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与WKI孤子族相应的完全可积的Bargmann系统*

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摘　要　本文给出一个新的有限维对合系，并由此证明WKI特征值问题在位势与谱函数之间的Bargmann约束下，被非线性化为一个Liouville完全可积的Hamilton系统．最后，我们由可换流的对合解获得WKI方程族的每一方程解的表示．

关键词　Bargmann系统，WKI族，对合解．

让 $(E,F)=\sum_{j=1}^{N}(\frac{\partial E}{\partial q_j}\frac{\partial F}{\partial p_j}-\frac{\partial E}{\partial p_j}\frac{\partial F}{\partial q_j})$ 表示辛流形（R^{2N}，$dp\wedge dq$）上的标准Poisson括号[1]．设 $\lambda_1<\cdots<\lambda_N$，令

$$\Gamma_k=\sum_{j=1,j\neq k}^{N}\frac{\lambda_k\lambda_j B_{kj}^2}{\lambda_k-\lambda_j},B_{kj}=p_kq_j-q_kp_j \tag{1}$$

则易算得：

$$\left.\begin{array}{l}(\Gamma_k,\Gamma_l)=0\\ (p_kq_k,p_lq_l)=0,(<\Lambda p,q>,p_kq_k)=0\\ (<\Lambda q,q>,p_kq_k)=2\lambda_kq_k^2,(<\Lambda p,p>,p_kq_k)=-2\lambda_kp_k^2\\ (\Gamma_k,p_lq_l)=\dfrac{2\lambda_k\lambda_l}{\lambda_k-\lambda_l}(p_kp_l+q_lq_k)B_{kl}\\ (\Gamma_k,<\Lambda q,q>)=4\lambda_k<\Lambda q,p>q_k^2-4\lambda_k<\Lambda q,q>p_kq_k\\ (\Gamma_k,<\Lambda q,p>)=-4\lambda_k<\Lambda q,p>p_k^2+4\lambda_k<\Lambda p,p>p_kq_k\\ (\Gamma_k,<\Lambda p,q>)=2\lambda_k<\Lambda p,p>q_k^2-2\lambda_k<\Lambda q,q>p_k^2\end{array}\right\} \tag{2}$$

这里 $q=(q_1,\cdots,q_N)^T$，$p=(p_1,\cdots p_N)^T$，$\Lambda=\mathrm{diag}(\lambda_1,\cdots,\lambda_N)$，<·，·>表示 R_N 中的标准内积．

使用上述公式及Poisson括号的性质（Poisson括号是斜称的、双线性的并且满足Jcaobi恒等式和Leibniz规则：(EF，H)＝E(F，H)＋F(E，H)），经一系列详细计

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算．我们有：

命题 1　如下定义的函数系 $E_1, \cdots, E_N$ 构成一个新的有限维对合系：

$$E_k=-\langle\Lambda p,q\rangle p_kq_k+\sqrt{1+\langle\Lambda p,p\rangle\langle\Lambda q,q\rangle}p_kq_k+\frac{1}{2}\Gamma_k, k=1,2,\cdots,N \tag{3}$$

