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An integrable (2+1)-dimensional Camassa-Holm hierarchy with peakon solutions

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Abstract

In this letter, we propose a (2+1)-dimensional generalized Camassa-Holm (2dgCH) hierarchy with both quadratic and cubic nonlinearity. The Lax representation and peakon solutions for the 2dgCH system are derived.

Keywords: Camassa-Holm (CH) equation, peakon, Lax representation

(Some figures may appear in colour only in the online journal)

1. Introduction

In recent years, the Camassa–Holm (CH) equation [1]

$$m_t - \alpha u_x + 2mu_x + m_x u = 0, \quad m = u - u_{xx}, \tag{1}$$

has attracted a great deal of attention in the theory of integrable systems and solitons. This equation was derived as a model for the propagation of shallow water waves over a flat bed [1, 11]. In the literature, this equation was implied in the paper of Fuchssteiner and Fokas on hereditary symmetries as a very special case [2]. Since the work of Camassa and Holm [1], various remarkable studies on this equation have been developed [6–14]. The most remarkable feature of the CH equation (1) is that it admits peaked soliton (peakon) solutions in the case of $\alpha = 0$ [1, 3]. A peakon is a weak solution in some Sobolev space with a corner at its crest. The stability and interaction of peakons were discussed in several references [9–14].

In addition to the CH equation being an integrable model with peakon solutions, other integrable peakon models have been found, including the Degasperis–Procesi (DP) equation [15] whose Lax pair, bi-Hamiltonian formulation and peakon solutions were discovered in [16, 17], the cubic nonlinear peakon equations [6, 18–20], and a generalized CH equation (gCH) with both quadratic and cubic nonlinearity [4, 5, 21]

$$m_t = \frac{1}{2}k_1\left[m\left(u^2 - u_x^2\right)\right]_x + \frac{1}{2}k_2\left(2mu_x + m_x u\right),$$
$$m = u - u_{xx}, \tag{2}$$

where k_1 and k_2 are two arbitrary constants. Through some appropriate rescaling, equation (2) could be transformed to the one in the papers of Fokas and Fuchssteiner [4, 5], where it was derived from the motion of a two-dimensional, inviscid, incompressible fluid over a flat bottom. In [21], the Lax pair, bi-Hamiltonian structure, peakons, weak kinks, kink-peakon interactional and smooth soliton solutions of equation (2) are presented.

It is an interesting task to study the (2+1)-dimensional generalizations of the peakon equations. For example, in [22, 23] the authors provided a (2+1)-dimensional extension of the CH hierarchy, and they further studied the hodograph transformations and peakon solutions for their (2+1)-dimensional CH equation. In this paper, we generalize the gCH equation (2) to the whole integrable hierarchies in (1+1) and (2+1)-dimensions. We show that the gCH hierarchies admit Lax representations and construct a relation between the gCH hierarchies in (1+1) and (2+1)-dimensions. Moreover, we derive the single-peakon solution and the multi-peakon dynamic system for the (2+1)-dimensional gCH equation.

This paper is organized as follows. In section 2, we review the CH hierarchies in (1+1) and (2+1)-dimensions. In section 3, we present the gCH hierarchies in (1+1) and (2+1)-dimensions. In particular, we give their Lax representations. In section 4, we derive the peakon solutions to the (2+1)-dimensional gCH equation. Conclusions are drawn in section 5.

2. Overviews

In this section, we review the (1+1) and (2+1)-dimensional CH hierarchies presented in [8, 22, 23]. The new results we find are a relation between the CH hierarchies in (1+1) and (2+1)-dimensions and isospectral Lax representations for the CH hierarchies.

