发展方程，保谱发展方程，换位表示

Northeastern Math. J.
9(2)(1993), 215—223

两族非线性保谱发展方程的换位表示

(10)

Commutator Representations for Two Hierarchies of Nonlinear Isospectral Evolution Equations*)

215-223

0175.29

Qiao Zhijun (乔志军)
(Department of Mathematics, Liaoning University, Shenyang, 110036)

Abstract Following Cao's idea, we present commutator representations for two hierarchies of nonlinear isospetral evolutión equations associated with two isospectral problems studied by Hu Xingbiao.

Key Words and Phrases Isospectral Problem; The Pair of Lenard's Operators; Commutator Representation

It is an important topic to search for the commutator representations for nonlinear isospectral evolution equations in soliton theory. In recent years, a lot of results on commutator representations have been successively obtained (see [1—7]). In this paper, following Cao Cewen's idea about commutator representation theory (see [1]), we study two isospectral problems presented by Hu Xingbiao[8] and give commutator representations for the corresponding hierarchies of nonlinear isospectral evolution equations.

The two spectral problems (see [8])

$$\psi_x = \begin{pmatrix} r & 1+q\lambda^{-1} \\ \lambda+q & -r \end{pmatrix}\psi$$

and

$$\psi_x = \begin{pmatrix} q & 1+r\lambda^{-1} \\ \lambda-r & -q \end{pmatrix}\psi$$

can be rewritten in a unified form

$$\psi_x = U\psi \equiv \begin{pmatrix} r & 1+q\lambda^{-1} \\ \lambda+\varepsilon q & -r \end{pmatrix}\psi, \qquad \varepsilon = \pm 1, \tag{1}$$

where $\psi \equiv (\psi_1, \psi_2)^T$, λ is an eigenparameter, and the vector-valued function $u(x) = (q(x), r(x))^T$ is called the potential of (1). The underlying interval Ω is $(-\infty, +\infty)$ or $(0, T)$ under the decaying condition at infinity or periodic condition respectively. Let $u \to u + \varepsilon\delta u$.

Received Apr. 21, 1991.
*) Project supported by the Natural Science Foundation of Education Committee, Liaoning Province, China.

Proposition 1 Let λ be an eigenvalue of (1), and $(\psi_1, \psi_2)^T$ be the corresponding eigenfunction:

$$\begin{cases} \psi_{1x} = r\psi_1 + (1 + q\lambda^{-1})\psi_2, \\ \psi_{2x} = (\lambda + \varepsilon q)\psi_1 - r\psi_2. \end{cases} \tag{2}$$

Then the functional gradient $\nabla_u\lambda$ of the eigenvalue λ with regard to the potential u is

$$\nabla_u\lambda \triangleq \begin{pmatrix} \delta\lambda/\delta q \\ \delta\lambda/\delta r \end{pmatrix} = \begin{pmatrix} -\varepsilon\psi_1^2 + \lambda^{-1}\psi_2^2 \\ 2\psi_1\psi_2 \end{pmatrix} \cdot \left(\int_\Omega (\psi_1^2 + q\lambda^{-2}\psi_2^2)\mathrm{d}x\right)^{-1}. \tag{3}$$

Proof In Section II of [9], we choose $m_{11} = r$, $m_{12} = 1 + q\lambda^{-1}$, $m_{21} = \lambda + \varepsilon q$. Then we have

$$\int_\Omega [(-\varepsilon\psi_1^2 + \lambda^{-1}\psi_2^2)\delta q + 2\psi_1\psi_2\delta r]\mathrm{d}x = \delta\lambda\int_\Omega (\psi_1^2 + q\lambda^{-2}\psi_2^2)\mathrm{d}x$$

which implies (3).