证明　当 k=l 时，显然有 $(E_k, E_l)=0$．当 $k\neq l$ 时，(E_k, E_l) 为：

$$\begin{aligned}(E_k,E_l)=&\langle\Lambda p,q\rangle p_kq_k(\langle\Lambda p,q\rangle,p_lq_l)+\langle\Lambda p,q\rangle p_lq_l(p_kq_k,\langle\Lambda p,q\rangle)\\&+\langle\Lambda p,q\rangle^2(p_kq_k,p_lq_l)-p_kq_kp_lq_l(\langle\Lambda p,q\rangle,\sqrt{r})-p_kq_k\sqrt{r}(\langle\Lambda p,q\rangle,p_lq_l)\\&-\langle\Lambda p,q\rangle p_lq_l(p_kq_k,\sqrt{r})-\frac{1}{2}p_kq_k(\langle\Lambda p,q\rangle,\Gamma_l)-\frac{1}{2}\langle\Lambda p,q\rangle(p_kq_k\ \sqrt{l})\\&-p_kq_kp_lq_l(\sqrt{r},\langle\Lambda p,q\rangle)-p_kq_k\langle\Lambda p,q\rangle(\sqrt{r},p_lq_l)\\&-p_lq_l\sqrt{r}(p_kq_k,\langle\Lambda p,q\rangle)\\&+p_kq_k\sqrt{r}(\sqrt{r},p_lq_l)+p_lq_l\sqrt{r}(p_kq_k,\sqrt{r})+\frac{1}{2}p_kq_k(\sqrt{r},\Gamma_l)+\frac{1}{2}\sqrt{r}(p_kq_k,\Gamma_l)\\&-\frac{1}{2}p_lq_l(\Gamma_k,\langle\Lambda p,q\rangle)-\frac{1}{2}\langle\Lambda p,q\rangle(\Gamma_k,p_lq_l)\\&+\frac{1}{2}p_lq_l(\Gamma_k,\sqrt{r})+\frac{1}{2}\sqrt{r}(\Gamma_k,p_lq_l)+\frac{1}{4}(\Gamma_k,\Gamma_l)\end{aligned}$$

这里 $r=1+\langle\Lambda p,p\rangle\langle\Lambda q,q\rangle$，将(2)式代入上式，并利用 Poisson 括号的性质，即得 $(E_k,E_l)=0$.

定义 R^N 上的一个双线性函数[2]如下：

$$Q_Z(\xi,\eta)\triangleq\langle(z-\Lambda)^{-1}\xi,\eta\rangle=\sum_{k=1}^{N}(z-\lambda_k)^{-1}\xi_k\eta_k \tag{4}$$

那么对合系 $\{E_k\}$ 的发生函数是：

$$-\langle\Lambda p,q\rangle Q_z(p,q)+\sqrt{1+\langle\Lambda p,p\rangle\langle\Lambda q,q\rangle}Q_z(p,q)+\frac{1}{2}\begin{vmatrix}Q_z(\Lambda q,q) & Q_z(\Lambda q,p)\\ Q_z(\Lambda q,p) & Q_z(\Lambda p,p)\end{vmatrix}=\sum_{k=1}^{N}\frac{E_k}{z-\lambda_k} \tag{5}$$

命题 2　令 $F_m=\sum_{k=1}^{N}\lambda_k^mE_k, m=0,1,2,\cdots$．那么我们有

$$F_0=-\langle\Lambda p,q\rangle\langle p,q\rangle+\sqrt{1+\langle\Lambda p,p\rangle\langle q,q\rangle}\langle p,q\rangle \tag{6}$$

$$F_m=-\langle\Lambda p,q\rangle\langle\Lambda^m p,q\rangle+\sqrt{1+\langle\Lambda p,p\rangle\langle\Lambda q,q\rangle}\langle\Lambda^m p,q\rangle+\frac{1}{2}\sum_{j=0}^{m-1}\begin{vmatrix}\langle\Lambda^{j+1}q,q\rangle & \langle\Lambda^{j+1}p,q\rangle\\ \langle\Lambda^{m-j}p,q\rangle & \langle\Lambda^{m-j}p,q\rangle\end{vmatrix} \tag{7}$$

且　$(F_l,F_m)=0$，　$\forall\ l,m\in Z$.

证明　显然有 $(F_l,F_m)=0$，当 $|z|>\max\{|\lambda_1|,\cdots,|\lambda_N|\}$ 时，我们得到

$$(z-\lambda_k)^{-1}=\sum_{m=0}^{\infty}z^{-m-1}\lambda_k^m,\quad Q_z(\xi,\eta)=\sum_{m=0}^{\infty}\langle\Lambda^m\xi,\eta\rangle z^{-m-1}$$

将上面二幂级数展式代入(5)式的两端，进行计算整理；比较 z 的同次幂系数后，我们就有(6)和(7).

