On the Involutive Solutions of Solition Hierarchy

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Abstract The involutive solutions of soliton hierarchy are given in this paper. Through the nonlinearization of the lax system for solition hierarchy, it is proven that the involutive solution of compatible systems (H) (spatial part) and (F_m) (time part) is mapped into the soliton of solutin hierarohyby the mapping f which is determined by the constraint relation $u=f(\varphi)$ between potentials and eigenfunotions.

Keywords soliton hierarchy; involutive system; ivolutive solution

§ 1. Introduction

The nonlinear evolution equations(NLEEs), which possess the solutions soluitins as one of their properties, are generally applied into field theory, fluid mechanics, nonlinear potics, and other branches connected with the mordon physics. The studies in recent years are very vital. The theory of finite-dimensional integrable system-linouville-Arnol'd theory [1] is paid great attention to. Finding finite-dimensional involutive systems as many as possible and deeply discussung their properties related to the soliton epuations, are helpful not only to deal with the problem of judgment for integrable systems, but also to provide some applicant NLEEs.

The isospectral evolution equation of eigenvalue problem $L(u)\varphi=\lambda\varphi$ has the Lax form $L_t=[V,L]$ or $L\varphi=\lambda\varphi, \varphi_t=V\varphi$. In ref. [2], a correct skeleton of commutator (or Lax)representation for the soliton equation (vectorfields form $u_t=X(u)$ is presented. Based on this, a socalled "lax systemnonlinearization approach" is proposed by Cao Cewed [3], and many finitedimensional involutive systems has been succesfully obtained [4]. In this paper, for the spectral problem like the form $L(u)\varphi=\lambda\varphi$, we shall show that the finite-dimensional involutive system through the nonlineatization of lax system is actuallr generated by the nonlinearization of its time part. Moreover, it is proven that the involutive solution of compatible systems (H) (spatial part) and (F_m) (time part) is mapped into the solution of soliton hierarchy by the mapping f which is determined by the constraint relation $u=f(\varphi)$ betweem potentials and eigenfunctions.

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§ 2. Hamiltonian System and Involutive Solution

Consider the eigenvalue problem

$$L(u)y = \lambda y \tag{2.1}$$

where $L(u)$ is a matrix differential operator, λ is a eigenparameter. For (2.1), we take the following procedure:

1. Find the Lenard's operator pair K,J of (2.1) and define the Lenard's recursive sequence

$$G_j: KG_{j-1} = JG_j \quad (G_{-1} \in \mathrm{Ker}J,\ j = 0,1,2,\cdots).$$

Generally, G_j is a polynomial of $u(x)$ and its derivatives. $X_j = JG_j$ $(j=0,1,2,\cdots)$ are called the soliton vector fields of (2.1). Look for a solution $\tilde{A}_j(\lambda_j, \varphi_j)$ of the lineat equation $K\tilde{A}_j = c\lambda_j \cdot J\tilde{A}_j$, where φ_j is the eigenfunction corresponding to λ_j, c is a constant. In the light of the skeleton in ref. [2], we can obtain the Lax form of soliton equtation $u_t = X_m(u)$

$$L(u)y = \lambda y \tag{2.2}$$

$$y_t = W_m Y \tag{2.3}$$

In general, the matrix differential operator W_m has the following form

$$W_m = \sum_{j=0}^{m} V_{j-1}(cL)^{m-j} \tag{2.4}$$

where V_{j-1} is a matrix function depending $u(x)$ and the Lenard's recursive sequence G_{j-1}.

2. Let λ_j $(j=1,2,\cdots,N)$ be N different eigenvalues of (2.1), then

$$L(u)\varphi = A\varphi \tag{2.5}$$

where $A = \mathrm{diag}(\lambda_1,\cdots,\lambda_N)$; $\varphi = (\varphi_1,\cdots,\varphi_N)^T$; φ_j is the eigenfunction corresspinding to λ_j.

Consider the constraint relation[4]

$$u = f(\varphi) \tag{2.6}$$

Under (2.6), (2.5) is nonlinearized to be

$$L(f(\varphi))\varphi = A\varphi \tag{2.7}$$

which can be expressed as an intergrable Hamiltonian system $(R^{2N}, d\varphi_1 \wedge d\varphi_2, H)$ whose involutive system[4] $\{F_m\}$ is produced by the nonlinearization of the time part (2.3) of Lax form, and H is usually one of F_m.