2.1. The CH hierarchies in (1+1) and (2+1)-dimensions

Let us consider the Lenard operators pair [1]

$$J = \partial_x m + m\partial_x, \quad K = \frac{1}{2}\left(\partial_x^3 - \partial_x\right). \tag{3}$$

The Lenard gradients b_{-k} are defined recursively by

$$Kb_{-k} = Jb_{-k+1}, \quad Kb_0 = 0, \quad k \in \mathbb{Z}^+. \tag{4}$$

Taking an initial value $b_0 = -\frac{1}{2}$, one may generate the negative CH hierarchy [8]

$$\begin{cases} m_{t_{-n}} = Jb_{-n}, & \\ Kb_{-j} = Jb_{-j+1}, & 1 \leqslant j \leqslant n. \end{cases} \tag{5}$$

For $n = 1$, (5) becomes

$$\begin{cases} m_{t_{-1}} = \left(mb_{-1}\right)_x + mb_{-1,x}, \\ \frac{1}{2}\left(b_{-1,xxx} - b_{-1,x}\right) = -\frac{1}{2}m_x, \end{cases} \tag{6}$$

which is nothing but the CH equation (1) with $\alpha = 0$ [1]. For $n = 2$, we arrive at

$$\begin{cases} m_{t_{-2}} = \left(mb_{-2}\right)_x + mb_{-2,x}, \\ \frac{1}{2}\left(b_{-2,xxx} - b_{-2,x}\right) = \left(mb_{-1}\right)_x + mb_{-1,x}, \\ \frac{1}{2}\left(b_{-1,xxx} - b_{-1,x}\right) = -\frac{1}{2}m_x. \end{cases} \tag{7}$$

In what follows, we call equation (7) the 2nd CH equation. For the general n, we refer to (5) as the nth CH equation.

In [22, 23], the authors proposed a (2+1)-dimensional CH equation

$$\begin{cases} m_t = \left(mb_{-2}\right)_x + mb_{-2,x}, \\ \frac{1}{2}\left(b_{-2,xxx} - b_{-2,x}\right) = m_y. \end{cases} \tag{8}$$

In general, a (2+1)-dimensional generalization of the CH hierarchy could be written as [22, 23]

$$\begin{cases} m_{t_{-n}} = Jb_{-n}, & \\ Kb_{-j} = Jb_{-j+1}, & 3 \leqslant j \leqslant n. \\ Kb_{-2} = m_y, & \end{cases} \tag{9}$$

In [22, 23], the authors also studied the hodograph transformations and the peakon solutions of the (2+1)-dimensional CH equation.

2.2. Lax representation

Let

$$U = \begin{pmatrix} 0 & 1 \\ \frac{1}{4} + \lambda m & 0 \end{pmatrix},$$

$$V^{(-n)} = -\frac{1}{2}U + \sum_{i+j=n,\ 0\leqslant i\leqslant n-1,\ 1\leqslant j\leqslant n} \lambda^{-i}\tilde{V}^{(-j)}, \tag{10}$$

where

$$\tilde{V}^{(-j)} = \begin{pmatrix} -\frac{1}{2}b_{-j,x} & b_{-j} + \frac{1}{2} - \frac{1}{2\lambda} \\ m\left(b_{-j} + \frac{1}{2}\right)\lambda - \frac{1}{2}b_{-j,xx} + \frac{1}{4}\left(b_{-j} + \frac{1}{2}\right) - \frac{1}{2}m - \frac{1}{8\lambda} & \frac{1}{2}b_{-j,x} \end{pmatrix}, \tag{11}$$

λ is the eigenparameter and b_j is defined through equation (4).

By a direct calculation, we obtain the following result.

Proposition 1. *The nth CH equation (5) admits the Lax representation*

$$U_{t_{-n}} - V_x^{(-n)} + \left[U, V^{(-n)}\right] = 0, \tag{12}$$

where the Lax pair U and $V^{(-n)}$ given by (10).

As $n = 1$, we recover the Lax pair of the well-known CH equation (1) with $\alpha = 0$ [1]

$$U = \begin{pmatrix} 0 & 1 \\ \frac{1}{4} + \lambda m & 0 \end{pmatrix},$$

$$V^{(-1)} = \begin{pmatrix} -\frac{1}{2}b_{-1,x} & b_{-1} - \frac{1}{2\lambda} \\ mb_{-1}\lambda - \frac{1}{2}b_{-1,xx} + \frac{1}{4}b_{-1} - \frac{1}{2}m - \frac{1}{8\lambda} & \frac{1}{2}b_{-1,x} \end{pmatrix}. \tag{13}$$