定理 1　Hamilton 系统

$$(F_m)\qquad q_{tm}=\frac{\partial F_m}{\partial P},\quad P_{tm}=-\frac{\partial F_m}{\partial q}, m=0,1,2,\cdots \tag{8}$$

在 Liouville 意义下是完全可积的.

考虑 WKI 特征值问题

$$y_x=My,\quad M=\begin{pmatrix}-i\lambda & \lambda u\\ \lambda v & i\lambda\end{pmatrix}. \tag{9}$$

让 $y=(y_1,y_2)^T$ 满足(9),令 $\nabla\lambda=(\lambda y_2^2,-\lambda y_1^2)^T$,则 $\nabla\lambda$ 满足(见文[3]):

$$K\nabla\lambda=\lambda\cdot J\nabla\lambda \tag{10}$$

其中,K,J 是两个斜称算子($\partial=\partial/\partial x,\partial\partial^{-1}=\partial^{-1}\partial=1$):

$$K=\frac{1}{2i}\begin{pmatrix}-\frac{1}{2}\partial^2\frac{u}{w}\partial^{-1}\frac{u}{w}\partial^2 & \partial^3+\frac{1}{2}\partial^2\frac{u}{w}\partial^{-1}\frac{v}{w}\partial^2\\ \partial^3+\frac{1}{2}\partial^2\frac{v}{w}\partial^{-1}\frac{u}{w}\partial^2 & -\frac{1}{2}\partial^2\frac{v}{w}\partial^{-1}\frac{v}{w}\partial^2\end{pmatrix},J=\begin{pmatrix}0 & -\partial^2\\ \partial^2 & 0\end{pmatrix},w=\sqrt{1-uv}. \tag{11}$$

算子 K,J 称为 WKI 特征值问题(9)的 Lenard 算子对.

定义 Lenald 递推叙列 $\{G_j\}$[3]: $G_{-1}=(1,1)^T,G_0=(i\frac{v}{w},i\frac{u}{w})^T,KG_{j-1}=JG_j(j=0,1,2,\cdots)$. WKI 族孤子方程由向量场 $X_m\triangleq JG_m(m=0,1,2,\cdots)$ 产生,即

$$(u,v)_t^T=X_m(u,v)\qquad m=0,1,2,\cdots. \tag{12}$$

特别指出,当 $v=-1$ 时,令 $1+u=s$,则 $(u,v)_t^T=X_1(u,v)$ 便化为著名的 Harry－Dym 方程

$$s_t=-\left(\frac{1}{\sqrt{s}}\right)_{xxx}.$$

文[3]还证明了方程族(12)具有换位表示:

$$L_t=[V_m,L]\qquad m=0,1,2\cdots \tag{13}$$

其中,算子 L,V_m 分别为:

$$L=\frac{1}{1-uv}\begin{pmatrix}i & -u\\ -v & -i\end{pmatrix}\partial \tag{14}$$

$$V_m=\sum_{j=0}^{m}\left[\begin{pmatrix}0 & F(G_{j-1})\\ E(G_{k-1}) & 0\end{pmatrix}+\begin{pmatrix}-i & u\\ v & i\end{pmatrix}\cdot iA(G_{j-1})L\right]\cdot L^{m+1-j} \tag{15}$$

这里,$A(G_{j-1})=\frac{1}{2iw}\partial^{-1}(\frac{u}{w}G_{j-1,xx}^{(1)}-\frac{v}{w}G_{j-1xx}^{(2)}),E(G_{j-1})=\frac{1}{2i}G_{j,xx}^{(1)}+\frac{1}{2}(vA(G_{j-1}))_x$,

$F(G_{j-1})=\frac{1}{2i}G_{j-1,xx}^{(2)}-\frac{1}{2}(uA(G_{j-1}))_x;G_{j-1}=(G_{j-1}^{(1)},G_{j-1}^{(2)})^T$……是 lenard 递推序列;(15)式是由文[3]的(7)式及 $V_{m-1}=\sum_{j=0}^{m-1}W_{j-1}\cdot L^{m-j}$ 并整理(7)式而来的.

