Physics Letters A ••• (••••) •••–•••

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Physics Letters A

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Integrable generalization of the associated Camassa–Holm equation ☆

Lin Luo [a,1], Zhijun Qiao [b], Juan Lopez [b]

[a] Department of Mathematics, Shanghai Second Polytechnic University, Shanghai 201209, PR China
[b] Department of Mathematics, University of Texas-Pan American, Edinburg, TX 78539, USA

ARTICLE INFO

Article history:
Received 27 August 2013
Accepted 24 November 2013
Available online xxxx
Communicated by R. Wu

Keywords:
The ACH–KdV equation
Bäcklund transformation
Conservation law
Darboux transformation

ABSTRACT

In this paper, we study an integrable generalization of the associated Camassa–Holm equation. The generalized system is shown to be integrable in the sense of Lax pair and the bilinear Bäcklund transformations are presented through the Bell polynomial technique. Meanwhile, its infinite conservation laws are constructed, and conserved densities and fluxes are given in explicit recursion formulas. Furthermore, a Darboux transformation for the system is derived with the help of the gauge transformation between two Lax pairs. As an application, soliton and periodic wave solutions are given through the Darboux transformation.

1. Introduction

The Camassa–Holm (CH) equation

$$U_T + 2k^2 U_X - U_{XXT} + 3UU_X = 2U_X U_{XXX} + UU_{XXX}, \tag{1.1}$$

where the real constant $k > 0$, was derived as a model for shallow water waves by Camassa and Holm in 1993 [1]. This equation is integrable with the following Lax pair:

$$\psi_{xx} = \lambda\left(u - u_{xx} + k^2\right)\psi + \frac{1}{4}\psi,$$
$$\psi_t = \left(\frac{1}{2\lambda} - u\right)\psi_x + \frac{1}{2}u_x\psi. \tag{1.2}$$

Considerable interest was paid on the CH equation in recent decades about its integrability and various kind of exact solutions [2–17]. Schiff and Fisher showed that the Camassa–Holm equation possessed the Bäcklund transformations and an infinite number of local conserved quantities by using the Loop group approach [5, 6]. Parker gave explicit multi-soliton solutions for the CH equation by taking the Hirota bilinear method and a coordinate transformation [12]. Its structure and dynamics were investigated in the different parameter regimes. According to the Ref. [12], there is a reciprocal transformation, $(T, X) \to (t, x)$, such that

$$dx = R\,dX - UR\,dT, \quad dt = dT, \quad R = \sqrt{U - U_{XX} + k^2}. \tag{1.3}$$

Let us apply the reciprocal transformation to the Lax pair (1.2) and define the following potential function $u(x,t)$

$$u = \frac{1}{2}\frac{R_{xx}}{R} - \frac{1}{4}\left(\frac{R_x}{R}\right)^2 + \frac{1}{4R^2} - \frac{1}{4k^2}, \tag{1.4}$$

then Eq. (1.1) is transformed into the following associated Camassa–Holm (ACH) equation

$$u_t + 2k^3 u_x + 4k^2 uu_t + 2k^2 u_x \partial_x^{-1} u_t - k^2 u_{xxt} = 0. \tag{1.5}$$

Hone showed in Ref. [18] how the ACH equation (1.5) is related to Schrodinger operators and the KdV equation, and described how to construct solutions of the ACH equation from tau-functions of the KdV hierarchy, including rational, N-soliton, and elliptic solutions.

Recently, integrable negative order flows, mixed equations and the relationship of different hierarchies attracted much attention, including continuous and discrete cases, such as the negative KdV, mixed KdV, and Volterra lattice equations [19–25]. Inspired by the above works, let us consider a new generalization of the associated Camassa–Holm equation, namely,

$$u_t + \alpha(u_{xxx} - 6uu_x)$$
$$+ \beta\left(2k^3 u_x + 4k^2 uu_t + 2k^2 u_x \partial_x^{-1} u_t - k^2 u_{xxt}\right) = 0, \tag{1.6}$$

where α, β are two arbitrary constants.