As $n = 2$, we obtain the Lax pair of the 2nd CH equation (7)

$$U = \begin{pmatrix} 0 & 1 \\ \frac{1}{4} + \lambda m & 0 \end{pmatrix},$$

$$V^{(-2)} = \begin{pmatrix} -\frac{1}{2}b_{-2,x} & b_{-2} - \frac{1}{2\lambda} \\ mb_{-2}\lambda - \frac{1}{2}b_{-2,xx} + \frac{1}{4}b_{-2} - \frac{1}{2}m - \frac{1}{8\lambda} & \frac{1}{2}b_{-2,x} \end{pmatrix} + \frac{1}{\lambda}\begin{pmatrix} -\frac{1}{2}b_{-1,x} & b_{-1} + \frac{1}{2} - \frac{1}{2\lambda} \\ m\left(b_{-1} + \frac{1}{2}\right)\lambda - \frac{1}{2}b_{-1,xx} + \frac{1}{4}\left(b_{-1} + \frac{1}{2}\right) - \frac{1}{2}m - \frac{1}{8\lambda} & \frac{1}{2}b_{-1,x} \end{pmatrix}. \quad (14)$$

It has been known that there exist some relations between integrable models in (1+1)-dimensions and ones in (2+1)-dimensions. For example, assembly of the first two 1+1 dimensional non-trivial members in the AKNS hierarchy: the coupled nonlinear Schrödinger equation and the coupled mKdV equation, yields the well-known (2+1)-dimensional KP equation [24–27]. The compatible solution of the first two members in the KdV hierarchy produces a special solution of the (2+1)-dimensional Sawada–Kotera equation [28–30]. In this paper, we have some similar results listed as follows.

Proposition 2. *Let $t_{-1} = y$, $t_{-2} = t$. Let $m(x, y, t)$ be a compatible solution of the CH equation (6) and the 2nd CH equation (7). Then $m(x, y, t)$ provides a special solution to the (2+1)-dimensional CH equation (8). In general, if $m(x, t_{-1}, t_{-n})$ is a compatible solution of the CH equation (6) and the nth CH equation (5), then the (2+1)-dimensional CH hierarchy (9) has a special solution $m(x, t_{-1}, t_{-n})$.*

Remark 1. Based on proposition 2, we may construct the algebraic-geometric solution of the (2+1)-dimensional CH hierarchy with the method developed in [8, 27, 28]. We will consider this topic in another publication.

3. The gCH hierarchies in (1+1)- and (2+1)-dimensions

Let us first introduce a pair of Lenard operators [21]

$$J = k_1\partial_x m\partial_x^{-1}m\partial_x + \frac{1}{2}k_2\left(\partial_x m + m\partial_x\right),$$
$$K = \partial_x - \partial_x^3, \quad (15)$$

and define the Lenard gradients b_{-k} recursively by

$$Kb_{-k} = Jb_{-k+1}, \quad Kb_0 = 0, \quad k \in \mathbb{Z}^+. \quad (16)$$

We define a gCH hierarchy in (1+1)-dimension as follows

$$\begin{cases} m_{t_{-n}} = Jb_{-n}, \\ Kb_{-j} = Jb_{-j+1}, \quad 2 \leqslant j \leqslant n. \\ Kb_{-1} = m_x, \end{cases} \quad (17)$$

The first member in (17) reads as

$$\begin{cases} m_{t_{-1}} = \frac{1}{2}k_1\left[m\left(b_{-1}^2 - b_{-1,x}^2\right)\right]_x + \frac{1}{2}k_2\left(2\,mb_{-1,x} + m_x b_{-1}\right), \\ m = b_{-1} - b_{-1,xx}, \end{cases} \quad (18)$$

which is nothing but the gCH equation (2). For $n = 2$, equation (17) is cast into the 2nd gCH equation in the gCH hierarchy (17)

$$\begin{cases} m_{t_{-2}} = k_1\left[m\partial_x^{-1}mb_{-2,x}\right]_x + \frac{1}{2}k_2\left(2\,mb_{-2,x} + m_x b_{-2}\right), \\ b_{-2,x} - b_{-2,xxx} = \frac{1}{2}k_1\left[m\left(b_{-1}^2 - b_{-1,x}^2\right)\right]_x + \frac{1}{2}k_2\left(2\,mb_{-1,x} + m_x b_{-1}\right), \\ m = b_{-1} - b_{-1,xx}. \end{cases} \quad (19)$$

For the general case $n \geqslant 2$, we refer to (17) as the nth gCH equation.