Proposition 2 Let λ be an eigenvalue of (1). Then for $\varepsilon = 1$ and $\varepsilon = -1$, $\nabla_u\lambda$ satisfies the linear relations

$$K\nabla_u\lambda = \lambda J\nabla_u\lambda \tag{4}$$

and

$$\hat{K}\nabla_u\lambda = \lambda\hat{J}\nabla_u\lambda \tag{5}$$

respectively, where K, J and $\hat{K}$, $\hat{J}$ are two pairs of skew-symmetric operators having the forms ($\partial = \partial/\partial x$, $\partial\partial^{-1} = \partial^{-1}\partial = 1$)

$$\begin{aligned} K &= \begin{pmatrix} \frac{1}{8}\partial\frac{q}{r}\partial\frac{q}{r}\partial - \frac{1}{2}\partial q\partial^{-1}q\partial & \frac{1}{8}\partial\frac{q}{r}\partial^2 - \frac{1}{2}\partial q\partial^{-1}r\partial - \frac{1}{2}\partial\frac{q^2}{r} \\ \frac{1}{8}\partial^2\frac{q}{r}\partial - \frac{1}{2}\partial r\partial^{-1}q\partial - \frac{1}{2}\frac{q^2}{r}\partial & \frac{1}{8}\partial^3 - \frac{1}{2}\partial r\partial^{-1}r\partial - \frac{1}{2}q\partial - \frac{1}{2}\partial q \end{pmatrix}, \\ J &= \begin{pmatrix} 0 & \frac{1}{2}\partial\frac{q}{r} \\ \frac{1}{2}\frac{q}{r}\partial & \frac{1}{2}\partial \end{pmatrix}, \end{aligned} \tag{6}$$

$$\begin{aligned} \hat{K} &= \begin{pmatrix} \frac{1}{8}\partial\frac{q}{r}\partial\frac{q}{r}\partial - \frac{1}{2}\partial q\partial^{-1}q\partial & \frac{1}{8}\partial\frac{q}{r}\partial^2 + \frac{1}{2}\partial q\partial^{-1}r\partial - \frac{1}{4}\partial\frac{q}{r}\partial\frac{q}{r} \\ \frac{1}{8}\partial^2\frac{q}{r}\partial - \frac{1}{2}\partial r\partial^{-1}q\partial - \frac{1}{4}\frac{q}{r}\partial\frac{q}{r}\partial & \frac{1}{8}\partial^3 - \frac{1}{2}\partial r\partial^{-1}r\partial - \frac{1}{2}\frac{q}{r}\partial\frac{q}{r} + \frac{1}{4}\frac{q}{r}\partial^2 - \frac{1}{4}\partial^2\frac{q}{r} \end{pmatrix}, \\ \hat{J} &= \begin{pmatrix} 0 & \frac{1}{2}\partial\frac{q}{r} \\ \frac{1}{2}\frac{q}{r}\partial & \frac{1}{2}\partial \end{pmatrix}, \end{aligned} \tag{7}$$

which are called the pair of Lenard's operators of (1) corresponding to $\varepsilon = 1$ and $\varepsilon = -1$, respectively.

Proof For $\varepsilon = 1$,

$$J^{-1}K = \begin{pmatrix} -\partial^{-1}q\partial & -\partial^{-1}r\partial \\ \frac{1}{4}\partial\frac{q}{r}\partial - r\partial^{-1}q\partial & \frac{1}{4}\partial^2 - r\partial^{-1}r\partial - q \end{pmatrix}.$$

Thus, in order to obtain (4) it suffices to prove

$$J^{-1}K\nabla_u\lambda = \lambda\nabla_u\lambda.$$

From (2) we get

$$(-\psi_1^2+\lambda^{-1}\psi_2^2)_x = -2r(\psi_1^2+\lambda^{-1}\psi_2^2),$$
$$(2\psi_1\psi_2)_x = 2(1+q\lambda^{-1})\psi_2^2+2(\lambda+q)\psi_1^2.$$

So,

$$-\partial^{-1}q\partial(-\psi_1^2+\lambda^{-1}\psi_2^2)-\partial^{-1}r\partial(2\psi_1\psi_2)$$
$$=-\partial^{-1}2r(\psi_2^2+\lambda\psi_1^2)=\lambda\cdot(-\psi_1^2+\lambda^{-1}\psi_2^2),$$
$$\left(\frac{1}{4}\partial\frac{q}{r}\partial-r\partial^{-1}q\partial\right)(-\psi_1^2+\lambda^{-1}\psi_2^2)+\left(\frac{1}{4}\partial^3-r\partial^{-1}r\partial-q\right)(2\psi_1\psi_2)$$
$$=\lambda\cdot(2\psi_1\psi_2),$$

which yield (8).

