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Algebraic structure of the operator related to stationary systems

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Abstract

The algebraic structure of the operator connected with the stationary systems of non-linear evolution equations (NLEEs) is presented in this paper. The relations between this kind of algebraic structure and stationary systems are further discussed.

Stationary equations of some soliton hierarchies (such as the KdV hierarchy, etc.) are higher-order nonlinear ODEs which often possess Lagrangian and Hamiltonian structure [1,2]. Such systems are known to be finite-dimensional systems and very often completely integrable [2]. In Ref. [3], Dickey extensively studied the stationary systems of the KdV hierarchy and the matrix hierarchy and gave their corresponding Hamiltonian structure, first integration, action-angle variables and Baker functions.

In 1987, Antonowicz, Fordy and Rauch-Wojciechowski [4] presented a Miura map between the finite-dimensional phase spaces of stationary flows of nonlinear evolution equations and used this to construct a finite bi-Hamiltonian structure for such systems. Afterwards, the so-called ''nonlinearization of Lax equations'' was proposed by Cao [5,6], and the constrained flows [7] of integrable PDEs or the ristricted flows of the soliton hierarchy were put forward by Antonowicz and Rauch-Wojciechowski. Based on their works, Rauch-Wojciechowski introduced the Newton representation for stationary flows of the KdV hierarchy in 1992. All these results are very interesting. There are very few understood algebraic or geometric structure properties connected with the stationary systems. In this paper, we try to construct the algebraic structure of an operator related to the stationary flows.

Recently, we have constructed a so-called operator pattern for generating hierarchies of generalized nonlinear evolution equations (GNLEEs), and have given the generalized Lax representation (GLR) [10]. Now, by making use of this kind of GLR, we further consider the algebraic structure of the operator related to the stationary systems of NLEEs.

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First, let us recall the procedure of GLR in Ref. [10]. Consider an ordinary $N \times N$ spectral problem

$$Ly \equiv L(u)\, y = \lambda y, \tag{1}$$

where $L = L(u)$ is a spectral operator, $u = (u_1, \dots, u_l)^{\mathrm{T}}$ is a potential vector function, λ is a spectral parameter, $y = (y_1, \dots, y_N)^{\mathrm{T}}$. According to the spectral gradient method (SGM) [11], we can always find a pair of operators $K = K(u)$, $J = J(u)$ (called the pair of Lenard's operators) or an integro-differential operator $\mathscr{L}$ (called the recursion operator) such that

$$K \nabla_u \lambda = \lambda^c \cdot J \nabla_u \lambda, \qquad c = \text{const.} \tag{2}$$

or

$$\mathscr{L} \nabla_u \lambda = \lambda^c \cdot \nabla_u \lambda, \qquad \mathscr{L} = J^{-1} K, \tag{2'}$$

where $\nabla_u \lambda = \delta\lambda / \delta u$ is the spectral gradient of the spectral parameter λ of (1) with respect to the potential u.

Denote the Gateaux derivative operator $L_*(\xi)$ of the spectral operator $L(u)$ in the direction ξ by [12]

$$L_*(\xi) \triangleq \left.\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0} L(u + \varepsilon\xi). \tag{3}$$

Throughout this paper, we always assume that L_* is an injective homomorphism. For two arbitrary given $N \times N$ matrix operators $M = (m_{ij})_{N \times N}$, $\tilde{M} = (\tilde{m}_{ij})_{N \times N}$, we construct a pair of operator equations with respect to the two undetermined vector functions $G_0 = (G_0^{(1)}, \dots, G_0^{(l)})^{\mathrm{T}}$ and $G_{-1} = (G_{-1}^{(1)}, \dots, G_{-1}^{(l)})^{\mathrm{T}}$,

$$L_*(JG_0) = M, \tag{4}$$

$$L_*(KG_{-1}) = \tilde{M}, \tag{5}$$

where K, J are the pair of Lenard's operators. Write the solution set of (4), (5) as $\mathscr{N}_J(M)$, $\mathscr{N}_K(\tilde{M})$ (generally, $\mathscr{N}_J(M) \neq \varnothing$, $\mathscr{N}_K(\tilde{M}) \neq \varnothing$), respectively.

Suppose $\mathscr{N}_J(M) \neq \varnothing$, $\mathscr{N}_K(\tilde{M}) \neq \varnothing$, choose $G_0 \in \mathscr{N}_J(M)$, $G_{-1} \in \mathscr{N}_K(\tilde{M})$, and define the Lenard's recursive sequence $(G_j)_{j=-\infty}^{\infty}$ as

$$G_0 \in \mathscr{N}_J(M), \qquad G_{j+1} = J^{-1} K G_j = \mathscr{L}^{j+1} G_0, \qquad j \geqslant 0, \tag{6}$$

$$G_{-1} \in \mathscr{N}_K(\tilde{M}), \qquad G_{j-1} = K^{-1} J G_j = \mathscr{L}^{j} G_{-1}, \qquad j < 0. \tag{7}$$