In the symplectic space $(R^{2N}, d\varphi_1 \wedge d\varphi_2)$, the Poisson bracket of functions F, G is defined by involutive

$$(F,G) = \sum_{j=1}^{N}\left(\frac{\partial F}{\partial \varphi_{2j}}\frac{\partial F}{\partial \varphi_{1j}} - \frac{\partial F}{\partial \varphi_{1j}}\frac{\partial G}{\partial \varphi_{2j}}\right)$$
$$= \left\langle\frac{\partial F}{\partial \varphi_2}, \frac{\partial G}{\partial \varphi_1}\right\rangle - \left\langle\frac{\partial F}{\partial \varphi_1}, \frac{\partial G}{\partial \varphi_2}\right\rangle. \tag{2.8}$$

F, G are called involutive if $(F,G) = 0$.

Proposition 2.1 $(F_m, F_n) = 0, \forall\, m,n$ if and only if $\left\langle\frac{\partial F_m}{\partial \varphi_2}, \frac{\partial F_n}{\partial \varphi_1}\right\rangle$ is symmetrical about m,n.

Proof $(F_m, F_n) = \left\langle\frac{\partial F_m}{\partial \varphi_2}, \frac{\partial F_n}{\partial \varphi_1}\right\rangle - \left\langle\frac{\partial F_m}{\partial \varphi_1}, \frac{\partial F_n}{\partial \varphi_2}\right\rangle = 0.$

Consider the canonical equation of F_m-flow

$$(F_m): \qquad \frac{\partial}{\partial t_m}\begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix} = \begin{pmatrix}-\partial F_m/\partial \varphi_2\\ -\partial F_m/\partial \varphi_1\end{pmatrix} \qquad (2.9)$$

Denote the solution operator of its initial-value problem by $g_m^{t_m}$, then we get

Proposition 2. 2 If $\langle\frac{\partial F_m}{\partial \varphi_2},\frac{\partial F_n}{\partial \varphi_1}\rangle$ is symmetrical about m,n then 1) any two canomical systems $(F_k),(F_l)$ are compatible; 2) the Hamiltonian phase-flows $g_k^{t_k},g_l^{t_l}$ commute.

3. Denote by $x=t_0,t_m$ the flow variables of $(H),(F_m)$ respectively. Define

$$\begin{pmatrix}\varphi_1(x,t_m)\\ \varphi_2(x,t_m)\end{pmatrix} = g_0^x g_m^{t_m}\begin{pmatrix}\varphi_1(0,0)\\ \varphi_2(0,0)\end{pmatrix} \qquad (2.10)$$

which is called the involutive solution[5] of consistent equations (H) and (F_m).

Proposition 2. 3 Let $(\varphi_1(x,t_m),\varphi_2(x,t_m))^T$ be an involutive solution of consistent systems (H) and (F_m). Let $u(x,t_m)=f(\varphi)$ $(\varphi=(\varphi_1,\varphi_2)^T)$, then

1) the two flow-equations $(H),(F_m)$ can be reduced to the spatial part (2. 11) and the time part (2. 12) respectively of Lax system for soliton equation (2. 13) with u as their potential.

$$L(u)\varphi = \Lambda\varphi \qquad \text{(spatial part)} \qquad (2.11)$$

$$\varphi_{t_m} = (W_m + c_1W_{m-1} + \cdots + c_mW_0)\varphi \qquad \text{(time part)} \qquad (2.12)$$

where c_i are independent of x; W_k $(k=0,1,2,\cdots,m)$ are defined by (2. 4).

2) $u(x,t_m)=f(\varphi)$ satisfies the soliton equation

$$U_{t_m} = X_m + c_1X_{m-1} + \cdots + c_mX_0 \qquad (2.13)$$

Remark. 2. 4 In the above sense , under the constraint $u=f(\varphi)$, the spatial part and time part of the Lax system for the soliton equation are nonlinearized to be the canonical equation (H), (F_m) respectively . They are compatible , and completely integrable in the Liouville's sense. Solving two times ordinary differential equations group (2. 11), (2. 12), and one time algebaric operation (2. 6), we can obtain the solution of the soliton equation (2. 13).