For $\varepsilon=-1$,

$$\hat{J}^{-1}\hat{K}=\begin{pmatrix}\frac{1}{2}\frac{q}{r}\partial & \frac{1}{2}\partial-\frac{q}{r}\\ \frac{1}{4}\partial\frac{q}{r}\partial-r\partial^{-1}q\partial & \frac{1}{4}\partial^3-r\partial^{-1}r\partial-\frac{1}{2}\partial\frac{q}{r}\end{pmatrix}.$$

Similarly, we can prove

$$\hat{J}^{-1}\hat{K}\nabla_u\lambda = \lambda\nabla_u\lambda. \tag{9}$$

(8), (9) imply (4), (5), respectively.

Proposition 3 The spectral problem (1) is equivalent to

$$L\psi=\lambda\psi,\qquad L=L(u,\varepsilon)=\begin{pmatrix}-\varepsilon q & r+\partial\\ \varepsilon(qr-q_x-q) & -q-r^2+r_x+\partial^2\end{pmatrix}. \tag{10}$$

Proof Obvious.

Definition 1 Let L: $u\to L(u,\varepsilon)$ be the mapping from a potential function into a differential operator. The Gateaux derivative of the mapping L in the direction ξ is defined by

$$L_{*u}(\xi)=\left.\frac{d}{d\eta}\right|_{\eta=0}L(u+\eta\xi). \tag{11}$$

Lemma 1 For the spectral problem (10), the Gateaux derivative of L is

$$L_{*u}(\xi)=\begin{pmatrix}-\varepsilon\xi_1 & \xi_2\\ \varepsilon(-\xi_{1x}+r\xi_1+q\xi_2) & \xi_{2x}-\xi_1-2r\xi_2\end{pmatrix}+\begin{pmatrix}0&0\\-\varepsilon\xi_1&0\end{pmatrix}\partial,$$
$$u=\begin{pmatrix}q\\r\end{pmatrix},\quad \xi=\begin{pmatrix}\xi_1\\\xi_2\end{pmatrix},\quad \varepsilon=\pm1, \tag{12}$$

and L_{*u} (simply written as L_* below) is an injective homomorphism.

Proof Directly calculate.

Consider the commutator $[V, L]$ of the two operators

$$V=V_1+V_2\partial,\qquad L=L(u,\varepsilon)=L_1+L_2\partial+L_3\partial^2,$$

where

$$L_1=\begin{pmatrix}-\varepsilon q & r\\ \varepsilon(qr-q_x) & -q-r^2+r_x\end{pmatrix},\quad L_2=\begin{pmatrix}0 & 1\\ -\varepsilon q & 0\end{pmatrix},\quad L_3=\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix};\tag{13}$$

$$V_1=\begin{pmatrix}0 & F\\ E & H\end{pmatrix},\qquad V_2=\begin{pmatrix}A & 0\\ 0 & D\end{pmatrix},\tag{14}$$

with A, D, E, F, H to be determined functions.

Through a series of calculations, we have

$$\begin{aligned}
&[V,L]=VL-LV\\
&=[V_1,L_1]-L_2V_{1x}+V_2L_{1x}-L_3V_{1xx}\\
&\quad+([V_1,L_2]+[V_2,L_1]-L_2V_{2x}+V_2L_{2x}-2L_3V_{1x}-L_3V_{2xx})\partial\\
&\quad+([V_2,L_2]+[V_1,L_3]+V_2L_{3x}-2L_3V_{2x})\partial^2\\
&=\begin{pmatrix}\varepsilon(-q_x+qr)F-\varepsilon q_xA-rE-E_x & (r_x-q+\varepsilon q-r^2)F-rH+r_xA-H_x\\ Z_1 & Z_2\end{pmatrix}\\
&\quad+\begin{pmatrix}-\varepsilon qF-E & -H+q(A-D)-D_x\\ -\varepsilon qH+\varepsilon(-q_x+qr)(D-A)+\varepsilon qA_x-\varepsilon q_xD-2E_x & E+\varepsilon qF-2H_x-D_{xx}\end{pmatrix}\partial\\
&\quad+\begin{pmatrix}0 & A-D+F\\ \varepsilon q(A-D)-E & -2D_x\end{pmatrix}\partial^2,
\end{aligned}\tag{15}$$

where

$$Z_1=\varepsilon(-q_x+qr)H-(r_x-q+\varepsilon q-r^2)E+\varepsilon(-q_{xx}+qr_x+rq_x)D-E_{xx},$$
$$Z_2=rE-\varepsilon(-q_x+qr)F+\varepsilon qF_x+(r_{xx}-q_x-2rr_x)D-H_{xx}.$$

In the following we shall seperately discuss (15) for $\varepsilon=1$ and $\varepsilon=-1$.