The generalized vector fields (GVF) $\{X_m\}_{m=-\infty}^{\infty}$ defined by

$$\begin{aligned} X_m = X_m(u) &= KG_m = K\mathscr{L}^m G_0, \qquad m \geqslant 0, \\ &= JG_m = J\mathscr{L}^{m+1} G_{-1}, \quad m < 0, \end{aligned} \tag{8}$$

produce the so-called hierarchy of generalized nonlinear evolution equations (GNLEEs) of (1):

$$u_{t_m} = X_m, \qquad m \in \mathbb{Z}. \tag{9}$$

Evidently, for different M and $\tilde{M}$, (9) is different, and for different G_0 and G_{-1}, (9) is also different. So, using the new pattern (4) and (5), we can generate many new hierarchies of NLEEs associated with (1), and for this reason, we call (9) the generalized nonlinear evolution equations. If $M = \tilde{M} = 0$, i.e. $G_0 \in \operatorname{Ker} J$, $G_{-1} \in \operatorname{Ker} K$, then (9) is exactly the isospectral hierarchy of (1) whose Lax representations have been studied in Ref. [13], and if $M = \tilde{M} = I_{N \times N}$ ($I_{N \times N}$ is an $N \times N$ unit operator) and $\mathscr{N}_J(I_{N \times N}) \neq \varnothing$, $\mathscr{N}_K(I_{N \times N}) \neq \varnothing$, then from Theorem 1 we can know that (9) is actually the non-isospectral hierarchy of (1).

Theorem 1. Let M, $\tilde{M}$ be two arbitary given $N\times N$ matrix operators. For the spectral problem (1), suppose
(i) $\mathcal{N}_J(M)\neq\emptyset$, $\mathcal{N}_K(\tilde{M})\neq\emptyset$;
(ii) for arbitary $G=(G^{(1)},\dots,G^{(1)})^{\mathrm{T}}$, the operator equation produced by the pair of Lenard's operators K, J,

$$[V,\ L]=L_*(KG)L^{\beta}-L_*(JG)L^{\alpha}, \tag{10}$$

possesses the operator solution $V=V(G)$, where α, $\beta=\text{const}$, and $\alpha>\beta$.

Then the hierarchy of GNLEEs (9) has the following Lax representation form (called the generalized Lax representation (GLR)

$$L_{t_m}=[W_m,\ L]+ML^{(m+1)\eta},\qquad m\geqslant 0, \tag{11}$$

$$L_{t_m}=[W_m,\ L]+\tilde{M}L^{m\eta},\qquad m<0, \tag{12}$$

$$\begin{aligned} W_m&=\sum_{j=0}^{m}V(G_j)L^{(m-j)\eta-\beta},\qquad m\geqslant 0,\\ &=-\sum_{j=m}^{-1}V(G_j)L^{(m-j)\eta-\alpha},\quad m<0, \end{aligned} \tag{13}$$

where $\eta=\text{const}=\alpha-\beta>0$. The operators W_m $(m\in\mathbb{Z})$ are called the generalized Lax operators (GLOs) of (9).

Proof. For $m\geqslant 0$,

$$\begin{aligned}[W_m,\ L]&=\sum_{j=0}^{m}\left[V(G_j),\ L\right]L^{(m-j)\eta-\beta}=\sum_{j=0}^{m}\left\{L_*(KG_j)L^{(m-j)\eta}-L_*(JG_j)L^{(m-j+1)\eta}\right\}\\ &=\sum_{j=0}^{m}\left\{L_*(JG_{j+1})L^{(m-j)\eta}-L_*(JG_j)L^{(m-j+1)\eta}\right\}=L_*(X_m)-L_*(JG_0)L^{(m+1)\eta}.\end{aligned}$$

Thus,

$$[W_m,\ L]=L_*(X_m)-ML^{(m+1)\eta}. \tag{14}$$

$L_*(u_{t_m})=L_{t_m}$ and L_* is injective. Hence

$$L_{t_m}=[W_m,\ L]+ML^{(m+1)\eta}\quad\Leftrightarrow\quad L_*(u_{t_m}-X_m)=0\quad\Leftrightarrow\quad u_{t_m}=X_m.$$

For $m<0$, (12) may be similarly proven.

By (14) and the injection of L_*, we immediately obtain

Corollary 1. The potential vector function u satisfies a stationary system

$$\sum_{i=-r}^{s}c_iX_i(u)=0,\qquad\forall r,\ s\in\mathbb{Z}^+, \tag{15}$$

if and only if

$$\left[\sum_{i=-r}^{s}c_iW_i,\ L\right]=-M\sum_{i=0}^{s}c_iL^{(i+1)\eta}-\tilde{M}\sum_{i=-r}^{-1}c_iL^{i\eta}, \tag{16}$$

where the constants c_i $(-r\leqslant i\leqslant s)$ are independent of x.