§ 3. Examples

1. Tu Guizhang eigenvalue priblem[6]

$$Ly \equiv (-\partial^2 + u + \lambda^{-1}v)y = \lambda y \qquad (3.1)$$

$$K = \begin{pmatrix}2\partial & 0\\ 0 & v\partial+\partial v\end{pmatrix},$$

$$J = \begin{pmatrix}0 & 2\partial\\ 2\partial & \frac{1}{2}\partial^3-(u\partial+\partial u)\end{pmatrix}.$$

$$KG_{j-1} = JG^j, G_{-2} = (1,0)^T,$$

$$G_{-1} = (\frac{1}{2}u,1)^T, j = 0,1,2,\cdots.$$

The Tu hierarchy of NLEEs $(u,v)^T_{t_m}=X_m(u,v)=JG_m$ possess the Lax representation

$$\begin{cases} Ly = \lambda y \\ y_{t_m} = \sum_{j=0}^{m}(-\frac{1}{2}G_{j-1,x}^{(2)} + G_{j-1}^{(2)}\partial)L^{m-j}y \end{cases} \tag{3.2}$$

where $G_{j-1}^{(2)}$ is the second component of G_{j-1}.

$\widetilde{A}_j = (\varphi_j^2(x), \lambda_j^{-1}\varphi_j^2(x))^T$ satisfies the equation $K\widetilde{A}_j = \lambda_j \cdot J\widetilde{A}_j$. The constraint $G_{-1} = \sum_{i=0}^{N}\widetilde{A}_i$ yields

$$\langle \Lambda^{-1}\varphi, \varphi\rangle = 1,$$
$$u = 2\langle \varphi, \varphi\rangle,$$
$$v = -\frac{\langle \varphi, \varphi\rangle + \langle \Lambda^{-1}\varphi, \varphi\rangle}{\langle \Lambda^{-2}\varphi, \varphi\rangle}, (\varphi = \varphi_x) \tag{3.3}$$

Under (3.3), the Lax pair (3.2) are nonlinearized as intrgrable Hamiltinian systems[4]

$$(R^{2N}, d\psi \wedge d\varphi|_{TQ^{N-1}}, H^* = H - \mu F),$$
$$(R^{2N}, d\psi \wedge d\varphi|_{TQ^{N-1}}, F_m^* = F_m - \mu_m F)$$

on the tangent boundle of ellispsoid sphere

$$TQ^{N-1} = \{(\psi, \varphi) \in R^{2N} | F = \frac{1}{2}(\langle \Lambda^{-1}\varphi, \varphi\rangle - 1) = 0, G = \langle \Lambda^{-1}\varphi, \varphi\rangle = 0\},$$

where

$$H = F_0$$
$$= \frac{1}{2}\langle \psi, \psi\rangle + \frac{1}{2}\langle \Lambda\varphi, \varphi\rangle - \frac{1}{2}\langle \varphi, \varphi\rangle^2,$$
$$F_m = \frac{1}{2}\langle \Lambda^m\psi, \psi\rangle + \frac{1}{2}\langle \Lambda^{m+1}\varphi, \varphi\rangle - \frac{1}{2}\langle \varphi, \varphi\rangle\langle \Lambda^m\varphi, \varphi\rangle$$
$$+ \frac{1}{2}\sum_{i+j=m-1}\begin{vmatrix} \langle \Lambda^i\varphi, \varphi\rangle & \langle \Lambda^i\psi, \varphi\rangle \\ \langle \Lambda^j\psi, \varphi\rangle & \langle \Lambda^j\psi, \psi\rangle \end{vmatrix}$$
$$\mu = \frac{(H, G)}{(F, G)}|_{TQ^{N-1}},$$
$$\mu_m = \frac{(F_m, G)}{(F, G)}|_{TQ^{N-1}}.$$

It isn't difficult to know $\langle \frac{\partial F_m^*}{\partial \varphi}, \frac{\partial F_n^*}{\partial \varphi}\rangle$ is symmetrical about m, n. So, we have $(F_m^*, F_n^*) = 0$ which implies that (H_*) and (F_m^*) are compatible. Hence, in proposition 2.3, let $f(\varphi)$ be defined by (3.3), we obtain

Conclusion Let $(\psi(x, t_m), \varphi(x, t_m))^T$ be an involutive solution of compatible system (H^*) and (F_m^*), then

$$u(x, t_m) = 2\langle \varphi, \varphi\rangle,$$
$$v(x, t_m) = -\frac{\langle \varphi, \varphi\rangle + \langle \Lambda^{-1}\psi, \psi\rangle}{\langle \Lambda^{-2}\varphi, \varphi\rangle}$$

satisfy a T_n equation

$$(u, v)_{t_m}^T = X_m + c_1X_{m-1} + \cdots + c_mX_0 \tag{3.4}$$

where c_j are independent of x.