I. $\varepsilon=1$.

We hope

$$[V,L]=L_*(KG)-L_*(JG)L,\tag{16}$$

i.e.,

$$\begin{aligned}
&[V,L]\\
&=\begin{pmatrix}-(KG)^{(1)} & (KG)^{(2)}\\ -(KG)_x^{(1)}+r(KG)^{(1)}+q(KG)^{(2)} & (KG)_x^{(2)}-2r(KG)^{(2)}-(KG)^{(1)}\end{pmatrix}\\
&\quad-\begin{pmatrix}-(JG)^{(1)} & (JG)^{(2)}\\ -(JG)_x^{(1)}+r(JG)^{(1)}+q(JG)^{(2)} & (JG)_x^{(2)}-2r(JG)^{(2)}-(JG)^{(1)}\end{pmatrix}L_1\\
&\quad+\begin{pmatrix}0 & 0\\ (JG)^{(1)} & 0\end{pmatrix}L_{1x}+\left\{\begin{pmatrix}0 & 0\\ -(KG)^{(1)} & 0\end{pmatrix}\right.\\
&\quad-\begin{pmatrix}-(JG)^{(1)} & (JG)^{(2)}\\ -(JG)_x^{(1)}+r(JG)^{(1)}+q(JG)^{(2)} & (JG)_x^{(2)}-2r(JG)^{(2)}-(JG)^{(1)}\end{pmatrix}L_2\\
&\quad\left.+\begin{pmatrix}0 & 0\\ (JG)^{(1)} & 0\end{pmatrix}(L_{1x}+L_{2x})\right\}\partial+\left\{\begin{pmatrix}0 & 0\\ (JG)^{(1)} & 0\end{pmatrix}L_2\right.\\
&\quad\left.-\begin{pmatrix}-(JG)^{(1)} & (JG)^{(2)}\\ -(JG)_x^{(1)}+r(JG)^{(1)}+q(JG)^{(2)} & (JG)_x^{(2)}-2r(JG)^{(2)}-(JG)^{(1)}\end{pmatrix}L_3\right\}\partial^2,
\end{aligned}\tag{17}$$

where K, J and $L=L(u,1)$ are defined by (6) and (10) respectively, $G(x)=(G^{(1)}(x),G^{(2)}(x))^T$, $G^{(1)}(x)$ and $G^{(2)}(x)$ are two arbitrary smooth functions on Ω, and $(\cdot)^{(i)}$ $(i=1,2)$

stands for the i-th component of (•).

In order to get (16), in (15) we should choose

$$
\begin{aligned}
A &= A(G) = -\frac{1}{2}\partial^{-1}(qG_x^{(1)} + rG_x^{(2)}) - \frac{1}{2}\frac{q}{r}G^{(2)} + \frac{1}{8}\frac{1}{r}\left(\frac{q}{r}G_x^{(1)} + G_x^{(2)}\right)_x,\\
D &= D(G) = -\frac{1}{2}\partial^{-1}(qG_x^{(1)} + rG_x^{(2)}) - \frac{1}{2}\frac{q}{r}G^{(2)} + \frac{1}{4}\left(\frac{q}{r}G_x^{(1)} + G_x^{(2)}\right),\\
E &= E(G) = -\frac{1}{4}\frac{q}{r}(qG_x^{(1)} + rG_x^{(2)}) + \frac{1}{8}\frac{q}{r}\left(\frac{q}{r}G_x^{(1)} + G_x^{(2)}\right)_x,\\
F &= F(G) = -\frac{1}{4}\left(\frac{q}{r}G_x^{(1)} + G_x^{(2)}\right) - \frac{1}{8}\frac{1}{r}\left(\frac{q}{r}G_x^{(1)} + G_x^{(2)}\right)_x,\\
H &= H(G) = \frac{1}{4}(qG_x^{(1)} + rG_x^{(2)}) - \frac{1}{8}\left(\frac{q}{r}G_x^{(1)} + G_x^{(2)}\right)_x.
\end{aligned}
\tag{18}
$$

