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The N-kink, bell-shape and hat-shape solitary solutions of b-family equation in the case of $b = 0$

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ABSTRACT

In this Letter, we study the solutions of equation $m_t = m_x u$, $m = u - u_{xx}$, which actually comes from the so-called b-family equation $m_t = m_x u + bmu_x$, $m = u - u_{xx}$ in the case of $b = 0$. We show that this equation admits both N-peakon and N-kink solutions. In particular, the bell-shape soliton, hat-shape soliton, single-kink, and two-kink solutions are presented in an explicit formula.

1. Introduction

In [1,2], the authors proposed the following so-called b-family equation

$$m_t = m_x u + bmu_x, \quad m = u - u_{xx}, \tag{1}$$

where b is an arbitrary constant. There are some distinguished special cases of this equation. For example, in the case of $b = 2$, this equation is reduced to the well-known Camassa–Holm (CH) equation, which was derived by Camassa and Holm [3] as a shallow water wave model [3]. The CH equation was found to be completely integrable with a Lax pair and associated bi-Hamiltonian structure [3,4]. The most interesting feature of the CH equation is that it admits peaked soliton (peakon) solutions [3,5]. A peakon is a weak solution in some Sobolev space with corner at its crest. The stability and interaction of peakons were discussed in several references [6–11]. In the case of $b = 3$, Eq. (1) becomes the Degasperis–Procesi (DP) equation, which is another important nonlinear model possessing peakon solutions [12–14]. The integrability of the Degasperis–Procesi equation was shown by constructing a Lax pair, and deriving two infinite sequences of conservation laws [13].

Recently, we presented an integrable equation with both quadratic and cubic nonlinearity [15]:

$$\begin{aligned} &m_t = \alpha u_x + \frac{1}{2}k_1\left[m\left(u^2 - u_x^2\right)\right]_x + \frac{1}{2}k_2(2mu_x + m_x u),\\ &m = u - u_{xx}, \end{aligned} \tag{2}$$

where α, k_1, and k_2 are three arbitrary constants. In the case of $\alpha = 0$, we derived the N-peakon solution. In the case of $\alpha \neq 0$ and $k_2 = 0$, we found that this equation allows single weak kink and kink–peakon interactional solutions. However, different from the N-peakon solutions in the form of linear superpositions of the single-peakon, Eq. (2) does not admit the N-kink solution in the form of the superpositions of single-kink. It is natural to ask whether there exists a nonlinear model that may admit both N-peakon and N-kink solutions.

In this Letter, we will show that Eq. (1) in the case of $b = 0$, namely,

$$m_t = m_x u, \quad m = u - u_{xx}, \tag{3}$$

possesses both N-peakon (N-peakon solutions were already presented in [1]) and N-kink solutions. In particular, the bell-shape soliton, hat-shape soliton, stationary kink, and two-kink solutions of Eq. (3) are presented in an explicit formula and plotted. Within our knowledge, this is probably the first nonlinear model that allows the N-peakon and N-kink solution at the same time.

2. The soliton and N-kink solutions of Eq. (3)

In [1,2], Holm, Staley, and Hone have deduced the N-peakon solutions for the b-family Eq. (1)

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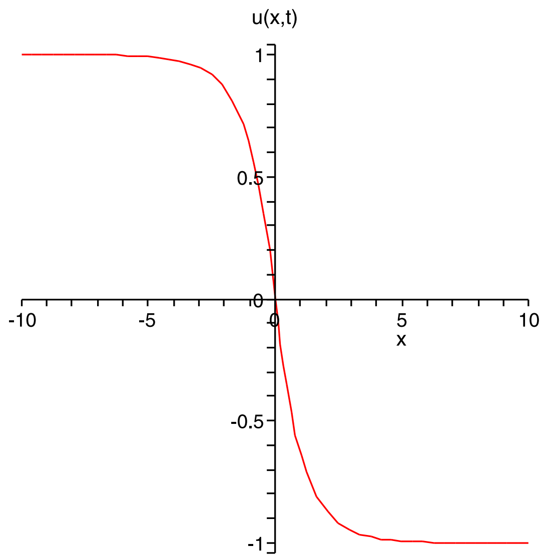

Fig. 1. The stationary single-kink solution $u(x,t)$ given by (10) with $c_1 = 1$ and $c = 0$.