Similar to the (2+1)-dimensional generalization of the CH equation, we extend the (1+1)-dimensional gCH equation (2) to the (2+1)-dimensional system as follows:

$$\begin{cases} m_t = k_1\left[m\partial_x^{-1}mb_{-2,x}\right]_x + \frac{1}{2}k_2\left(2\,mb_{-2,x} + m_x b_{-2}\right), \\ m_y = b_{-2,x} - b_{-2,xxx}. \end{cases} \quad (20)$$

Furthermore, we may define the (2+1)-dimensional gCH hierarchy in the following form:

$$\begin{cases} m_{t_{-n}} = Jb_{-n}, \\ Kb_{-j} = Jb_{-j+1}, \quad 3 \leqslant j \leqslant n. \\ m_y = Kb_{-2}, \end{cases} \quad (21)$$

In particular, as $k_1 = 0$ and $k_2 = 2$, our (2+1)-dimensional gCH hierarchy (21) is reduced to the (2+1)-dimensional CH hierarchy (9).

Let us now show that the gCH hierarchies admit Lax representations. Let

$$U = \frac{1}{2}\begin{pmatrix} -1 & \lambda m \\ -k_1\lambda m - k_2\lambda & 1 \end{pmatrix},$$
$$V^{(-n)} = U + \sum_{0\leqslant j\leqslant n-1} \lambda^{-2j}\tilde{V}^{-(n-j)}, \tag{22}$$

where

$$\tilde{V}^{(-j)} = -\frac{1}{2}\begin{pmatrix} A & B \\ C & -A \end{pmatrix}, \tag{23}$$

with

$$A = \lambda^{-2} + k_1\partial^{-1}mb_{-j,x} + \frac{1}{2}k_2\left(b_{-j} - b_{-j,x}\right) - 1,$$
$$B = -\lambda^{-1}\left(m - b_{-j,x} + b_{-j,xx}\right) + \lambda m\left(- k_1\partial^{-1}mb_{-j,x} - \frac{1}{2}k_2b_{-j} + 1\right),$$
$$C = \lambda^{-1}\left[k_1\left(m + b_{-j,xx} + b_{-j,x}\right) + k_2\right] - \lambda\left(k_1m + k_2\right) \times\left(- k_1\partial^{-1}mb_{-j,x} - \frac{1}{2}k_2b_{-j} + 1\right). \tag{24}$$

Direct calculations lead to the following proposition.

Proposition 3. *The gCH hierarchy (17) possesses the Lax representation*

$$U_{t_{-n}} - V_x^{(-n)} + \left[U, V^{(-n)}\right] = 0,$$

with the Lax pair U and $V^{(-n)}$ given by (22).

In particular, the Lax pair of the gCH equation (18) is given by

$$U = \frac{1}{2}\begin{pmatrix} -1 & \lambda m \\ -k_1\lambda m - k_2\lambda & 1 \end{pmatrix}, V^{(-1)} = \begin{pmatrix} A_1 & B_1 \\ C_1 & -A_1 \end{pmatrix}, \tag{25}$$

with

$$A_1 = \lambda^{-2} + \frac{1}{2}k_1\left(b_{-1}^2 - b_{-1,x}^2\right) + \frac{1}{2}k_2\left(b_{-1} - b_{-1,x}\right),$$
$$B_1 = -\lambda^{-1}\left(b_{-1} - b_{-1,x}\right) - \frac{1}{2}\lambda m\left[k_1\left(b_{-1}^2 - b_{-1,x}^2\right) + k_2b_{-1}\right],$$
$$C_1 = \lambda^{-1}\left[k_1\left(b_{-1} + b_{-1,x}\right) + k_2\right] + \frac{1}{2}\lambda\left[k_1^2\, m\left(b_{-1}^2 - b_{-1,x}^2\right) + k_1k_2\left(mb_{-1} + b_{-1}^2 - b_{-1,x}^2\right) + k_2^2b_{-1}\right]. \tag{26}$$

The Lax pair of the 2nd gCH equation (19) is given by

$$U = \frac{1}{2}\begin{pmatrix} -1 & \lambda m \\ -k_1\lambda m - k_2\lambda & 1 \end{pmatrix},$$
$$V^{(-2)} = U + \tilde{V}^{(-2)} + \lambda^{-2}\tilde{V}^{(-1)}, \tag{27}$$

where $\tilde{V}^{(-1)}$ and $\tilde{V}^{(-2)}$ are defined by (23) and (24).