Hence, we have

Theorem 1 Let $G^{(1)}(x)$ and $G^{(2)}(x)$ be two given smooth functions on Ω, and $G \triangleq (G^{(1)}, G^{(2)})^T$. Then the operator equation determined by the pair of Lenard's operators K, J and the spectral operator $L=L(u, 1)$,

$$[V, L] = L_*(KG) - L_*(JG)L \tag{19}$$

possesses the operator solution

$$V = V(G) = \begin{pmatrix} 0 & F(G) \\ E(G) & H(G) \end{pmatrix} + \begin{pmatrix} A(G) & 0 \\ 0 & D(G) \end{pmatrix}\partial, \tag{20}$$

where $A(G)$, $D(G)$, $E(G)$, $F(G)$, $H(G)$ are defined by (18).

Proof Substituting the expressions (18) of $A(G)$, $D(G)$, $E(G)$, $F(G)$, $H(G)$ into the right-hand side of (15) and noticing $\varepsilon=1$, through a lenthy calculations we find that the calculated result is equal to the right-hand side of (17). So, Theorem 1 holds.

Now, for $\varepsilon=1$, we recursively define the Lenard's gradient sequence G_j of (1) as follows:

$$
\begin{aligned}
&G_{-1} = (0, 0)^T, \qquad G_0 = (2, 2r)^T,\\
&JG_{j+1} = KG_j, \qquad j = -1, 0, 1, \cdots
\end{aligned}
\tag{21}
$$

$X_m = JG_m$ $(m=0,1,2,\cdots)$ are called the vector fields of the spectral problem (1) with $\varepsilon=1$, the first few results of calculations being

$$
\begin{aligned}
&X_0 = (q_x, r_x)^T, \qquad G_0 = (2, 2r)^T;\\
&X_1 = \left(\left(\frac{1}{4}\frac{q}{r}q_{xx} - \frac{1}{2}qr^2 - q^2\right)_x, -qr_x + \left(\frac{1}{4}r_{xx} - \frac{1}{2}r^3 - qr\right)_x\right)^T,\\
&G_1 = \left(-r^2, \frac{1}{2}r_{xx} - r^2 - 2qr\right)^T.
\end{aligned}
$$

The hierarchy of evolution equations associated with (1) for $\varepsilon=1$ are produced by the vector field X_m, i. e.,

$$u_t \equiv (q, r)_t^T = X_m(q, r), \qquad m = 0,1,2,\cdots \tag{22}$$

with the representative equation

$$(q, r)_t = X_1(q, r)$$

$$=\left(\left(\frac{1}{4}\frac{q}{r}q_{xx}-\frac{1}{2}qr^2-q^2\right)_x,\ -qr_x+\left(\frac{1}{4}r_{xx}-\frac{1}{2}r^3-qr\right)_x\right)^T, \tag{23}$$

which can be reduced to the well-known Mkdv equation

$$r_t=\frac{1}{4}r_{xxx}-\frac{3}{2}r^2r_x \tag{24}$$

as $q=0$.

Ⅱ. $\varepsilon=-1$.

We hope

$$[V,\ L]=L_*(\hat{K}G)-L_*(\hat{J}G)L, \tag{25}$$

where $\hat{K}$, $\hat{J}$ and $L=L(u,\ -1)$ are defined by (7) and (10) respectively, $G(x)=(G^{(1)}(x),\ G^{(2)}(x))^T$, $G^{(1)}(x)$ and $G^{(2)}(x)$ are two arbitrary smooth functions on Ω.