Remark. Ordering $M = \tilde{M} = 0$, (11) and (12) can be put together as $L_{t_m} = [W_m, L]$, $m \in \mathbb{Z}$ which is actually the standard Lax representation of the isospectral ($\lambda_t = 0$) hierarchy of (1) (see Ref. [13]). Ordering $M = \tilde{M} = I_{N\times N}$, as $\mathscr{N}_J(I_{N\times N}) \neq \varnothing$, $\mathscr{N}_K(I_{N\times N}) \neq \varnothing$, (11) and (12) become

$$L_{t_m} = [W_m, L] + L^{(m+1)\eta}, \qquad m \geqslant 0, \tag{17}$$

$$L_{t_m} = [W_m, L] + L^{m\eta}, \qquad m < 0, \tag{18}$$

which are exactly the Lax representations of the non-isospectral ($\lambda_t = \lambda^{(m+1)\eta}$, $m \geqslant 0$; $\lambda_t = \lambda^{m\eta}$, $m < 0$; $\eta = \alpha - \beta > 0$) hierarchy,

$$\begin{aligned} U_{t_m} &= KG_m = K\mathscr{L}^m G_0, \qquad G_0 \in \mathscr{N}_J(I_{N\times N}), \qquad m \geqslant 0, \\ &= JG_m = J\mathscr{L}^{m+1} G_{-1}, \qquad G_{-1} \in \mathscr{N}_K(I_{N\times N}), \qquad m < 0. \end{aligned} \tag{19}$$

Now, we construct the algebraic structure of the operator associated with (16). First, we give some basic symbols and notations. Let $x \in \mathbb{R}^P$, $t \in \mathbb{R}$, and $\mathscr{B}$ stands for all complex (or real) functions $P = P(x, t, u)$ which are C^∞-differentiable with respect to x, t and C^∞-Gateaux differentiable with respect to $u(x)$. $\mathscr{B}^N = \{(P_1, P_2, \dots, P_N)^T \mid P_i \in \mathscr{B}, 1 \leqslant i \leqslant N\}$. $\mathscr{V}^N$ stands for all linear operators $\Phi = \Phi(x, t, u)$: $\mathscr{B}^N \to \mathscr{B}^N$ which are C^∞-differentiable with respect to x, t and C^∞-Gateaux differentiable with respect to $u(x)$. For a given operator $L \in \mathscr{V}^N$, by $\mathscr{V}_L^N$ we denote all matrix differentiable operators S with the form $S = \sum_{\alpha \in \mathbb{Z}^+} P_\alpha(u) L^\alpha$, where $P_\alpha(u) \in \mathscr{B}$, $\sum_{\alpha \in \mathbb{Z}^+}$ is a finite sum. [,] stands for the commutator.

Definition 1. For an arbitary given integer $m \in \mathbb{Z}$ ($m \neq 0$), and spectral operator $L \in \mathscr{V}^N$, if there exists a pair of operators A, M such that

$$[A, L] = -ML^m, \qquad A \in \mathscr{V}^N, \qquad M \in \mathscr{V}^N \tag{20}$$

then (A, M) is called the pair of stationary operators of L. We denote by S_L^m all these pairs (A, M) satisfying (20).

Definition 2. For $m \in \mathbb{Z}$ ($m \neq 0$), let $(A, M), (B, N) \in S_L^m$. We define the product of (A, M) and (B, N) as

$$[\![(A, M), (B, N)]\!] = ([A, B], [\![M, N]\!]), \tag{21}$$

where

$$[\![M, N]\!] = [M, B] - [N, A] + \sum_{j=0}^{m-1} [ML^{m-1-j}, \quad NL^j], \qquad m \geqslant 0, \tag{22}$$

$$= [M, B] - [N, A] + \sum_{j=m}^{-1} [NL^{m-1-j}, \quad ML^j], \qquad m < 0. \tag{23}$$

Apparently, the multiplication operation (21) is a skew-symmetric and bilinear binary operation. In order to prove that operation (21) is closed in S_L^m, we present several Lemmas.

Lemma 1. Let $(A, M) \in S_L^m$. Then for arbitary $k \in \mathbb{Z}$ ($k \neq 0$)

$$[A, L^k] = -\sum_{j=0}^{k-1} L^{k-1-j} M L^{j+m}, \qquad k > 0, \tag{24}$$

$$= \sum_{j=k}^{-1} L^{k-1-j} M L^{j+m}, \qquad k < 0. \tag{25}$$

Proof. For $k>0$, we use the mathematical induction method. As $k=1$, (24) obviously holds. Suppose (24) is correct for all integers $l \leqslant k$, then

$$[A, L^{k+1}] = L[A, L^k] + [A, L^k]L - L[A, L^{k-1}]L$$
$$= -\sum_{j=0}^{k-1} (L^{k-j}ML^{j+m} + L^{k-1-j}ML^{j+1+m}) + \sum_{j=0}^{k-2} L^{k-1-j}ML^{j+1+m}$$
$$= -\sum_{j=0}^{k-1} L^{k-j}ML^{j+m} - ML^{m+k} = -\sum_{j=0}^{k} L^{k-j}ML^{j+m}.$$

For $k<0$,

$$[A, L^k] = -L^k[A, L^{-k}]L^k = L^k \sum_{j=0}^{-k-1} L^{-k-1-j}ML^{j+m}L^k = \sum_{j=0}^{-k-1} L^{-1-j}ML^{j+k+m} = \sum_{j=k}^{-1} L^{k-1-j}ML^{j+m}.$$

Lemma 2. Let $(A, M), (B, N) \in S_L^m$ $(m \in \mathbb{Z},\ m \neq 0)$, then we have

$$M[B, L^m] - N[A, L^m] = -\sum_{j=0}^{m-1} [ML^{m-1-j}, NL^j]L^m, \qquad m>0, \tag{26}$$
$$= -\sum_{j=m}^{-1} [NL^{m-1-j}, ML^j]L^m, \qquad m<0. \tag{27}$$

Proof. For $m>0$,

$$M[B, L^m] - N[A, L^m] = -\sum_{j=0}^{m-1} (ML^{m-1-j}NL^{j+m} - NL^{m-1-j}ML^{j+m}) = -\sum_{j=0}^{m-1} [ML^{m-1-j}, NL^j]L^m.$$

For $m<0$, (27) is similarly proven.

