1}\operatorname{ch} u\partial^{-1}G \\ \partial^{-1}\operatorname{ch} u\partial^{-1}G & -\partial^{-1}\operatorname{sh} u\partial^{-1}G \end{pmatrix}$$
$$-\frac{1}{8}\partial^{-1}(\operatorname{ch} u\partial^{-1}\operatorname{ch} u - \operatorname{sh} u\partial^{-1}\operatorname{sh} u)\partial^{-1}G\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} L. \tag{35}$$

Theorem 4. Let $\{G_j\}_{j=-1}^{-\infty}$ be defined by Eq. (29). Then the sinh-Gordon hierarchy (30) possesses the Lax representations (operator form)

$$L_{t_m} = [W_m, L], m = -1, -2, \dots,$$
$$W_m = -\sum_{j=m}^{-1} V(G_j)L^{2(m-j)-1}, \tag{36}$$

where $V(G_j)$ is the expression (35) with $G = G_j$, and L is determined by Eq. (32).
As $m = -1$, setting $t_{-1} = t$, we get

Proposition 3. The SHGE (2) has the following new Lax representation

$$L_t = [W_{-1}, L], \tag{37}$$

where the operator L and W_{-1} are

$$L = 4\begin{pmatrix} \operatorname{ch} u & -\operatorname{sh} u \\ \operatorname{sh} u & -\operatorname{ch} u \end{pmatrix}\partial, \tag{38$_1$}$$

$$W_{-1} = -V(G_{-1})L^{-1} = \frac{1}{4}\begin{pmatrix} \partial^{-1}\operatorname{ch} u & -\partial^{-1}\operatorname{sh} u + 2(\partial^{-1}\operatorname{sh} u) \\ -\partial^{-1}\operatorname{sh} u + 2(\partial^{-1}\operatorname{sh} u) & \partial^{-1}\operatorname{ch} u \end{pmatrix}, \tag{38$_2$}$$

where $(\partial^{-1}\,\mathrm{sh}\,u)$ is a function (i.e. $(\partial^{-1}\,\mathrm{sh}\,u)f = f(\partial^{-1}\,\mathrm{sh}\,u)$, for an arbitrary function f), and $\partial^{-1}\,\mathrm{ch}\,u\,\partial^{-1}\,\mathrm{sh}\,u$ are the two operators (i.e. $\partial^{-1}\,\mathrm{ch}\,u\,f = \partial^{-1}(f\,\mathrm{ch}\,u)$, $\partial^{-1}\,\mathrm{sh}\,u f = \partial^{-1}(f\,\mathrm{sh}\,u)$, for an arbitrary function f).

Proof. Direct calculations. □

Remark 2. Eqs. $(38)_1$ and $(38)_2$ can be considered to generate the nonlinearized integrable system [28] of SEHE (2) under some constraints between the potential function u and the eigenfunction vector y of Eq. (25), which can be developed.

4. Discussion and explanation

1. In this article, following the example of SEGE (1) and SHGE (2), we present a kind of method on how to generate a new hierarchy of NLEEs and obtain new Lax pair from the old Lax pair. The spatial part of new Lax pair is actually given by the time part of old Lax pair (see Eqs. $(23)_1$ and $(3)_2$, Eqs. $(38)_1$ and $(24)_2$). The time part of new Lax pair is different from the spatial part of old Lax pair (see Eqs. $(23)_2$ and $(3)_1$, Eqs. $(38)_2$ and $(24)_1$). Denote the old Lax pair by L_0, M_0, new Lax pair by L_1, M_1 (here $L_1 = M_0$). Then from L_1 and M_1, in light of the method descripted as above, we can further obtain another new Lax pair L_2, M_2 ($L_2 = M_1$)? For the sine-Gordon and sinh-Gordon equation, the answer is "No" (because the associated spectral gradient is very difficult to calculate). For other soliton equations, this problem is still open.

2. The recursion operators $\mathscr{L} = K^{-1}J$ (see Eqs. (9) and (29)) of the sine-Gordon and sinh-Gordon hierarchy are apparently either different from (/inequivalent to) the recursion operator $\mathscr{D} = D^2 + u_x^2 - u_x D^{-1} - u_{xx}$ ($D = \partial/\partial x$) used in Ref. [16] or different from (/inequivalent to) the inverse operator of the recursion operator $\Phi = D^2 - 4u^2 - 4u_x D^{-1} u$ ($D^{-1}D = DD^{-1} = 1$, $D = \partial/\partial x$) presented in Ref. [19]. Additionally, the new Lax pairs (22) and (37) of SEGE (1) and SHGE (2) are different from (/inequivalent to) the Lax pairs obtained in Ref. [17] and Ref. [18], respectively, and Eqs. (22), (37) are simpler than the Lax pairs in Refs. [17,18] in their expressions.

3. Because the Lenard's operators K, J (see Eqs. (8) and (28)) of the sine-Gordon and sinh-Gordon hierarchy are skew-symmetric, we may further induce the Hamiltonian property of K, J, and prove that the sine-Gordon and sinh-Gordon hierarchy possess the bi-Hamiltonian attributes.

4. The operators $W_m = -\sum_{j=m}^{-1} V(G_j)L^{2(m-j)-1}$, which are reckoned by solving an operator equation according to the frame structure of commutator representations [25,26], are called the Lax operators [29]. We can prove that the Lax operators W_m compose an infinite-dimensional Lie operator algebra.