In fact, acting with the operator $(J^{-1}K)^k$ upon $G_{-1} = \sum_{j=1}^{N}\widetilde{A}_j$, we find that there exist some constants $c_2, \cdots, c_{k+2}$ such that

$$\begin{aligned} A_k &= \begin{pmatrix} A_k^{(1)} \\ A_k^{(2)} \end{pmatrix} \\ &\triangleq \begin{pmatrix} \langle \Lambda^{k+1}\varphi, \varphi\rangle \\ \langle \Lambda^k \varphi, \varphi\rangle \end{pmatrix} \\ &= G_k + c_2 G_{k-2} + \cdots + c_k G_0 + c_{k+1} G_{-1} + c_{k+2} G_{-2} \qquad (3.5) \\ u_{t_m} &= 4\langle \varphi, \Lambda^m \varphi\rangle \\ &= 2\partial A_m^{(2)} \\ v_{t_m} &= 4\langle \varphi, \Lambda^{m+1}\varphi\rangle + 2\langle \phi_x, \Lambda^m \varphi\rangle - 2u\langle \phi, \Lambda^m\varphi\rangle \\ &\quad + v_x\langle \varphi, \Lambda^{m-1}\varphi\rangle + 2v\langle \phi, \Lambda^{m-1}\varphi\rangle - 2\langle \phi, \Lambda^m \varphi\rangle \\ &= 2\partial A_m^{(1)} + [\frac{1}{2}\partial^3 - (u\partial + \partial u)]A_m^{(2)} \end{aligned}$$

Thus, $(uv)^T_{t_m} = JA_m = X_m + c_1 X_{m-1} + \cdots + c_m X_0$.

2. Jaulent-Miodek eigenvalue problem[7]

$$\begin{aligned} Ly &\equiv (-\lambda^{-1}\partial^2 + v + \lambda^{-1}u)y \\ &= \lambda y \end{aligned} \qquad (3.6)$$

$$K = \begin{bmatrix} -\frac{1}{4}\partial^3 + \frac{1}{2}(u\partial + \partial u) & 0 \\ 0 & \partial \end{bmatrix}$$

$$J = \begin{bmatrix} -\frac{1}{2}(v\partial + \partial v) & \partial \\ \partial & 0 \end{bmatrix}$$

$$KG_{j-1} = JG_j, G_{-2} = (1,0)^T, G_{-1} = (1, \frac{1}{2}v)^T,$$

$$G_0 = (\frac{1}{2}v, \frac{1}{2}u + \frac{3}{8}v^2)^T, j = 0, 1, 2, \cdots.$$

The JM hierarchy of equations $(u,v)^T_{t_m} = X_m(u,v) = JG_m$ have the Lax pair

$$\begin{cases} Ly = \lambda y \\ y_{t_m} = \sum_{j=0}^{m} (-\frac{1}{4}G^{(1)}_{j-1,x} + \frac{1}{2}G^{(1)}_{j-1}\partial)L^{m-j}y \end{cases} \qquad (3.7)$$

where $G^{(1)}_{j-1}$ is the first component of G_{j-1}.

Consider the constraint

$$\begin{cases} u = \langle \varphi, \Lambda\varphi\rangle - \frac{3}{4}\langle \varphi, \varphi\rangle^2 \\ v = \langle \varphi, \varphi\rangle \end{cases} \qquad (3.8)$$