Apparently, Eq. (1.6) is reduced to the ACH equation (1.5) when we take $\alpha = 0$, $\beta = 1$. For $\alpha = 1, \beta = 0$, Eq. (1.6) gives the KdV equation. So, Eq. (1.6) may be called the ACH–KdV equation. In this paper, we show that (1.6) is integrable in the sense of Lax pair. We use the Bell polynomial [26] and Hirota's bilinear methods [27]

E-mail addresses: luolin@sspu.edu.cn (L. Luo), qiao@utpa.edu (Z. Qiao), jflopezz@utpa.edu (J. Lopez).
[1] Fax number: +86 21 50214982.

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http://dx.doi.org/10.1016/j.physleta.2013.11.029

to provide the bilinear Bäcklund transformation and infinite conservation laws [28–40]. Furthermore, we construct (1.6)'s Darboux transformation through Lax pair, and apply it to obtain the exact solutions of the ACH–KdV equation (1.6).

Our paper is organized as follows. In Section 2, by using Bell polynomials, we study the bilinear Bäcklund transformations of the ACH–KdV equation, and the Lax pair is also recovered. In Section 3, infinite conservation laws of the ACH–KdV equation are derived by virtue of the Lax equations. All conserved densities and fluxes are recursively determined in an explicit formula. In Sections 4 and 5, a Darboux transformation of the ACH–KdV equation is presented through its Lax pair, and soliton and periodic wave solutions are obtained through the Darboux transformation. Some conclusions are given in the last section.

2. The bilinear Bäcklund transformation

In this section, we focus on the bilinear Bäcklund transformation of the ACH–KdV equation (1.6). Let

$$u = -q_{xx}, \tag{2.1}$$

then substituting (2.1) into Eq. (1.6), yields

$$\begin{aligned} &q_{2x,t} + \alpha(q_{5x} + 6q_{2x}q_{3x}) \\ &\quad + \beta\big(2k^3 q_{3x} - 4k^2 q_{2x}q_{2x,t} - 2k^2 q_{3x}q_{x,t} - k^2 q_{4x,t}\big) = 0, \end{aligned} \tag{2.2}$$

where $q_{r_1x_1,\ldots,r_\ell x_\ell} = \partial_{x_1}^{r_1}\cdots\partial_{x_\ell}^{r_\ell} q$.
Simultaneously, an auxiliary independent variable τ is able to be determined through the following equation

$$q_{4x} + 3q_{2x}^2 = -q_{x,\tau}. \tag{2.3}$$

Substituting (2.3) into Eq. (2.2), and integrating once with respect to x, we obtain

$$\begin{aligned} E(q) &\equiv q_{x,t} + \alpha\big[q_{4x} + 3q_{2x}^2\big] \\ &\quad + \beta\left[2k^3 q_{2x} - \frac{2}{3}k^2(q_{3x,t} + 3q_{2x}q_{x,t}) + \frac{1}{3}k^2 q_{\tau,t}\right] = 0. \end{aligned} \tag{2.4}$$

Let $q = 2\ln g$ and $q' = 2\ln f$ be two different solutions of the above equation (2.4). The corresponding two-field condition is

$$\begin{aligned} &E(q') - E(q) \\ &\quad = (q'-q)_{x,t} + \alpha\big[(q'-q)_{4x} + 3q_{2x}'^2 - 3q_{2x}^2\big] \\ &\qquad + \beta\bigg[2k^3(q'-q)_{2x} + \frac{1}{3}k^2(q'-q)_{\tau,t} - \frac{2}{3}k^2(q'-q)_{3x,t} \\ &\qquad - 2k^2\big(q'_{2x}q'_{x,t} - q_{2x}q_{x,t}\big)\bigg] \\ &\quad = 0. \end{aligned} \tag{2.5}$$