设 $\lambda_j(j=1,\cdots,N)$ 是(9)的 N 个互异的特征值,$y=(q_j,p_j)^T$ 为相应于 λ_j 的特征函数,考虑 Bargmann 约束[4]: $G_0=\sum_{j=1}^{N}\nabla\lambda_j$,经简单计算及 $G_0=(i\frac{v}{w},i\frac{u}{w})^T,w=\sqrt{1-uv}$ 即有:

$$u=\frac{i\langle\Lambda q,q\rangle}{\sqrt{1+\langle\Lambda q,q\rangle\langle\Lambda p,p\rangle}},\quad v=\frac{-i\langle\Lambda p,p\rangle}{\sqrt{1+\langle\Lambda p,p\rangle\langle\Lambda q,q\rangle}} \tag{16}$$

在 Bargmann 约束(16)下 WKI 特征值问题(9)被非线性化为一个 Bargmann 系统:

$$(B):\quad \begin{cases} q_x=-i\Lambda q+\dfrac{i<\Lambda q,q>}{\sqrt{1+<\Lambda q,q><\Lambda p,p>}}\Lambda p \\ p_x=i\Lambda p-\dfrac{i<\Lambda p,p>}{\sqrt{1+<\Lambda q,q><\Lambda p,p>}}\Lambda q \end{cases} \tag{17}$$

命题3 Bargmann系统(17)可表为Hamilton结构：

$$(H):\quad \begin{cases} q_x=\dfrac{\partial H}{\partial p} \\ p_x=-\dfrac{\partial H}{\partial q} \end{cases} \tag{18}$$

其中，Hamilton函数 $H=-i<\Lambda p,q>+i\sqrt{1+<\Lambda q,q><\Lambda p,p>}$.

证明 直接验证即知(18)就是(17).

命题4 (18)的Hamilton函数H与F_m对合，即

$$(H,F_m)=0,m=0,1,2\cdots. \tag{19}$$

证明 通过直接计算Poisson括号

$$(H,F_m)=<\frac{\partial H}{\partial q},\frac{\partial F_m}{\partial p}>-<\frac{\partial H}{\partial p},\frac{\partial F_m}{\partial q}>$$

我们便知(19)式成立．

由命题4立即得到

定理2 Bargmann系统(17)在Liouvilie意义下是完全可积的．

定理3 设(q,p)是Bargmann系统(17)的一个解，那么由(16)所决定的函数u,v满足一个定态的非线性WKI系统：

$$X_N+\sum_{l=0}^{N-1}\beta_{N-l}X_l=0 \tag{20}$$

这里β_j由$\lambda_1,\cdots,\lambda_N$确定；$X_j\triangleq JG_j$是WKI向量场．

证明 用算子$(J^{-1}K)^l$作用等式$G_0=\sum_{j=1}^{N}\nabla\lambda_j$，并注意到(10)式及$KG_{j-1}=JG_j$，我们有

$$(J^{-1}K)^lG_0=\sum_{j=1}^{N}\lambda_j^l\nabla\lambda_j \tag{21}$$

考察多项式($\beta_0=1$)：

$$p(\lambda)=\prod_{j=1}^{N}(\lambda-\lambda_j)=\beta_0\lambda^N+\beta_1\lambda^{N-1}\cdots+\beta_N. \tag{22}$$

让算子$J\sum_{l=0}^{N}\beta_{N-l}$作用(21)的两端，并使用(22)式及$X_l=JG_l=J(J^{-1}K)^lG_0$，我们即可获得(20).