Under (3.8), the Lax pair (3.7) are nonlinearized as completely integrable Hamiltonian systems $(R^{2N}, d\phi\wedge d\varphi, H)$ and $(R^{2N}, d\phi\wedge d\varphi, F_m)$, where

$$\begin{aligned} H &= 2F_0 \\ &= \frac{1}{2}\langle \phi, \phi\rangle + \frac{1}{2}\langle \Lambda^2\varphi, \varphi\rangle \end{aligned}$$

$$-\frac{1}{2}\langle\Lambda\varphi,\varphi\rangle\langle\varphi,\varphi\rangle+\frac{1}{8}\langle\varphi,\varphi\rangle^{3}\quad(\psi=\varphi_{x})\tag{3.9}$$

$$\begin{aligned}F_{m}=&\frac{1}{4}\langle\Lambda^{m}\psi,\psi\rangle+\frac{1}{4}\langle\Lambda^{m+2}\varphi,\varphi\rangle\\&-\frac{1}{8}\langle\Lambda\varphi,\varphi\rangle\langle\Lambda^{m}\varphi,\varphi\rangle-\frac{1}{8}\langle\varphi,\varphi\rangle\langle\Lambda^{m+1}\varphi,\varphi\rangle\\&+\frac{1}{16}\langle\varphi,\varphi\rangle^{2}\langle\Lambda^{m}\varphi,\varphi\rangle\\&+\frac{1}{8}\sum_{i+j=m-1}\begin{vmatrix}\langle\Lambda^{i}\varphi,\varphi\rangle&\langle\Lambda^{i}\psi,\varphi\rangle\\\langle\Lambda^{j}\psi,\varphi\rangle&\langle\Lambda^{j}\psi,\psi\rangle\end{vmatrix}\end{aligned}\tag{3.10}$$

This result is given by Mou Weihua. Easy to see that $\langle\frac{\partial F_m}{\partial\varphi},\frac{\partial F_n}{\partial\psi}\rangle$ is symmetrical about m,n. So we have $(F_m,F_n)=0$ which implies that $(H)$and (F_m) are compatible . Hence, let $f(\varphi)$ be defined by (3.8) in proposition 2.3, we know that

$$u(x,t_m)=\langle\Lambda\varphi,\varphi\rangle-\frac{3}{4}\langle\varphi,\varphi\rangle^{2},$$
$$v(x,t_m)=\langle\varphi,\varphi\rangle$$

are an involutive solution of the *JM* equation $(u,v)^T_{t_m}=X_m+c_1X_{m-1}+\cdots+c_mX_0$, where c_j are constant.

3. Under the constraint[3]

$$\begin{cases}q=-\langle\varphi_1,\varphi_1\rangle\\r=\langle\varphi_2,\varphi_2\rangle\end{cases}\tag{3.11}$$

the Lax pair of the AKNS hiearachy are nonlinearized as completely integrable systems $(R^{2N},d\varphi_1\wedge d\varphi_2,H)$ and $(R^{2N},d\varphi_1\wedge d\varphi_2,F_m)$, where

$$H=i\langle\Lambda\varphi_1,\varphi_2\rangle+\frac{1}{2}\langle\varphi_1,\varphi_1\rangle\langle\varphi_2,\varphi_2\rangle\tag{3.12}$$

$$\begin{aligned}F_m=&\,i\langle\varphi_1,\Lambda^{m}\varphi_2\rangle\\&+\frac{1}{2}\sum_{j=1}^{m}\begin{vmatrix}\langle\varphi_1,\Lambda^{m-j}\varphi_1\rangle&\langle\varphi_1,\Lambda^{m-j}\varphi_2\rangle\\\langle\varphi_1,\Lambda^{j-1}\varphi_2\rangle&\langle\varphi_2,\Lambda^{j-1}\varphi_2\rangle\end{vmatrix},\quad m=1,2,\cdots.\end{aligned}\tag{3.13}$$

$\langle\frac{\partial F_m}{\partial\varphi_2},\frac{\partial F_n}{\partial\varphi_1}\rangle=\langle\frac{\partial F_n}{\partial\varphi_2},\frac{\partial F_m}{\partial\varphi_1}\rangle$ implies $(F_m,F_n)=0$. Thus the AKNS hiertarchy of NLEEs

$$\begin{pmatrix}q\\r\end{pmatrix}_{t_m}=J(J^{-1}K)^{m}\begin{pmatrix}r\\q\end{pmatrix},m=0,1,2,\cdots,\tag{3.14}$$

$$J=2i\begin{pmatrix}0&-1\\1&0\end{pmatrix},$$

$$K=\begin{bmatrix}2q\partial^{-1}q&\partial-2q\partial^{-1}r\\\partial-2r\partial^{-1}q&2r\partial^{-1}r\end{bmatrix}$$

have the involutive solution

$$\begin{aligned}q(x,t_m)&=-\langle\varphi_1,\varphi_1\rangle,\\r(x,t_m)&=\langle\varphi_2,\varphi_2\rangle\end{aligned}\tag{3.15}$$

where $(\varphi_1(x,t_m),\varphi_2(x,t_m))^T$ is the inbolutive solution of compatible systems $(H),(F_m)$.