To find the bilinear Bäcklund transformation of the ACH–KdV equation (1.6), we define the following two variables

$$v = (q'-q)/2 = \ln\frac{f}{g}, \qquad w = (q'+q)/2 = \ln(fg). \tag{2.6}$$

Then, inserting (2.6) into Eqs. (2.3) and (2.5) yields

$$\begin{aligned} &\frac{1}{2}\big(E(q') - E(q)\big) \\ &\quad = v_{x,t} + \alpha(v_{4x} + 6w_{2x}v_{2x}) + \beta\bigg[2k^3 v_{2x} + \frac{1}{3}k^2 v_{\tau,t} - \frac{2}{3}k^2 v_{3x,t} \\ &\qquad - 2k^2(v_{2x}w_{x,t} + v_{x,t}w_{2x})\bigg] = 0, \end{aligned} \tag{2.7}$$

and

$$v_{x,\tau} + v_{4x} + 6v_{2x}w_{2x} = 0. \tag{2.8}$$

Let us impose a constraint equation

$$w_{2x} + v_x^2 = \lambda, \tag{2.9}$$

where λ is a parameter. We substitute (2.9) into Eq. (2.8) and integrate in x to obtain

$$v_\tau = 2v_x^3 - v_{3x} - 6\lambda v_x. \tag{2.10}$$

Then, the two-field condition (2.7) becomes

$$\begin{aligned} &v_{x,t} + \alpha(v_{4x} + 6w_{2x}v_{2x}) + \beta\bigg[2k^3 v_{2x} - 2k^2 v_x^2 v_{x,t} - \frac{1}{3}k^2 v_{3x,t} \\ &\quad + 6k^2\lambda v_{x,t} - 2k^2(v_{2x}w_{x,t} + v_{x,t}w_{2x})\bigg] = 0, \end{aligned} \tag{2.11}$$

i.e.

$$\begin{aligned} &\partial_x\big[v_t + \alpha\big(v_{3x} + 3v_x w_{2x} + v_x^3\big) + 3\alpha\lambda v_x + 2\beta k^3 v_x \\ &\quad - \beta k^2\big(v_{2x,t} + w_{2x}v_t + 2w_{x,t}v_x + v_x^2 v_t\big) - 3\beta k^2\lambda v_t\big] = 0. \end{aligned} \tag{2.12}$$

Based on the Bell polynomial theory, we have

$$\begin{aligned} &\partial_x\big[\mathcal{Y}_t(v,w) + \alpha\big(\mathcal{Y}_{3x}(v,w) + 3\lambda\mathcal{Y}_x(v,w)\big) \\ &\quad + \beta\big(2k^3\mathcal{Y}_x(v,w) - k^2\mathcal{Y}_{2x,t}(v,w) - 3k^2\lambda\mathcal{Y}_t(v,w)\big)\big] = 0, \end{aligned} \tag{2.13}$$

where we have taken advantage of the following formula [26]:

$$\begin{aligned} &\mathcal{Y}_{n_1x_1,\ldots,n_\ell x_\ell}(v,w) \\ &\quad = \mathcal{Y}_{n_1,\ldots,n_\ell}(f)\Big|_{f_{r_1x_1,\ldots,r_\ell x_\ell} = \begin{cases} v_{r_1x_1,\ldots,r_\ell x_\ell}, & r_1+\cdots+r_\ell \text{ is odd,} \\ w_{r_1x_1,\ldots,r_\ell x_\ell}, & r_1+\cdots+r_\ell \text{ is even.} \end{cases}} \end{aligned} \tag{2.14}$$

Integrating Eq. (2.13) with respect to x and setting the integral constant equal to zero, we obtain the following coupled system of $\mathcal{Y}(v,w)$ polynomials

$$\begin{aligned} &\mathcal{Y}_{2x}(v,w) = \lambda, \\ &\mathcal{Y}_t(v,w) + \alpha\big(\mathcal{Y}_{3x}(v,w) + 3\lambda\mathcal{Y}_x(v,w)\big) \\ &\quad + \beta\big(2k^3\mathcal{Y}_x(v,w) - k^2\mathcal{Y}_{2x,t}(v,w) - 3k^2\lambda\mathcal{Y}_t(v,w)\big) = 0. \end{aligned} \tag{2.15}$$