5. The sine-Gordon and sinh-Gordon hierarchy obtained in this article are actually the second hierarchy [26, 1994] or the negative-order [30] of NLEEs related to Eqs. (4) and (25). It is well-known that, the "nonlinearization technique [31] of Lax pair"

is valid for much more soliton systems in finding completely integrable systems in the Liouville sense. However, in these famous soliton systems (such as KdV, MKdV, Bousinessq, etc.) discussed before, only the sine-Gordon and sinh-Gordon systems have not been considered. Now, we have obtained a new Lax representations (operator form) of the sine-Gordon and sinh-Gordon hierarchy, then we hope that their nonlinearized systems and associated finite-dimensional completely integrable system are developed, which is in study. If that will do, then it is very likely to find a new approach for getting the exact solutions of sine-Gordon and sinh-Gordon hierarchy.

Acknowledgements

The author would like to express his sincere thanks to Prof. Gu Chaohao and Prof. Hu Hesheng for their encouragement and instructions, and is also grateful to Dr. Zhou Zixiang and Dr. Zhou Ruguang for their discussions. The author also acknowledges part support by Science and Technology Foundation of Liaoning Province and Science Research Foundation of Education Committee, Liaoning Province, China.

References

[1] M.J. Ablowitz, H. Segur, Soliton and the Inverse Scattering Transformation, SIAM, Philadelphia, 1981.
[2] G.L. Lamb, Elements of Soliton Theory, Wiley-Interscience, New York, 1980.
[3] L.P. Eisenhart, A Treatise on the Differential Geometry of Curves and Surfaces, Ginn, Boston, reprinted 1960, Dover, New York, 1909, pp. 284.
[4] R. Steuerwald, Abh. Bayer. Akad. Wiss (Muench.) 40 (1936) 1.
[5] M.H. Amsler, Math. Ann. 130 (1995) 234.
[6] H.T. Davis, Introduction to Nonlinear Differential and Integral Equations, United States Atomic Energy Commission, US Government Printing Office, Washington, DC, reprinted, Dover, New York 1960, pp. 185.
[7] W.F. Ames, Nonlinear Partial Differential Equations in Engineering, Academic Press, New York, 1965, pp. 133.
[8] E.L. Ince, Ordinary Differential Equations, New York, reprinted 1956, Dover, New York, 1969, pp. 345.
[9] E. Goursat, Mem. Sci. Math. Fasc., Vol. 6, Gauthier–Villars, Paris, 1925.
[10] H. Wahlquist, F.B. Estabrook, Phys. Rev. Lett. 31 (1973) 1386.
[11] R.M. Miura, Lecture Notes in Math. 515 (1976) 77.
[12] C.H. Gu, Lett. Math. Phys. 12 (1986) 31.
[13] C.H. Gu et al. Soliton Theory and its Applications, Ch. 3, Springer, Berlin, 1995.
[14] V.E. Zakharov, A.B. Shabat, Sov. Phys. JETP 34 (1972) 62.
[15] M.J. Ablowitz, D.J. Kaup, A.C. Newell, H. Segur, Stud. Appl. Math. 53 (1974) 249.
[16] P.J. Olver, J. Math. Phys. 18 (1977) 1212.
[17] M. Bruschi, D. Levi, O. Ragnisco, Phys. Lett. A 88 (1982) 379.
[18] M. Bruschi, O. Ragnisco, Il Nuovo Cimento B 88 (1988) 119.
[19] S. Lou, Phys. Lett. B 302 (1993) 261.
[20] H.W. Capel, F.W. Nijhoff, Integrable lattice equation, in: A.S. Fokas, V.E. Zakharov (Eds.), Important Developments in Soliton Theory, Lecture Notes in Nonlinear Dynamics, Berlin, 1993, pp. 38.

[21] F.W. Nijhoff, H.W. Capel, Acta Appl. Math. 39 (1995) 133.
[22] F.W. Nijhoff, H.W. Capel, G.L. Wiersma, G.R.W. Quispel, Phys. Lett. A 105 (1984) 267.
[23] F.W. Nijhoff, V.G. Papageorgiou and H.W. Capel, Integrable time-discrete systems: lattices and mappings, in: P.P. Kulish (Ed.), Quantum Groups, Lecture Notes in Mathematics, vol. 1510, Springer, Berlin, 1992, pp. 312.
[24] F.W. Nijhoff, O. Ragnisco, V.B. Kuznetsov, Commun. Math. Phys. 176 (1996) 681.
[25] Z. Qiao, Acta Math. Appl. Sinica 18 (1995) 287 (in Chinese).
[26] Z. Qiao, Phys. Lett. A 206 (1995) 347; Phys. Lett. A 195 (1994) 319.
[27] C. Cao, Chin. Sci. Bull. 34 (1989) 723.
[28] Z. Qiao, The nonlinearized integrable systems for the sine-Gordon and sinh-Godon equation, in preparation.
[29] W. Ma, J. Phys. A 25 (1992) 5329.
[30] R. Zhou, J. Math. Phys. 36 (1995) 4220.
[31] C. Cao, Sci. China A 33 (1990) 528 (1990).