One may easily check the following results.

Proposition 4. *Let $t_{-1} = y$, $t_{-2} = t$. Let $m(x, y, t)$ be a compatible solution of the gCH equation (18) and the 2nd gCH equation (19). Then $m(x, y, t)$ provides a special solution to (2+1)-dimensional gCH equation (20). In general, if $m(x, t_{-1}, t_{-n})$ is a compatible solution of the gCH equation (18) and the nth gCH equation (17), then the (2+1)-dimensional gCH hierarchy (21) has a special solution $m(x, t_{-1}, t_{-n})$.*

4. Peakon solutions to the 2dgCH equation (20)

Assume that the single-peakon solution of the (2+1)-dimensional gCH equation (20) is given in the form of

$$b_{-2} = p(y, t)e^{-|x-q(y,t)|}, \quad m = 2r(y, t)\delta(x - q(y, t)), \tag{28}$$

where $p(y, t)$, $q(y, t)$ and $r(y, t)$ are to be determined. Substituting (28) into (20) and integrating against the test function with support around the peak, we finally arrive at

$$\begin{cases} r_y = r_t = 0, \\ q_y = -\dfrac{p}{r}, \\ q_t = -\dfrac{1}{3}k_1rp - \dfrac{1}{2}k_2p, \end{cases} \tag{29}$$

which yields

$$\begin{cases} r = c, \\ q = F\left(y + \left(\dfrac{1}{3}k_1c^2 + \dfrac{1}{2}k_2c\right)t\right), \\ p = -cq_y, \end{cases} \tag{30}$$

where c is an arbitrary constant and F is an arbitrary smooth function. Thus, the single-peakon solution of equation (20) is given by

$$b_{-2} = -cF_y\left(y + \left(\frac{1}{3}k_1c^2 + \frac{1}{2}k_2c\right)t\right) \times e^{-\left|x-F\left(y+\left(\frac{1}{3}k_1c^2+\frac{1}{2}k_2c\right)t\right)\right|},$$
$$m = 2c\delta\left(x - F\left(y + \left(\frac{1}{3}k_1c^2 + \frac{1}{2}k_2c\right)t\right)\right). \tag{31}$$

As $k_1 = 0$, $k_2 = 2$, we recover the single-peakon solution of the (2+1)-dimensional CH equation proposed in [22].

In particular, if we take $F(y + (\frac{1}{3}k_1c^2 + \frac{1}{2}k_2c)t) = y + (\frac{1}{3}k_1c^2 + \frac{1}{2}k_2c)t$, then the single-peakon solution of equation (20) becomes

$$b_{-2} = -ce^{-\left|x-y-\left(\frac{1}{3}k_1c^2+\frac{1}{2}k_2c\right)t\right|},$$
$$m = 2c\delta\left(x - y - \left(\frac{1}{3}k_1c^2 + \frac{1}{2}k_2c\right)t\right). \tag{32}$$

See figure 1 for the graph of the single-peakon solution $b_{-2}(x, y, t)$ at $t = 0$. If we take $F(y + (\frac{1}{3}k_1c^2 + \frac{1}{2}k_2c)t) = \left(y + (\frac{1}{3}k_1c^2 + \frac{1}{2}k_2c)t\right)^2$, then the single-peakon solution (31)

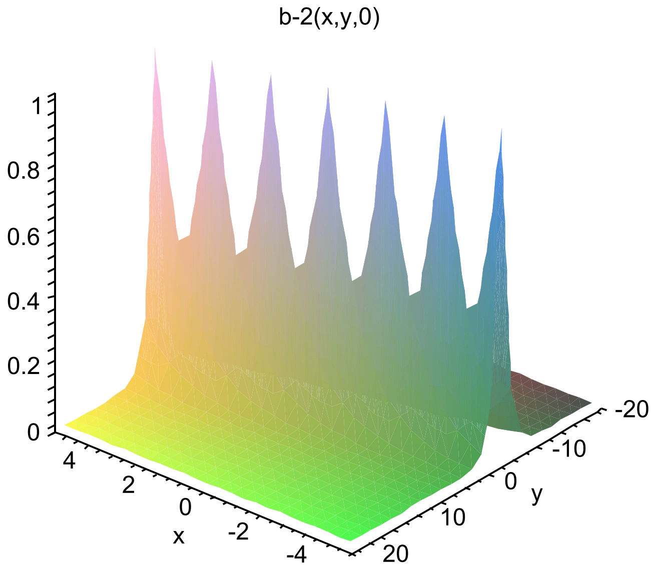

Figure 1. Single-peakon solution $b_{-2}(x, y, t)$ in (32) with $c = -1$ at $t = 0$.