Theorem 2. Let $(A, M), (B, N) \in S_L^m$ $(m \in \mathbb{Z},\ m \neq 0)$, then $[\![(A, M), (B, N)]\!] \in S_L^m$. Thus S_L^m constitutes an algebra under the multiplication operation (21).

Proof. Since the commutator $[\ ,\ ]$ satisfies the Jacobi identity, we have

$$[[A, B], L] = -[[B, L], A] + [[A, L], B] = [NL^m, A] - [ML^m, B]$$
$$= N[L^m, A] - M[L^m, B] + ([B, M] + [N, A])L^m$$
$$\begin{cases} = -\left(\sum_{j=0}^{m-1} [ML^{m-1-j}, NL^j] + [M, B] - [N, A]\right) L^m, & m>0, \\ = -\left(\sum_{j=m}^{-1} [NL^{m-1-j}, ML^j] + [M, B] - [N, A]\right) L^m, & m<0, \end{cases}$$
$$= -[\![M, N]\!]L^m.$$

Hence, $[\![(A, M), (B, N)]\!] \in S_L^m$.

From Corollary 1, we know that u satisfies

$$X_i(u)=0, \qquad i\geqslant 0 \quad \Leftrightarrow \quad (W_i,\ M)\in S_L^{(i+1)\eta} \qquad (i\geqslant 0),$$

$$X_i(u)=0, \qquad i<0 \quad \Leftrightarrow \quad (W_i,\ M)\in S_L^{i\eta} \qquad (i<0).$$

Thus, by Theorem 2, for two different $(W_i^1,\ M_1)$, $(W_i^2,\ M_2)\in S_L^{(i+1)\eta}$, $i\geqslant 0$ (or $S_L^{i\eta}$, $i<0$), the potential u satisfies $[X_i^1(u),\ X_i^2(u)]=0$ iff

$$\left(\left[W_i^1,\ W_i^2\right],⟦M_1,\ M_2⟧\right)\in S_L^{(i+1)\eta}, \qquad i\geqslant 0 \qquad \left(\text{or } S_L^{i\eta}, \qquad i<0\right),$$

where

$$\left[X_i^1(u),\ X_i^2(u)\right]=\frac{\partial}{\partial\varepsilon}\bigg|_{\varepsilon=0}\left[X_i^1\left(u+\varepsilon X_i^2\right)-X_i^2\left(u+\varepsilon X_i^1\right)\right].$$

In the following, for a fixed matrix operator $M=(m_{ij})_{N\times N}\in\mathscr{V}^N$ we discuss the algebraic structure of the operator related to the ordinary stationary system (15).

Definition 3. Let $L\in\mathscr{V}^N$ be an arbitary given spectral operator. For a fixed matrix operator $M=(m_{ij})_{N\times N}\in\mathscr{V}^N$, if there exists a pair of operators $A,\ P\in\mathscr{V}_L^N$ such that

$$[A,\ L]=-MP, \qquad A, \qquad P\in\mathscr{V}_L^N \tag{28}$$

then $(A,\ P)$ is called the pair of stationary operators of L and M. We denote by S_L^M all these pairs $(A,\ P)$ satisfying (28).

Definition 4. For a fixed $M\in\mathscr{V}^N$, suppose that M is invertible, i.e. $M^{-2}M=MM^{-1}=I_{N\times N}$ ($I_{N\times N}$ is an $N\times N$ unit operator) and let $(A,\ P)$, $(B,\ Q)\in S_L^M$. We define the product of $(A,\ P)$ and $(B,\ Q)$ as

$$⟦(A,\ P),(B,\ Q)⟧=([A,\ B],⟦(P,\ Q)⟧), \tag{29}$$

where

$$⟦P,\ Q⟧=[P,\ B]-[Q,\ A]+M^{-1}[M,\ B]P-M^{-1}[M,\ A]Q. \tag{30}$$

Theorem 3. Let $(A,\ P)$, $(B,\ Q)\in S_L^M$, then $⟦(A,\ P),(B,\ Q)⟧\in S_L^M$. Hence, S_L^M constitutes an algebra under the multiplication operation (29).

Proof. Since $(A,\ P)$, $(B,\ Q)\in S_L^M$, we have

$$[A,\ L]=-MP, \qquad [B,\ L]=-MQ.$$

Thus

$$\begin{aligned}[][[A,\ B],\ L]&=-[[B,\ L],\ A]+[[A,\ L],\ B]=[MQ,\ A]-[MP,\ B]\\&=M[Q,\ A]-M[P,\ B]+[M,\ A]Q-[M,\ B]P=-M⟦P,\ Q⟧.\end{aligned}$$

So, $([A,\ B],⟦(P,\ Q)⟧)\in S_L^M$.