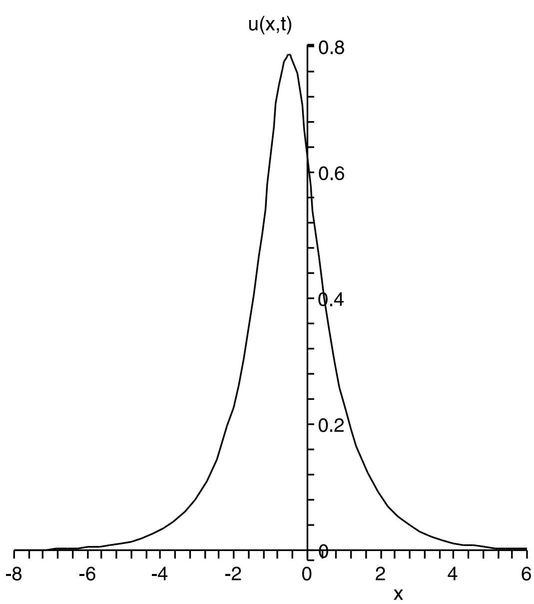

Fig. 2. The bell-shape solution $u(x,t)$ determined by (13) with $B_1 = c_1 = 1$ at the moment of $t = 0$.

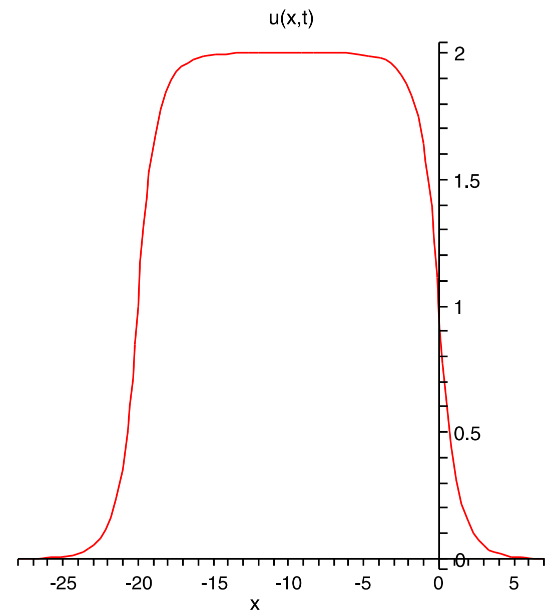

Fig. 3. The hat-shape solution $u(x,t)$ determined by (13) with $B_1 = 20$ and $c_1 = 1$ at the moment of $t = 0$.

$$u = \sum_{j=1}^{N} p_j(t) e^{-|x-q_j(t)|}, \tag{4}$$

where $p_j(t)$ and $q_j(t)$ evolve according to the dynamical system

$$q_{j,t} = \sum_{k=1}^{N} p_k e^{-|q_j-q_k|},$$
$$p_{j,t} = (b-1)\sum_{k=1}^{N} p_j p_k \operatorname{sgn}(q_j - q_k) e^{-|q_j-q_k|}. \tag{5}$$

The N-peakon dynamical system of Eq. (3) is just (5) with the case of $b = 0$. See [1] for the details of the peakon solutions of b-family Eq. (1).

Let us now derive the N-kink solution of Eq. (3). We suppose the N-kink solution as the form

$$u = \sum_{j=1}^{N} c_j \operatorname{sgn}\big(x - q_j(t)\big)\big(e^{-|x-q_j(t)|} - 1\big), \tag{6}$$

where c_j are arbitrary constants and $q_j(t)$ are to be determined. It is easy to check that

$$u_x = -\sum_{j=1}^{N} c_j e^{-|x-q_j|}, \qquad u_t = \sum_{j=1}^{N} c_j q_{j,t} e^{-|x-q_j|}. \tag{7}$$

The second order and higher order partial derivatives of (6) do not exist at $x = q_j(t)$. But in the distribution sense, we have

$$m_x = -2\sum_{j=1}^{N} c_j \delta(x - q_j), \qquad m_t = 2\sum_{j=1}^{N} c_j q_{j,t} \delta(x - q_j). \tag{8}$$

Substituting (6)–(8) into Eq. (3) and integrating against test function with compact support, we obtain that $q_j(t)$ evolve according to the system

$$q_{j,t} = -\sum_{i=1}^{N} c_i \operatorname{sgn}(q_j - q_i)\big(e^{-|q_j-q_i|} - 1\big), \quad 1 \leqslant j \leqslant N. \tag{9}$$

For $N = 1$, we have $q_{1,t} = 0$, which yields $q_1 = c$, where c is an arbitrary constant. Thus the single-kink solution is stationary and it reads

$$u = c_1 \operatorname{sgn}(x - c)\big(e^{-|x-c|} - 1\big). \tag{10}$$

See Fig. 1 for the profile of this stationary kink solution.