4. The TD hierarchy[8] of NLEEs

$$\begin{pmatrix} q \\ r \end{pmatrix}_{t_m} = J(J^{-1}K)^{m+1}\begin{pmatrix} q \\ 0 \end{pmatrix}, m = 0,1,2,\cdots, \tag{3.16}$$

$$J = \begin{pmatrix} 0 & q^{-1}\partial \\ \partial q^{-1} & 0 \end{pmatrix},$$

$$K = \frac{1}{2}\begin{pmatrix} \partial & q^{-1}r\partial \\ \partial r q^{-1} & -4\partial + \partial q^{-1}\partial q^{-1}\partial \end{pmatrix}$$

possess the Lax pair

$$L(q,r)y \equiv \begin{bmatrix} -\partial + \frac{1}{2}r & q \\ -q & \partial + \frac{1}{2}r \end{bmatrix} y = \lambda y \tag{3.17}$$

$$y_{t_m} = \sum_{j=0}^{m} \begin{bmatrix} \frac{1}{2}q^{-1}\partial q^{-1}G^{(2)}_{j-1,x} - G^{(2)}_{j-1} - \frac{1}{2}rq^{-1}G^{(1)}_{j-1} + q^{-1}G^{(1)}_{j-1}\partial & -\frac{1}{2}(G^{(1)}_{j-1} + q^{-1}G^{(2)}_{j-1,x}) \\ \frac{1}{2}(G^{(1)}_{j-1} + q^{-1}G^{(2)}_{j-1,x}) & -G^{(2)}_{j-1} \end{bmatrix} \times L^{m-j}y \tag{3.18}$$

which under the constraint relation[4]

$$q = 2\sqrt{\langle \varphi_1, \varphi_2 \rangle},$$
$$r = \frac{\langle \varphi_1, \varphi_1 \rangle - \langle \varphi_2, \varphi_2 \rangle}{\sqrt{\langle \varphi_1, \varphi_2 \rangle}} \tag{3.19}$$

are nonlinearized as integrable Hamiltonian systems(H), (F_m) with

$$H = F_0$$
$$= -\langle \Lambda\varphi_1, \varphi_2 \rangle + \sqrt{\langle \varphi_1, \varphi_2 \rangle}(\langle \varphi_1, \varphi_1 \rangle - \langle \varphi_2, \varphi_2 \rangle)$$
$$F_m = -\langle \Lambda^{m+1}\varphi_1, \varphi_2 \rangle + \sqrt{\langle \varphi_1, \varphi_2 \rangle}(\langle \Lambda^m\varphi_1, \varphi_1 \rangle - \langle \Lambda^m\varphi_2, \varphi_2 \rangle)$$
$$- \sum_{i+j=m-1} \begin{vmatrix} \langle \Lambda^i\varphi_1, \varphi_1 \rangle & \langle \Lambda^i\varphi_1, \varphi_2 \rangle \\ \langle \Lambda^j\varphi_1, \varphi_2 \rangle & \langle \Lambda^j\varphi_2, \varphi_2 \rangle \end{vmatrix}, m = 1,2,\cdots. \tag{3.20}$$

In (3.18) $G_{j-1} = (G^{(1)}_{j-1}, G^{(2)}_{j-1})^T$ is the Lenard's recursive sequence; $KG_{j-1} = JG_j = JG_j$, $G_{-1} = (q, 0)^T$, $j = 0,1\cdots$. (H) and (F_m) are completely compatible in virtue of $(H, F_m) = 0$. Let $(\varphi_1(x,t_m), \varphi_2(x,t_m))^T$ be aninvolutive solution of systems (H) and (F_m), then

$$q(x,t_m) = 2\sqrt{\langle \varphi_1, \varphi_2 \rangle},$$
$$r(x,t_m) = \frac{\langle \varphi_1, \varphi_1 \rangle - \langle \varphi_2, \varphi_2 \rangle}{\sqrt{\langle \varphi_1, \varphi_2 \rangle}}$$

satisfy (3.16).