Theorem 4. The multiplication operation (29) is a skew-symmetric and bilinear binary operation, and satisfies the Jacobi identity,

$$⟦⟦(A_1,\ P_1),(A_2,\ P_2)⟧,(A_3,\ P_3)⟧+\text{cycle}\left((A_1,\ P_1),(A_2,\ P_2),(A_3,\ P_3)\right)=0. \tag{31}$$

Thus, by Theorem 3 S_L^M composes a Lie algebra under the multiplication operation (29).

Proof. The skew-symmetric and bilinear binary properties are obvious. Let $(A_1, P_1), (A_2, P_2), (A_3, P_3) \in S_L^M$, then

$$M[\![[\![(P_1, P_2]\!], P_3]\!] = [M[\![P_1, P_2]\!], A_3] - [MP_3, [A_1, A_2]]$$
$$= [[MP_1, A_2] - [MP_2, A_1], A_3] - [MP_3, [A_1, A_2]]$$
$$= [[MP_1, A_2], A_3] - [[MP_2, A_1], A_3] - [MP_3, [A_1, A_2]].$$

So

$$M([\![[\![(P_1, P_2]\!], P_3]\!] + \text{cycle}\,(P_1, P_2, P_3)) = 0.$$

Because M is invertible, $[\![[\![(P_1, P_2]\!], P_3]\!] + \text{cycle}\,(P_1, P_2, P_3) = 0$. Hence,

$$[\![[\![(A_1, P_1), (A_2, P_2)]\!], (A_3, P_3)]\!] + \text{cycle}\,((A_1, P_1), (A_2, P_2), (A_3, P_3))$$
$$= ([[A_1, A_2], A_3] + \text{cycle}\,(A_1, A_2, A_3), [\![[\![(P_1, P_2]\!], P_3]\!] + \text{cycle}\,(P_1, P_2, P_3)]\!]) = 0.$$

Thus, S_L^M composes a Lie algebra under the multiplication operation (29).

Theorem 5. Let $M = I_{N\times N}$ ($I_{N\times N}$ is an $N\times N$ unit operator), and $L \in \mathscr{V}^N$ be a given spectral operator. Suppose that for arbitary $i \in \mathbb{Z}$, there exists an operator W_i, such that $(W_i, L^i) \in S_L^{I_{N\times N}}$. Then

$$[\![L^i, L^j]\!] = (|i| - |j|) L^{i+j-1}, \qquad \forall i, j \in \mathbb{Z}. \tag{32}$$

Thus,

$$[W_i, W_j] = (|i| - |j|) W_{i+j-1}, \qquad \forall i, j \in \mathbb{Z}. \tag{33}$$

Proof. For $(W_i, L^i), (W_j, L^j) \in S_L^{I_{N\times N}}$, we have

$$[W_i, L] = -L^i, \qquad [W_j, L] = -L^j.$$

Hence, for $i, j \geqslant 0$,

$$L^i, L^j = [L^i, W_j] - [L^j, W_i] = \sum_{k=0}^{i-1} L^{i-1-k} \cdot I_{N\times N} \cdot L^{k+j} - \sum_{k=0}^{j-1} L^{j-1-k} \cdot I_{N\times N} \cdot L^{k+i}$$
$$= i \cdot L^{i+j-1} - j \cdot L^{i+j-1} = (i-j) L^{i+j-1},$$

for $i \geqslant 0$, $j \leqslant 0$,

$$[\![L^i, L^j]\!] = [L^i, W_j] - [L^j, W_i] = \sum_{k=0}^{i-1} L^{i-1-k} \cdot I_{N\times N} \cdot L^{k+j} - \sum_{k=j}^{-1} L^{j-1-k} \cdot I_{N\times N} \cdot L^{k+i}$$
$$= i \cdot L^{i+j-1} + j \cdot L^{i+j-1} = (i+j) L^{i+j-1}.$$

Similarly, for $i \leqslant 0$, $j \leqslant 0$ and $i \leqslant 0$, $j \geqslant 0$, we obtain

$$[\![L^i, L^j]\!] = (j-i) L^{i+j-1}, \qquad [\![L^i, L^j]\!] = -(i+j) L^{i+j-1}.$$

So, (32) holds.

Write $\mathscr{L}$ = linear span(L^m, $m \in \mathbb{Z}$). Theorem 5 shows that $\langle \mathscr{L}, [\![\ ,\ ]\!] \rangle$ constitutes an infinite-dimensional Lie algebra and its generators L^m ($m \in \mathbb{Z}$) satisfy (32). Write $\mathscr{W}$ = linear span(W_m, $m \in \mathbb{Z}$), then according to (33), $\mathscr{W}$ also composes an infinite-dimensional Lie algebra which is called the non-isospectral Lax operator algebra of the spectral problem $Ly = \lambda y$.

By Corollary 1 and Theorem 5, we promptly have

Theorem 6. Let $M=\tilde{M}=I_{N\times N}$. For a given spectral operator L, suppose that the conditions in Theorem 1 are satisfied. Then

(i) u satisfies $\sum_{i=-r}^{s} c_i X_i(u)=0$, $\forall r,\ s\in\mathbb{Z}^+$, iff $\sum_{i=-r}^{s+1} c_i(W_i,\ L^{i\eta})\in S_L^{I_{N\times N}}$.