By virtue of the identity between the $\mathcal{Y}(v,w)$ polynomials and the bilinear operator

$$\begin{aligned} &\mathcal{Y}_{n_1x_1,\ldots,n_\ell x_\ell}(v = \ln f/g, w = \ln fg) = (fg)^{-1} D_{x_1}^{n_1}\cdots D_{x_\ell}^{n_\ell} f\cdot g, \\ &\quad n_1 + n_2 + \cdots + n_\ell \geqslant 1, \end{aligned} \tag{2.16}$$

where D is the Hirota derivatives defined by

$$\begin{aligned} &D_{x_1}^{n_1}D_{x_2}^{n_2} f\cdot g \\ &\quad = (\partial_{x_1} - \partial_{x_1'})^{n_1}(\partial_{x_2} - \partial_{x_2'})^{n_2} f(x_1,x_2)g(x_1',x_2')\big|_{x_1'=x_1, x_2'=x_2}. \end{aligned}$$

Thus we immediately obtain the following bilinear Bäcklund transformation for the ACH–KdV equation (1.6):

$$\begin{aligned} &\big[D_x^2 - \lambda\big] f\cdot g = 0, \\ &\big[(1 - 3k^2\lambda\beta)D_t + \alpha\big(D_x^3 + 3\lambda D_x\big) + \beta\big(2k^3 D_x - k^2 D_x^2 D_t\big)\big] f\cdot g \\ &\quad = 0. \end{aligned} \tag{2.17}$$

Using the Hopf–Cole transformation $v = \ln\varphi$ and formula

$$(fg)^{-1} D_{x_1}^{n_1} \cdots D_{x_l}^{n_l} f \cdot g \,\big|_{g=\exp(q/2),\, f/g=\varphi} = \varphi^{-1} \sum_{r_1+\cdots+r_l=\text{even}} \sum_{r_1=0}^{n_1} \cdots \sum_{r_l=0}^{n_l} \prod_{i=1}^{l} \binom{n_i}{r_i} P_{r_1 x_1,\ldots,r_l x_l}(q) \times \varphi_{(n_1-r_1)x_1,\ldots,(n_l-r_l)x_l}, \tag{2.18}$$

we may transform (2.17) into the following linear system,

$$\begin{aligned} &M\varphi \equiv \varphi_{2x} + q_{2x}\varphi = \lambda\varphi, \\ &N\varphi \equiv \left(1 - 3\beta k^2\lambda - \beta k^2 q_{2x}\right)\varphi_t + \alpha\varphi_{xxx} - \beta k^2\varphi_{xx,t} \\ &\quad + \left(3\alpha q_{2x} - 2\beta k^2 q_{x,t} + 2\beta k^3 + 3\alpha\lambda\right)\varphi_x = 0, \end{aligned} \tag{2.19}$$

which is the Lax pair of Eq. (2.2) with Lax operators

$$\begin{aligned} &M = \partial_x^2 + q_{2x}, \\ &N = \left(1 - 3\beta k^2\lambda - \beta k^2 q_{2x}\right)\partial_t + \alpha\partial_x^3 - \beta k^2\partial_x^2\partial_t \\ &\quad + \left(3\alpha q_{2x} - 2\beta k^2 q_{x,t} + 2\beta k^3 + 3\alpha\lambda\right)\partial_x. \end{aligned} \tag{2.20}$$

Combining with $u = -q_{2x}$, we have the Lax pair for the ACH–KdV equation (1.6)

$$\begin{aligned} &\varphi_{xx} - u\varphi = \lambda\varphi, \\ &\left(1 - 3\beta k^2\lambda + \beta k^2 u\right)\varphi_t + \alpha\varphi_{xxx} - \beta k^2\varphi_{xx,t} \\ &\quad + \left(-3\alpha u + 2\beta k^2\partial_x^{-1}u_t + 2\beta k^3 + 3\alpha\lambda\right)\varphi_x = 0. \end{aligned} \tag{2.21}$$

By a long calculation, it is not difficult to get that (1.6) is exactly the compatibility condition of (2.21).