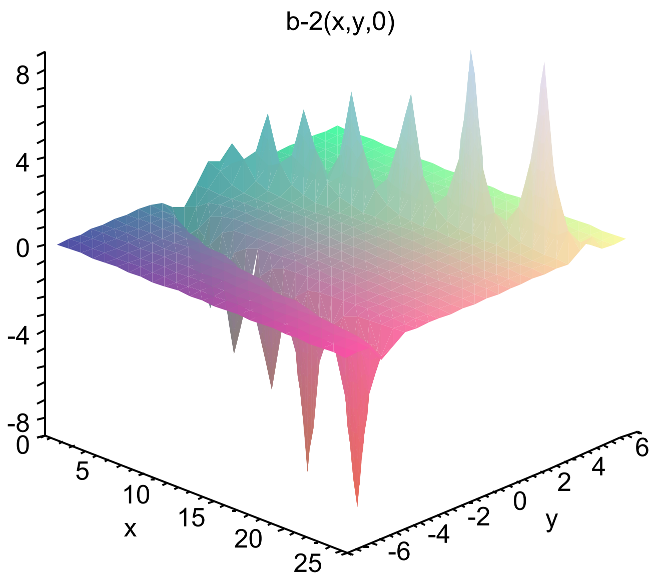

Figure 2. Single-peakon solution $b_{-2}(x, y, t)$ in (33) with $c = -1$ at $t = 0$.

becomes

$$b_{-2} = -2c\left(y + \left(\frac{1}{3}k_1c^2 + \frac{1}{2}k_2c\right)t\right) \times e^{-\left|x-\left(y+\left(\frac{1}{3}k_1c^2+\frac{1}{2}k_2c\right)t\right)^2\right|},$$
$$m = 2c\delta\left(x - \left(y + \left(\frac{1}{3}k_1c^2 + \frac{1}{2}k_2c\right)t\right)^2\right). \tag{33}$$

See figure 2 for the graph of $b_{-2}(x, y, t)$ in (33) at $t = 0$.

In general, let us suppose that the N-peakon has the following form

$$b_{-2} = \sum_{j=1}^{N} p_j(y, t)e^{-\left|x-q_j(y,t)\right|},$$
$$m = 2\sum_{j=1}^{N} r_j(y, t)\delta\left(x - q_j(y, t)\right). \tag{34}$$

Similar to the cases of one-peakon but with a lengthy calculation, we are able to obtain the following N-peakon dynamical system

$$r_{j,y} = 0,$$
$$r_{j,t} = -\frac{1}{2}k_2r_j\sum_{k=1}^{N}p_k\, sgn\left(q_j - q_k\right)e^{-\left|q_j-q_k\right|},$$
$$p_j = -r_jq_{j,y},$$
$$q_{j,t} = \frac{1}{6}k_1r_jp_j - \frac{1}{2}k_2\sum_{k=1}^{N}p_ke^{-\left|q_j-q_k\right|} + \frac{1}{2}k_1\sum_{i,k=1}^{N}r_ip_k\left(sgn\left(q_j - q_i\right)sgn\left(q_j - q_k\right) - 1\right) \times e^{-\left|q_j-q_i\right|-\left|q_j-q_k\right|}. \tag{35}$$

5. Conclusion

In this paper, we have extended the gCH equation to the hierarchies in (1+1)-dimensions and (2+1)-dimensions. We first show the gCH hierarchies admit Lax representation. Then we show that the (2+1)-dimensional gCH equation possesses a single peakon solution as well as multi-peakon solutions. Other topics, such as smooth soliton solutions, cuspons, peakon stability, and algebra-geometric solutions, remain to be developed.

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