(ii) u satisfies $\sum_{i-j\in\mathbb{Z}} c_i c_j[X_i,\ X_j]=0$ ($\sum_{i,j\in\mathbb{Z}}$ is a finite sum) iff

$$\sum_{i,j\in\mathbb{Z}} c_i c_j(|i|-|j|)\left(W_{i+j-1},\ L^{i+j-1}\right)\in S_L^{I_{N\times N}}.$$

As an application of the operator algebra described as above, we discuss the WKI hierarchy. Consider the WKI spectral problem [14]

$$y_x=\begin{pmatrix}-\mathrm{i}\lambda & \lambda q\\ \lambda r & \mathrm{i}\lambda\end{pmatrix}y,\qquad \mathrm{i}^2=-1, \tag{34}$$

where λ is a spectral parameter, $u=(q,\ r)^{\mathrm{T}}$ is the scalar potential vector function, $y=(y_1,\ y_2)^{\mathrm{T}}$.

It is easy to calculate the spectral gradient $\nabla_u\lambda$,

$$\nabla_u\lambda\triangleq\begin{pmatrix}\delta\lambda/\delta q\\ \delta\lambda/\delta r\end{pmatrix}=\begin{pmatrix}\lambda y_2^1\\ -\lambda y_1^2\end{pmatrix}. \tag{35}$$

Notice that

$$\left(y_1^2\right)_x=-2\mathrm{i}\lambda y_1^2+2\lambda q y_1 y_2,\qquad \left(y_2^2\right)_x=2\mathrm{i}\lambda y_2^2+2\lambda r y_1 y_2,$$

$$(y_1 y_2)_x=\lambda q y_2^2+\lambda r y_1^2,\qquad \left(\frac{r}{p}\right)_x q+\left(\frac{q}{p}\right)_x r=\left(\frac{2}{p}\right)_x.$$

Here $p=\sqrt{1-qr}$. Thus, only choosing

$$K=\frac{1}{2\mathrm{i}}\begin{pmatrix}-\frac{1}{2}\partial^2\frac{q}{p}\partial^{-1}\frac{q}{p}\partial^2 & \partial^3+\frac{1}{2}\partial^2\frac{q}{p}\partial^{-1}\frac{r}{p}\partial^2\\ \partial^3+\frac{1}{2}\partial^2\frac{r}{p}\partial^{-1}\frac{q}{p}\partial^2 & -\frac{1}{2}\partial^2\frac{r}{p}\partial^{-1}\frac{r}{p}\partial^2\end{pmatrix},\qquad J=\begin{pmatrix}0 & -\partial^2\\ \partial^2 & 0\end{pmatrix},\qquad \partial=\frac{\partial}{\partial x},$$

$$\partial\partial^{-1}=\partial^{-1}\partial=1 \tag{36}$$

as a pair of Lenard's operators of (34), we are sure to have

$$K\nabla_u\lambda=\lambda\cdot J\nabla_u\lambda. \tag{37}$$

Apparently, K and J are skew-symmetric, and J is a symplectic operator. The recursion operator $\mathscr{L}=J^{-1}K$ is

$$\mathscr{L}=\frac{1}{2\mathrm{i}}\begin{pmatrix}\partial+\frac{r}{2p}\partial^{-1}\frac{q}{p}\partial^2 & -\frac{r}{2p}\partial^{-1}\frac{r}{p}\partial^2\\ \frac{q}{2p}\partial^{-1}\frac{q}{p}\partial^2 & -\partial-\frac{q}{2p}\partial^{-1}\frac{r}{p}\partial^2\end{pmatrix}. \tag{38}$$

Proposition 1. The WKI spectral problem (34) is equivalent to

$$Ly=\lambda y,\qquad L=L(u)=\frac{1}{1-qr}\begin{pmatrix}\mathrm{i} & -q\\ -r & -\mathrm{i}\end{pmatrix}\partial \tag{39}$$

and the Gateaux derivative operator $L_*(\xi)$ of the spectral operator L in the direction $\xi=(\xi_1,\ \xi_2)^{\mathrm{T}}$ is

$$L_*(\xi)=\frac{1}{1-qr}\begin{pmatrix} q\xi_2 & -\mathrm{i}\xi_1 \\ \mathrm{i}\xi_2 & r\xi_1 \end{pmatrix}L. \tag{40}$$

Proposition 2. The invertible operators of L, J, K, and $\mathscr{L}$ are

$$L^{-1}=\begin{pmatrix} -\mathrm{i}\partial^{-1} & \partial^{-1}q \\ \partial^{-1}r & \mathrm{i}\partial^{-1} \end{pmatrix},\qquad J^{-1}=\begin{pmatrix} 0 & \partial^{-2} \\ -\partial^{-2} & 0 \end{pmatrix}, \tag{41}$$

$$K^{-1}=2\mathrm{i}\begin{pmatrix} \frac{1}{2}\partial^{-1}r\partial^{-1}r\partial^{-1} & \partial^{-3}-\frac{1}{2}\partial^{-1}r\partial^{-1}q\partial^{-1} \\ \partial^{-3}-\frac{1}{2}\partial^{-1}q\partial^{-1}r\partial^{-1} & \frac{1}{2}\partial^{-1}q\partial^{-1}q\partial^{-1} \end{pmatrix}, \tag{42}$$

$$\mathscr{L}^{-1}=2\mathrm{i}\begin{pmatrix} \partial^{-1}-\frac{1}{2}\partial^{-1}r\partial^{-1}q\partial & -\frac{1}{2}\partial^{-1}r\partial^{-1}r\partial \\ \frac{1}{2}\partial^{-1}q\partial^{-1}q\partial & -\partial^{-1}+\frac{1}{2}\partial^{-1}q\partial^{-1}r\partial \end{pmatrix}, \tag{43}$$

respectively.