In order to solve V from (25) by using the approach used in case Ⅰ, in (15) we should make choice of

$$\begin{aligned}
A=\hat{A}(G)&=-\frac{1}{2}\partial^{-1}(qG_x^{(1)}+rG_x^{(2)})-\frac{1}{4}\frac{1}{r}(\frac{q}{r}G^{(2)})_x+\frac{1}{8}\frac{1}{r}(\frac{q}{r}G_x^{(1)}+G_x^{(2)})_x,\\
D=\hat{D}(G)&=-\frac{1}{2}\partial^{-1}(qG_x^{(1)}+rG_x^{(2)})+\frac{1}{4}(\frac{q}{r}G_x^{(1)}+G_x^{(2)}),\\
E=\hat{E}(G)&=\frac{1}{4}\frac{q}{r}(qG_x^{(1)}+rG_x^{(2)})-\frac{1}{8}\frac{q}{r}(\frac{q}{r}G_x^{(1)}+G_x^{(2)})_x+\frac{1}{4}\frac{q}{r}(\frac{q}{r}G^{(2)})_x,\\
F=\hat{F}(G)&=-\frac{1}{4}(\frac{q}{r}G_x^{(1)}+G_x^{(2)})-\frac{1}{8}\frac{1}{r}(\frac{q}{r}G_x^{(1)}+G_x^{(2)})_x+\frac{1}{4}\frac{1}{r}(\frac{q}{r}G^{(2)})_x,\\
H=\hat{H}(G)&=\frac{1}{4}(qG_x^{(1)}+rG_x^{(2)})-\frac{1}{8}(\frac{q}{r}G_x^{(1)}+G^{(2)})_x+\frac{1}{4}\left(\frac{q}{r}G^{(2)}\right)_x.
\end{aligned} \tag{26}$$

So, we get

Theorem 2 Let $G^{(1)}(x)$ and $G^{(2)}(x)$ be two given smooth functions on Ω, and $G\triangleq(G^{(1)},\ G^{(2)})^T$. Then the operator equation determined by the pair of Lenard's operators $\hat{K}$, $\hat{J}$ and the spectral operator $L=L(u,\ -1)$,

$$[V,\ L]=L_*(\hat{K}G)-L_*(\hat{J}G)L \tag{27}$$

has the operator solution

$$\hat{V}=\hat{V}(G)=\begin{pmatrix}0 & \hat{F}(G)\\ \hat{E}(G) & \hat{H}(G)\end{pmatrix}+\begin{pmatrix}\hat{A}(G) & 0\\ 0 & \hat{D}(G)\end{pmatrix}\partial, \tag{28}$$

where $\hat{A}(G)$, $\hat{D}(G)$, $\hat{E}(G)$, $\hat{F}(G)$, $\hat{H}(G)$ are defined by (26).

Proof Substituting (26) into (15) and noticing $\varepsilon=-1$, by directly calculating (15) and decomposing (25) into the form like (17), we see that the operator equation (27) has the operator solution (28).

For $\varepsilon=-1$, the Lenard's recursive gradient sequence $\hat{G}_j$ of (1) are defined by

$$\begin{aligned}
&\hat{G}_{-1}=(0,\ 0)^T, \qquad \hat{G}_0=(0,\ 2r)^T,\\
&\hat{J}\hat{G}_{j+1}=\hat{K}\hat{G}_j, \qquad j=-1,\ 0,\ 1,\ \cdots
\end{aligned} \tag{29}$$

$\hat{X}_m=\hat{J}\hat{G}_m$ $(m=0,\ 1,\ 2,\ \cdots)$ are called the vector fields of the spectral problem (1) with $\varepsilon=-1$, the first few results being

$$\hat{X}_0 = (q_x, r_x)^T, \qquad \hat{G}_0 = (0, 2r)^T;$$

$$\hat{X}_1 = \left(\left(\frac{1}{4}\frac{q}{r}r_{xx} - \frac{1}{2}\frac{q}{r}q_x - \frac{1}{2}qr^2\right)_x, \frac{1}{2}\frac{q}{r}(r_{xx} - 2q_x) + \left(\frac{1}{4}r_{xx} - \frac{1}{2}r^3 - \frac{1}{2}q_x\right)_x\right)^T,$$

$$\hat{G}_1 = (r_x - 2q, \frac{1}{2}r_{xx} - r^3 - q_x)^T.$$

The hierarchy of evolution equations associated with (1) for $\varepsilon=-1$ are given by the vector fields $\hat{X}_m$, i. e. ,