既然Poisson括号$(H,F_m)=0$，那么Hamilton系统(H)与(F_m)是相容的，记g_0^x、g_m^{tm}分别是(H)、(Fm)初值问题的解算子，则流g_0^x、g_m^{tm}可换(见文[1])．因而相容方程组(H)、(F_m)的对合解

$$\begin{pmatrix} q(x,t_m) \\ p(x,t_m) \end{pmatrix}=g_0^xg_m^{tm}\begin{pmatrix} q(0,0) \\ p(0,0) \end{pmatrix}$$

是(x,t_m)的二元光滑函数．

定理 4　设$(q(x,t_m),p(x,t_m))^T$是相容系统(H)与(F_m)的一个对合解，那么

$$u(x,t_m)=\frac{i\langle\Lambda q,q\rangle}{\sqrt{1+\langle\Lambda q,q\rangle\langle\Lambda p,p\rangle}},\quad v(x,t_m)=\frac{-i\langle\Lambda p,p\rangle}{\sqrt{1+\langle\Lambda p,p\rangle\langle\Lambda q,q\rangle}} \tag{23}$$

满足非线性 WKI 发展方程

$$4i(u,v)^T_{t_{m+2}}=J(J^{-1}K)^mG_0\qquad m=0,1,2,\cdots. \tag{24}$$

其中，J,K 如(11)式所示．

证明　设$Q=\sqrt{1+\langle\Lambda p,p\rangle\langle\Lambda q,q\rangle}$，由(17)式，我们得到：

$$\begin{aligned}\partial Q^{-1}&=-Q^{-3}(\langle\Lambda p,p\rangle\langle\Lambda q,q_x\rangle+\langle\Lambda q,q\rangle\langle\Lambda p,p_x\rangle)\\&=iQ^{-3}(\langle\Lambda^2q,q\rangle\langle\Lambda p,p\rangle-\langle\Lambda^2p,p\rangle\langle\Lambda q,q\rangle)\end{aligned} \tag{25}$$

$$\begin{aligned}\partial\langle\Lambda^{m+1}q,q\rangle&=-2i\langle\Lambda^{m+2}q,q\rangle+2iQ^{-1}\langle\Lambda q,q\rangle\langle\Lambda^{m+2}p,q\rangle\\
\partial^2\langle\Lambda^{m+1}q,q\rangle&=-4i\langle\Lambda^{m+2}q,q_x\rangle+2i\langle\Lambda q,q\rangle\langle\Lambda^{m+2}p,q\rangle\partial Q^{-1}\\
&\quad+2iQ^{-1}[2\langle\Lambda q,q_x\rangle\langle\Lambda^{m+2}p,q\rangle+\langle\Lambda q,q\rangle(\langle\Lambda^{m+2}p,q_x\rangle\\
&\quad+\langle\Lambda^{m+2}q,p_x\rangle)]\\
&=-4\langle\Lambda^{m+3}q,q\rangle+4Q^{-1}\langle\Lambda q,q\rangle\langle\Lambda^{m+3}p,q\rangle+4Q^{-1}\langle\Lambda^2q,q\rangle\langle\Lambda^{m+2}p,q\rangle\\
&\quad-4Q^{-2}\langle\Lambda q,q\rangle\langle\Lambda^2p,q\rangle\langle\Lambda^{m+2}p,q\rangle-2Q^{-2}\langle\Lambda q,q\rangle^2\langle\Lambda^{m+3}p,p\rangle\\
&\quad+2Q^{-2}\langle\Lambda p,p\rangle\langle\Lambda q_1,q\rangle\langle\Lambda^{m+3}q,q\rangle-2Q^{-3}\langle\Lambda q,q\rangle\langle\Lambda p,p\rangle\langle\Lambda^2q,\\
&\quad q\rangle\langle\Lambda^{m+2}p,q\rangle+2Q^{-3}\langle\Lambda q,q\rangle^2\langle\Lambda^2p,p\rangle\langle\Lambda^{m+2}p,q\rangle\end{aligned} \tag{26}$$