Proof. (41) is obvious. By virtue of the identity $r\partial(q/p)+q\partial(r/p)=2[\partial(1/p)-p\partial]$, we can prove that $K^{-1}K=I_{2\times 2}$ ($I_{2\times 2}$ is the 2×2 unit operator).

Proposition 3. Let $A=A(x,\ t,\ q,\ r)$, $B=B(x,\ t,\ q,\ r)$, $C=C(x,\ t,\ q,\ r)$, $D=D(x,\ t,\ q,\ r)$ be four arbitary given C^∞-functions. Then iff

$$M=\frac{1}{1-qr}\begin{pmatrix} q & -\mathrm{i} \\ \mathrm{i} & r \end{pmatrix}\begin{pmatrix} A & 0 \\ 0 & -B \end{pmatrix}L,\qquad \tilde{M}=\frac{1}{1-qr}\begin{pmatrix} q & -\mathrm{i} \\ \mathrm{i} & r \end{pmatrix}\begin{pmatrix} D & 0 \\ 0 & C \end{pmatrix}L \tag{44}$$

the operator equations (4) and (5) have solutions

$$G_0^{(1)}=\partial^{-2}A,\qquad G_0^{(2)}=\partial^{-2}B, \tag{45}$$

$$G_{-1}^{(1)}=2\mathrm{i}\partial^{-3}D+\mathrm{i}\partial^{-1}r\partial^{-1}\left(r\partial^{-1}C-q\partial^{-1}D\right),\qquad G_{-1}^{(2)}=2\mathrm{i}\partial^{-3}C-\mathrm{i}\partial^{-1}q\partial^{-1}\left(r\partial^{-1}C-q\partial^{-1}D\right). \tag{46}$$

Proof. By (40) and Proposition 2, we obtain (45) and (46).

Let $G_0=(G_0^{(1)},\ G_0^{(2)})^{\mathrm{T}}$, $G_{-1}=(G_{-1}^{(1)},\ G_{-1}^{(2)})^{\mathrm{T}}$ be defined by (45) and (46), respectively. Recursively define the Lenard's sequence $\{G_j\}_{j=-\infty}^{\infty}$:

$$G_j=\mathscr{L}^jG_0,\qquad j\geqslant 0, \tag{47}$$

$$G_j=\mathscr{L}^{j+1}G_{-1},\qquad j<0. \tag{48}$$

Then the vector fields

$$\begin{aligned} X_m(q,\ r)&=KG_m=K\mathscr{L}^mG_0,\qquad m\geqslant 0,\\ &=JG_m=J\mathscr{L}^{m+1}G_{-1},\quad m<0 \end{aligned} \tag{49}$$

yield the hierarchy of generalized NLEEs,

$$(q,\ r)_{t_m}=X_m(q,\ r),\qquad m\in\mathbb{Z}. \tag{50}$$

For $m\geqslant 0$, (50) is the hierarchy of higher-order NLEEs, and for $m<0$, (50) is the hierarchy of lower-order NLEEs.

As $A = B = 0$, $G_0 = (ax + b, cx + d)^{\mathrm{T}} \in \operatorname{Ker} J$ (a, b, c, d are constants) and $G_1 = \mathscr{L} G_0 = (1/2\mathrm{i})(a - 2r/p, -b - 2q/p)^{\mathrm{T}}$. Thus, $X_0 = JG_1 = (-\mathrm{i}(q/p)_{xx}, \mathrm{i}(r/p)_{xx})^T$, $X_1 = KG_1 = \frac{1}{2}((q_x/p^3)_{xx}, (r_x/p^3)_{xx})^{\mathrm{T}}$. As $m = 1$, (50) becomes

$$q_{t_1} = \frac{1}{2}\left(\frac{q_x}{p^3}\right)_{xx}, \qquad r_{t_1} = \frac{1}{2}\left(\frac{r_x}{p^3}\right)_{xx}, \tag{51}$$

which can be further reduced to the well-known Harry–Dym equation

$$s_{t_1} = -\left(\frac{1}{\sqrt{s}}\right)_{xxx}, \tag{52}$$

when $r = -1$, $1 + q = s$. So, as A, B are chosen as zero, the hierarchy of higher-order NLEEs is actually the Harry–Dym hierarchy, which corresponds to the higher-order isospectral ($\lambda_t = 0$) NLEEs of (39).