$$u_t \equiv (q, r)_t^T = \hat{X}_m(q, r), \qquad m = 0,1,2,\cdots \tag{30}$$

with the representative equation

$$(q, r)_t = \hat{X}_1(q, r)$$
$$= \left(\left(\frac{1}{4}\frac{q}{r}r_{xx} - \frac{1}{2}\frac{q}{r}q_x - \frac{1}{2}qr^2\right)_x, \frac{1}{2}\frac{q}{r}(r_{xx} - 2q_x) + \left(\frac{1}{4}r_{xx} - \frac{1}{2}r^3 - \frac{1}{2}q_x\right)_x\right)^T, \tag{31}$$

which can be also reduced to the remarkable Mkdv equation

$$r_t = \frac{1}{4}r_{xxx} - \frac{3}{2}r^2r_x$$

as $q=0$.

Combining I ($\varepsilon=1$) with II ($\varepsilon=-1$), we have two theorems below, which describe the close connection between the commutator representations for the hierarchies of evolution equations (22), (30) and the operator solutions of the operator equation (19), (27).

Theorem 3 Let $G_j=(G_j^{(1)}, G_j^{(2)})^T$ and $\hat{G}_j=(\hat{G}_j^{(1)}, \hat{G}_j^{(2)})^T$ be the Lenard's recursive gradient sequences of (1) for $\varepsilon=1$ and $\varepsilon=-1$, respectively. Let $V_j=V(G_j)$ and $\hat{V}_j=\hat{V}(\hat{G}_j)$ be seperately determined by (20) with $G=G_j$ and (28) with $G=\hat{G}_j$. Then

$$[W_m, L] = L_*(X_m), \qquad m = 0,1,2,\cdots \tag{32}$$

$$[\hat{W}_m, L] = L_*(\hat{X}_m), \qquad m = 0,1,2,\cdots \tag{33}$$

where $W_m = \sum_{j=0}^{m} V_{j-1}L^{m-j}$, $\hat{W}_m = \sum_{j=0}^{m}\hat{V}_{j-1}L^{m-j}$, $L = L(u, 1)$ in (32), $L=L(u, -1)$ in (33).

Proof From Theorem 1 and (21), we have

$$[W_m, L] = \sum_{j=0}^{m}[V_{j-1}, L]L^{m-j}$$
$$= \sum_{j=0}^{m}(L_*(KG_{j-1}) - L_*(JG_{j-1})L)L^{m-j}$$
$$= \sum_{j=0}^{m}(L_*(JG_j)L^{m-j} - L_*(JG_{j-1})L^{m-j+1})$$
$$= L_*(JG_m) - L_*(JG_{-1})L^{m+1} = L_*(X_m).$$

Similarly, from Theorem 2 and (29) we can obtain (33).

Theorem 4 The two hierarchies of evolution equations (22) and (30) possess the commutator representations

$$L_t = [W_m, L], \quad L = L(u, 1), \qquad m = 0,1,2,\cdots \tag{34}$$

and

$$L_t = [\hat{W}_m, L], \quad L = L(u, -1), \qquad m = 0,1,2,\cdots \tag{35}$$

respectively.

Proof

$$L_t = \begin{pmatrix} -\varepsilon q_t & r_t \\ \varepsilon(-q_{xt} + rq_t + qr_t) & r_{xt} - q_t - 2rr_t \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ -\varepsilon q_t & 0 \end{pmatrix}\partial = L_*(u_t).$$

For $\varepsilon=1$,

$$L_t - [W_m, L] = L_*(u_t) - L_*(X_m) = L_*(u_t - X_m).$$

For $\varepsilon=-1$,

$$L_t - [\hat{W}_m, L] = L_*(u_t) - L_*(\hat{X}_m) = L_*(u_t - \hat{X}_m).$$

In addition, noting that L_* is injective, we obtain $u_t - X_m = 0$, $u_t - \hat{X}_m = 0$ if and only if $L_t = [W_m, L]$, $L_t = [\hat{W}_m, L]$, respectively. Those are the disired rusults.

Corollary 1 (22) and (30) are the natural compatible conditions of $L(u, 1)\psi = \lambda\psi$, $\psi_t = W_m\psi$ and $L(u, -1)\psi = \lambda\psi$, $\psi_t = \hat{W}_m\psi$, respectively.