5. The L-C-Z hierarchy[9] of NLEEs

$$\begin{pmatrix} q \\ r \end{pmatrix}_{t_m} = J(J^{-1}K)\begin{pmatrix} r \\ q \end{pmatrix}, m = 0,1,2,\cdots, \tag{3.21}$$

$$J = 2i\begin{bmatrix} 0 & \frac{1}{2}\partial \\ \frac{1}{2}\partial + q & -r \end{bmatrix},$$

$$K = \begin{bmatrix} -\frac{1}{2}\partial^2 + 2\partial q\partial^{-1}q & \partial r - 2\partial q\partial^{-1}r \\ \partial r + r\partial + 2\partial r\partial^{-1}q + qr & -\frac{1}{2}\partial^2 - 2\partial r\partial^{-1}r - q\partial - 2\partial \end{bmatrix}$$

have the Lax representation

$$L(q,r)y \equiv -i\begin{pmatrix} r-\partial & r+q \\ r-q & r+\partial \end{pmatrix} y = \lambda y \tag{3.22}$$

$$V_m = \sum_{j=0}^{m} \begin{bmatrix} -\frac{1}{2}G^{(2)}_{j-1,x} + [G^{(1)}_{j-1} + 2\partial^{-1}(qG^{(1)}_{j-1} - rG^{(2)}_{j-1})]\partial & -\frac{1}{2}(G^{(1)}_{j-1} + G^{(2)}_{j-1})_x - qG^{(1)}_{j-1} + rG^{(2)}_{j-1} \\ -\frac{1}{2}(G^{(1)}_{j-1} - G^{(2)}_{j-1})_x - qG^{(1)}_{j-1} + rG^{(2)}_{j-1} & \frac{1}{2}G^{(2)}_{j-1,x} + [G^{(2)}_{j-1} + 2\partial^{-1}(qG^{(1)}_{j-1} - rG^{(2)}_{j-1})]\partial \end{bmatrix} \times L^{m-j}y \tag{3.23}$$

where $G_{j-1} = (G^{(1)}_{j-1}, G^{(2)}_{j-1})^T$ is the Lenard's recursive sequence; $KG_j = JG_{j+1}, G_0 = (r,q)^T, j = 0,1,\cdots$. Under the constraint

$$q = \langle\varphi_1,\varphi_1\rangle - \langle\varphi_2,\varphi_2\rangle,$$
$$r = \langle\varphi_1,\varphi_1\rangle + \langle\varphi_2,\varphi_2\rangle, \langle\varphi_1,\varphi_2\rangle = 0 \tag{3.24}$$

(3.22) and (3.23) are nonlinearized to be homiltonian systems $(H = -iF_0), (F_m)$ with

$$H = -iF_0 = i\langle\Lambda\varphi_1,\varphi_2\rangle - \langle\varphi_1+\varphi_2,\varphi_1+\varphi_2\rangle\langle\varphi_1,\varphi_2\rangle - \langle\varphi_1,\varphi_1\rangle\langle\varphi_2,\varphi_2\rangle + \langle\varphi_1,\varphi_2\rangle^2$$

$$F_m = -\langle\Lambda^{m+1}\varphi_1,\varphi_2\rangle - i\langle\Lambda^m(\varphi_1+\varphi_2),(\varphi_1+\varphi_2)\rangle\langle\varphi_1,\varphi_2\rangle - i\sum_{j=0}^{m}\begin{vmatrix} \langle\varphi_1,\Lambda^j\varphi_1\rangle & \langle\varphi_2,\Lambda^j\varphi_2\rangle \\ \langle\varphi_1,\Lambda^{m-j}\varphi_2\rangle & \langle\varphi_2,\Lambda^{m-j}\varphi_2\rangle \end{vmatrix}, m = 0,1,2,\cdots. \tag{3.25}$$

which are completely integrable in the Lioubille's sense.

Let $(\varphi_1(x,t_m),\varphi_2(x,t_m))^T$ be a solution of the consistent systems $(H),(F_m)$, then

$$q(x,t_m) = \langle\varphi_1,\varphi_2\rangle - \langle\varphi_2,\varphi_2\rangle,$$
$$r(x,t_m) = \langle\varphi_1,\varphi_1\rangle + \langle\varphi_2,\varphi_2\rangle$$

is an involutive solution of the equation (3.21).