$$\begin{aligned}\partial\langle\Lambda^{m+1}p,p\rangle&=2i\langle\Lambda^{m+2}p,p\rangle-2iQ^{-1}\langle\Lambda p,p\rangle\langle\Lambda^{m+2}p,q\rangle\\
\partial^2\langle\Lambda^{m+1}p,p\rangle&=-4i\langle\Lambda^{m+2}q,q_x\rangle-2i\langle\Lambda p,p\rangle\langle\Lambda^{m+2}p,q\rangle\partial Q^{-1}\\
&\quad-2iQ^{-1}[2\langle\Lambda p,p_x\rangle\langle\Lambda^{m+2}p,q\rangle+\langle\Lambda p,p\rangle(\langle\Lambda^{m+2}p,q_x\rangle\\
&\quad+\langle\Lambda^{m+2}q,p_x\rangle)]\\
&=-4\langle\Lambda^{m+3}p,p\rangle+4Q^{-1}\langle\Lambda p,p\rangle\langle\Lambda^{m+3}p,q\rangle\\
&\quad+4Q^{-1}\langle\Lambda^2p,p\rangle\langle\Lambda^{m+2}p,q\rangle\\
&\quad-4Q^{-2}\langle\Lambda p,p\rangle\langle\Lambda^2p,q\rangle\langle\Lambda^{m+2}p,q\rangle-2Q^{-2}\langle\Lambda p,p\rangle^2\langle\Lambda^{m+3}q,q\rangle\\
&\quad+2Q^{-2}\langle\Lambda q,q\rangle\langle\Lambda p,p\rangle\langle\Lambda^{m+3}p,p\rangle-2Q^{-3}\langle\Lambda p,p\rangle\langle\Lambda q,q\rangle\langle\Lambda^2p,\\
&\quad p\rangle\langle\Lambda^{m+2}p,q\rangle+2Q^{-3}\langle\Lambda p,p\rangle^2\langle\Lambda^2q,q\rangle\langle\Lambda^{m+2}p,q\rangle\end{aligned} \tag{27}$$

从(23)式，我们有

$$\begin{pmatrix}u\\v\end{pmatrix}_{t_{m+2}}=iQ^{-3}\langle\Lambda q,q_{t_{m+2}}\rangle\begin{pmatrix}2+\langle\Lambda p,p\rangle\langle\Lambda q,q\rangle\\\langle\Lambda p,p\rangle^2\end{pmatrix}-iQ^{-3}\langle\Lambda p,p_{t_{m+2}}\rangle]\begin{pmatrix}\langle\Lambda q,q\rangle^2\\2+\langle\Lambda p,p\rangle\langle\Lambda q,q\rangle\end{pmatrix} \tag{28}$$

将(8)式代入(28)，并注意到(26)、(27)和(21)，我们即可得到

$$4i\begin{pmatrix}u\\v\end{pmatrix}_{t_{m+2}}=\begin{pmatrix}0&-\partial^2\\\partial^2&0\end{pmatrix}\begin{pmatrix}\langle\Lambda^{m+1}p,p\rangle\\-\langle\Lambda^{m+1}q,q\rangle\end{pmatrix}=J\sum_{k=1}^{N}\lambda_k^m\nabla\lambda_k=J(J^{-1}K)^mG_0.$$

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Completely Integrable Bargmann System Associated with the WKI Soliton Hierarchy

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ABSTRACT A new finite - dimensional involutive system is presented, and the WKI hierarchy of nonlinear evolution equations and their commutator representations are discussed in this article. By this finite－dimensional involutive system, it is proven that under thd Bargmann constraints thc WKI eigenvalue problem is nonlinearized as a completely integrable Hamiltonian system in the Liouville sense. Moreover, the involutive representation of solutions of each equation in the WKI hierarchy is obtained by making use of the solution of two compatible systems.

KEY WORDS Bargmann system; WKI hierarchy; Involutive solutions.