From (32) and (33), we get the results immediately.

Corollary 2 The potential vector $u=(q, r)^T$ is a finite gap, namely, it satisfies some stationary nonlinear evolution equation

$$\sum_{k=0}^{N}\alpha_k X_{N-k} = 0 \quad \text{or} \quad \sum_{k=0}^{N}\beta_k \hat{X}_{N-k} = 0 \qquad (N \geqslant 0), \tag{36}$$

if and only if

$$\Big[\sum_{k=0}^{N}\alpha_k W_{N-k}, L\Big] = 0, \qquad L = L(u, 1)$$

or

$$\Big[\sum_{k=0}^{N}\beta_k \hat{W}_{N-k}, L\Big] = 0, \quad L = L(u, 1) \qquad (N \geqslant 0), \tag{37}$$

where α_k, β_k $(0 \leqslant k \leqslant N)$ are some constants.

As a special case of Theorem 4, we obtain the commutator representations for the Mkdv hierarchy if letting $q=0$.

Corollary 3 The Mkdv hierarchy of equations

$$r_t = J\mathscr{L}^m r, \qquad m = 0,1,2,\cdots \tag{38}$$

have the commutator representations

$$L_t = [W_m, L], \qquad m = 0,1,2,\cdots \tag{39}$$

with

$$L = \begin{pmatrix} 0 & r + \partial \\ 0 & -r^2 + r_x + \partial^2 \end{pmatrix}, \tag{40}$$

$$W_m = \sum_{j=0}^{m}\left\{\begin{pmatrix} 0 & -\frac{1}{4}G_{j-1,x} - \frac{1}{8}\frac{1}{r}G_{j-1,xx} \\ 0 & \frac{1}{4}rG_{j-1,x} - \frac{1}{8}G_{j-1,xx} \end{pmatrix}\right.$$

$$+\left(\begin{array}{cc} -\frac{1}{2}\partial^{-1}rG_{j-1,x}+\frac{1}{8}\frac{1}{r}G_{j-1,xx} & 0 \\ 0 & -\frac{1}{2}\partial^{-1}rG_{j-1,x}+\frac{1}{4}G_{j-1,x} \end{array}\right)\partial\Bigg\}L^{m-j}, \qquad (41)$$

where $J=\partial$, $\mathscr{L}=\frac{1}{4}\partial^2-\partial r\partial^{-1}r\partial$, G_{j-1} $(j=0,1,\cdots,m)$ is recursively determined by the following relations: $G_j=\mathscr{L}G_{j-1}$ $(j=0,1,2,\cdots)$, $G_{-1}=0$, $G_0=r$.

Remark On the nonlinearization of the spectral problem (1) and its Lax operator algebra, we have got some results, which are left to a forthcoming paper.

Acknowledgement The author would like to express his sincere thanks to the referees for their precious opinions.

References

[1] Cao Cewen, Commutator representation of isospectral equation, *KEXUE TONGBAO*, 34 (1989), 723—724.

[2] Xu Taixi and Gu Zhuquan, Lax representations of the higher-order Heisenberg rotational chain equations, *KEXUE TONGBAO*, 34(1989), 1437.

[3] Qiao Zhijun, Lax Representations of the Levi hierarchy, *KEXUE TONGBAO*, 35(1990), 1353—1354.

[4] Ma Wenxiu, Commutator representations of Yang hierarchy of integrable evolution equations, *Chinese Sci. Bull.*, 36(1991), 1325—1329.

[5] Qiao Zhijun, Commutator representations of the D-AKNS hierarchy of evolution equations, *Mathematica Applicata*, 4(1991), 64—70.

[6] Qiao Zhijun, Commutator representations of the WKI hierarchy, *KEXUE TONGBAO*, 37(1992), 763—764.

[7] Qiao Zhijun, Commutator representations of three hierarchies of isospectral evolution equations, *Chinese Ann. Math.*, 14A(1993), 31—38.

[8] Hu Xingbiao, Three kinds of nonlinear evolution equations and their Hamiltonian structures, *Northeastern Math. J.*, 6(1990), 187—194.

[9] Cao Cewen, Nonlinearization of the Lax system for AKNS hierarchy, *Sci. China*, 33A(1990), 528—538.