6. The Levi hierarchy[10] of NLEEs

$$\begin{pmatrix} q \\ r \end{pmatrix}_{t_m} = J(J^{-1}K)^m\begin{pmatrix} r \\ q \end{pmatrix}, m = 0,1,2,\cdots, \tag{3.26}$$

$$J = \begin{pmatrix} 0 & \partial \\ \partial & 0 \end{pmatrix},$$

$$K = \begin{bmatrix} -\partial q - q\partial & -\partial^2 - r\partial + \partial q \\ \partial^2 - \partial r + q\partial & r\partial + \partial r \end{bmatrix}$$

is the consistent condition of

$$L(q,r)y \equiv \begin{bmatrix} -\partial + \frac{q-r}{2}r & q \\ -r & \partial + \frac{q-r}{2} \end{bmatrix} = \frac{\lambda}{2}y \tag{3.27}$$

$$y_{t_m} = \sum_{j=0}^{m} \begin{bmatrix} -\frac{1}{2}(G_{j-1}^{(2)} + G_{j-1}^{(1)})_x + (G_{j-1}^{(2)} - G_{j-1}^{(1)})\partial & -G_{j-1,x}^{(2)} \\ G_{j-1,x}^{(1)} & \frac{1}{2}(G_{j-1}^{(2)} + G_{j-1}^{(1)})_x + (G_{j-1}^{(2)} - G_{j-1}^{(1)})\partial \end{bmatrix} \times (2L)^{m-j} y \tag{3.28}$$

where $G_{j-1} = (G_{j-1}^{(1)}, G_{j-1}^{(2)})^T$ is the Lenard's recursive sequence: $KG_j = JG_{j+1}, G_0 = (r, q)^T, j = 0, 1, \cdots$. Under the constraint

$$q = -\langle \varphi_1 + \varphi_2, \varphi_1 \rangle,$$
$$r = \langle \varphi_1 + \varphi_2, \varphi_2 \rangle \tag{3.29}$$

(3. 27), (3. 28) are nonlinearized as Hamiltonian systems (H), (F_m) with

$$H = F_0$$
$$= \frac{1}{2}\langle \Lambda\varphi_1, \varphi_2 \rangle + \frac{1}{2}\langle \varphi_1 + \varphi_2, \varphi_1 + \varphi_2 \rangle \langle \varphi_1, \varphi_2 \rangle$$
$$+ \frac{1}{2}(\langle \varphi_1, \varphi_1 \rangle \langle \varphi_2, \varphi_2 \rangle - \langle \varphi_1, \varphi_2 \rangle^2)$$

$$F_m = \frac{1}{2}\langle \Lambda^{m+1}\varphi_1, \varphi_2 \rangle + \frac{1}{2}\langle \Lambda^m(\varphi_1 + \varphi_2), \varphi_1 + \varphi_2 \rangle \langle \varphi_1, \varphi_2 \rangle$$
$$+ \frac{1}{2}\sum_{j=0}^{m} \begin{vmatrix} \langle \Lambda^j\varphi_1, \varphi_1 \rangle & \langle \Lambda^j\varphi_1, \varphi_2 \rangle \\ \langle \Lambda^{m-j}\varphi_1, \varphi_2 \rangle & \langle \Lambda^{m-j}\varphi_2, \varphi_2 \rangle \end{vmatrix}, m = 0, 1, 2, \cdots. \tag{3.30}$$

Both of (H), (F_m) are completetly integrable , and their phase — flows commute. Thus, the Levi equation (3. 26) has the involutive solution

$$q(x, t_m) = -\langle \varphi_1 + \varphi_2, \varphi_1 \rangle,$$
$$r(x, t_m) = \langle s\varphi_1 + \varphi_2, \varphi_2 \rangle,$$

where $(\varphi_1(x, t_m), \varphi_2(x, t_m))^T$ is a solution of the compatible systems (H), (F_m).

Cao[5] has considered the case of Kdv hierarchy. Gu[11,12] gave the involutive solutions of MKDV and Boussinisq-Burger 's hierarchy. Of course, we, may look for the involutive solutions of other soliton hierarchies.

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