For $N = 2$, (9) is reduced to

$$\begin{cases} q_{1,t} = -c_2 \operatorname{sgn}(q_1 - q_2)\big(e^{-|q_1-q_2|} - 1\big), \\ q_{2,t} = c_1 \operatorname{sgn}(q_1 - q_2)\big(e^{-|q_1-q_2|} - 1\big). \end{cases} \tag{11}$$

If $c_1 + c_2 = 0$, we may obtain

$$\begin{cases} q_1(t) = c_1 \operatorname{sgn}(B_1)\big(e^{-|B_1|} - 1\big)t, \\ q_2(t) = q_1(t) - B_1, \end{cases} \tag{12}$$

where B_1 is an arbitrary constant. The solution becomes

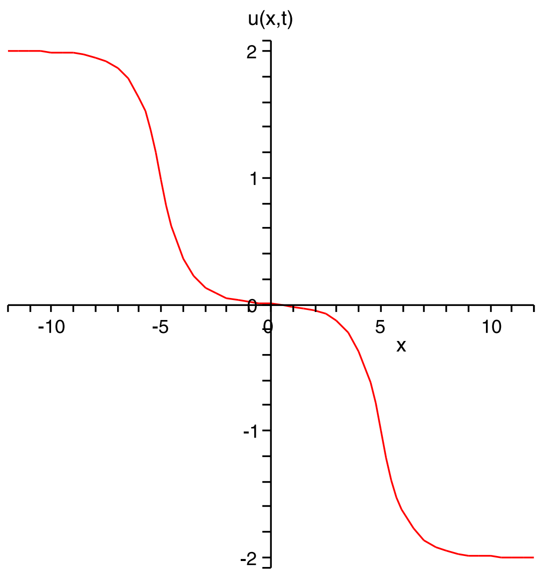

Fig. 4. The two-kink solution (15) at the moment of $t = 5$.

$$u(x,t) = c_1\big[\operatorname{sgn}(x-q_1)\big(e^{-|x-q_1|}-1\big) - \operatorname{sgn}(x-q_2)\big(e^{-|x-q_2|}-1\big)\big], \tag{13}$$

where q_1 and q_2 are given by (12). It is interesting that the above solution demonstrates the solitary wave shape. Explicitly, if the value $|B_1|$ is small, solution (13) presents the bell-shape. See Fig. 2 for this solution with $B_1 = 1$. If the value $|B_1|$ is a little large, solution (13) takes on the hat-shape. See Fig. 3 for this hat-shape solution with $B_1 = 20$.

If $c_1 + c_2 \neq 0$, from (11) we obtain

$$\begin{cases} q_1(t) = \dfrac{c_2}{c_1+c_2}\ln\big(A_1 e^{(c_1+c_2)t}+1\big), \\ q_2(t) = \dfrac{-c_1}{c_1+c_2}\ln\big(A_1 e^{(c_1+c_2)t}+1\big), \end{cases} \tag{14}$$

where $A_1 \geqslant 0$ is a constant. In particular, for $c_1 = c_2 = 1$ and $A_1 = 1$, the solution becomes

$$u(x,t) = \operatorname{sgn}\Big(x - \frac{1}{2}\ln\big(e^{2t}+1\big)\Big)\big(e^{-|x-\frac{1}{2}\ln(e^{2t}+1)|}-1\big) + \operatorname{sgn}\Big(x + \frac{1}{2}\ln\big(e^{2t}+1\big)\Big)\big(e^{-|x+\frac{1}{2}\ln(e^{2t}+1)|}-1\big), \tag{15}$$

which is two-kink solution. See Fig. 4 for the profile of this two-kink solution.

Acknowledgements

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