Let $G^{(1)} = G^{(1)}(x)$, $G^{(2)} = G^{(2)}(x)$ be two arbitary given smooth functions, $G = (G^{(1)}, G^{(2)})^{\mathrm{T}}$. For the spectral problem (39), we establish the following operator equation,

$$[V, L] = L_*(KG)L^{-1} - L_*(JG), \tag{53}$$

which corresponds to $\beta = -1$, $\alpha = 0$ in (10). Through a series of careful calculations, we can prove the following proposition

Proposition 4. The operator equation (53) possesses the operator solution

$$V = V(G) = \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix} + A\begin{pmatrix} -\mathrm{i} & q \\ r & \mathrm{i} \end{pmatrix} L, \tag{54}$$

where

$$A = A(G) = \frac{1}{2p}\partial^{-1}\left(\frac{q}{p}G^{(1)}_{xx} - \frac{r}{p}G^{(2)}_{xx}\right),$$

$$B = B(G) = \frac{1}{2\mathrm{i}}\left[G^{(2)}_{xx} - \frac{1}{2}\partial\frac{q}{p}\partial^{-1}\left(\frac{q}{p}G^{(1)}_{xx} - \frac{r}{p}G^{(2)}_{xx}\right)\right],$$

$$C = C(G) = \frac{1}{2\mathrm{i}}\left[G^{(1)}_{xx} + \frac{1}{2}\partial\frac{r}{p}\partial^{-1}\left(\frac{q}{p}G^{(1)}_{xx} - \frac{r}{p}G^{(2)}_{xx}\right)\right]. \tag{55}$$

Until here, for the spectral problem (34) and the two matrix operators (44), the two conditions of Theorem 1 hold. So, the hierarchy of GNLEEs (50) has the GLR

$$L_{t_m} = [W_m, L] + ML^{m+1}, \qquad m \geqslant 0, \tag{56}$$

$$L_{t_m} = [W_m, L] + ML^{m}, \qquad m < 0, \tag{57}$$

$$W_m = \sum_{j=0}^{m}\left\{\begin{pmatrix} 0 & B_j \\ C_j & 0 \end{pmatrix} + A_j\begin{pmatrix} -\mathrm{i} & q \\ r & \mathrm{i} \end{pmatrix}L\right\}L^{m-j+1}, \qquad m \geqslant 0, \tag{58}$$

$$= -\sum_{j=m}^{-1}\left\{\begin{pmatrix} 0 & B_j \\ C_j & 0 \end{pmatrix} + A_j\begin{pmatrix} -\mathrm{i} & q \\ r & \mathrm{i} \end{pmatrix}L\right\}L^{m-j}, \qquad m < 0, \tag{59}$$

where $A_j = A(G_j)$, $B_j = B(G_j)$, $C_j = C(G_j)$ are expressed by (55) with $G = G_j$ (G_j is the Lenard's sequence (47) and (48)).

By Corollary 1, we have

Corollary 2. The potentials q, r satisfy the stationary WKI system

$$\sum_{i=-l}^{s} c_i X_i(q, r) = 0, \qquad \forall l,\ s \in \mathbb{Z}^+, \tag{60}$$

if and only if

$$\left[\sum_{i=-l}^{s} c_i W_i,\ L\right] = -M \sum_{i=0}^{s} c_i L^{i+1} - \tilde{M} \sum_{i=-l}^{-1} c_i L^i, \tag{61}$$

where the constants c_i $(-l \leqslant i \leqslant s)$ are independent of x, $X_i(q, r)$ are the vector fields (49), L, W_1, M and $\tilde{M}$ are determined by (39), (58) for (or (59)) and (44), respectively.

Let L be the WKI spectral operator (39), then according to Theorems 2 and 4, for an arbitrary given integer $m \in \mathbb{Z}$ $(m \neq 0)$ S_L^m is an algebra under the multiplication operation (21), and for the fixed matrix operator M (or $\tilde{M}$) determined by (44) S_L^M composes a Lie algebra under the multiplication operation (29). According to Proposition 3, $M \neq I_{2\times 2}$, $M \neq I_{2\times 2}$ are obvious, thus, for the WKI spectral problem (44), Theorems 5 and 6 do not hold here. But for the KdV hierarchy, Theorems 5 and 6 hold, which will be discussed elsewhere.

Remark. Classical r-matrices discussed by Semenov [15] have been used for the construction of the integrable nonlinear equations [16]. At the beginning of this paper, we simply described the so-called operator pattern for producing hierarchies of GNLEEs and GLR as in Ref. [10]. Then, is there a relationship between the classical r-matrix formulation and Theorem 4 of this paper? It looks like no relationship exists. Because if the left-hand side of (30) is defined as a classical r-matrix, then P, Q satisfy $[\![P, Q]\!] = [P, Q]_r = [r(P), Q] + [P, r(Q)]$. Substituting (30) into the above equality, we have

$$[r(P), Q] + [P, r(Q)] = M^{-1}AMQ - QA + PB - M^{-1}BMP, \tag{62}$$

which implies

$$r(P)Q - Qr(P) = M^{-1}AMQ - QA, \qquad Pr(Q) - r(Q)P = PB - M^{-1}BMP. \tag{63}$$

Obviously, there is no r-matrix or linear map r satisfying equality (63) except for the case $M = I_{N\times N}$ ($I_{N\times N}$ is an $N \times N$ unit operator). So, for a general fixed invertible operator $M \neq I_{N\times N} \in \mathscr{V}^N$, it has not been seen that the Jacobi identity (31) is related to the classical r-matrix.

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