3. Conservation laws

This section is contributed to construct the infinite conservation laws for Eq. (2.2). We introduce a new potential function $\eta(x,t)$, such that

$$\eta = \frac{q'_x - q_x}{2}, \tag{3.1}$$

where q, q' is the same as in (2.15), then we have

$$v_x = \eta, \qquad w_x = \eta + q_x. \tag{3.2}$$

Substituting (3.2) into (2.9) yields a Riccati-type equation:

$$\eta_x + \eta^2 + q_{2x} = \lambda. \tag{3.3}$$

We set $\lambda = \epsilon^2 - \frac{1}{4k^2}$, and expand η as

$$\eta = \epsilon + \sum_{n=1}^{\infty} I^{(n)}\epsilon^{-n}, \tag{3.4}$$

which is inserted into the Riccati equation (3.3), and equating the coefficients of ϵ, we obtain the recursion relations for $I^{(n)}$ as follows:

$$\begin{aligned} &I^{(1)} = -\frac{1}{2}q_{2x} + \frac{1}{8k^2}, \qquad I^{(2)} = \frac{1}{4}q_{3x}, \\ &I^{(3)} = -\frac{1}{8}\left[q_{4x} + \left(q_{2x} - \frac{1}{4k^2}\right)^2\right], \\ &\ldots \\ &2I^{(n+1)} + I_x^{(n)} + \sum_{i+j=n,\, i,j\geqslant 1} I^{(i)}I^{(j)} = 0, \end{aligned} \tag{3.5}$$

where $I^{(n)}$ does not denote the conserved density of the ACH–KdV equation.

Taking advantage of (3.1) and (3.3), we are able to rewrite Eq. (2.7) as the divergence-type form

$$\partial_t\left[\left(1 - 4\beta k^2\lambda\right)\eta - \beta k^2\eta_{2x} - 2\beta k^2\eta\eta_x\right] + \partial_x\left[\alpha\left(\eta_{2x} - 2\eta^3 + 6\lambda\eta\right) + 2\beta k^3\eta - 2k^2\beta q_{x,t}\eta\right] = 0. \tag{3.6}$$

Substituting expansion (3.4) into (3.6) leads to

$$\begin{aligned} &\partial_t\Bigg[\left(1 - 4\beta k^2\lambda\right)\sum_{n=1}^{\infty} I^{(n)}\epsilon^{-n} - \beta k^2\sum_{n=1}^{\infty} I_{2x}^{(n)}\epsilon^{-n} \\ &\quad - 2\beta k^2\left(\epsilon + \sum_{i=1}^{\infty} I^{(i)}\epsilon^{-i}\right)\left(\sum_{j=1}^{\infty} I_x^{(j)}\epsilon^{-j}\right)\Bigg] \\ &\quad + \partial_x\Bigg[\alpha\sum_{n=1}^{\infty} I_{2x}^{(n)}\epsilon^{-n} - 2\alpha\left(\epsilon + \sum_{i=1}^{\infty} I^{(i)}\epsilon^{-i}\right) \\ &\quad \times \left(\epsilon + \sum_{j=1}^{\infty} I^{(j)}\epsilon^{-j}\right)\left(\epsilon + \sum_{l=1}^{\infty} I^{(l)}\epsilon^{-l}\right) \\ &\quad + 6\alpha\lambda\sum_{n=1}^{\infty} I^{(n)}\epsilon^{-n} + 2\beta k^3\sum_{n=1}^{\infty} I^{(n)}\epsilon^{-n} \\ &\quad - 2k^2\beta q_{x,t}\left(\epsilon + \sum_{n=1}^{\infty} I^{(n)}\epsilon^{-n}\right)\Bigg] = 0. \end{aligned} \tag{3.7}$$

Comparing the powers of ϵ in Eq. (3.7) provides the following infinite conservation laws

$$F_t^{(n)} + G_x^{(n)} = 0, \quad n = -1, 0, 1, 2, \ldots, \tag{3.8}$$

where $F^{(n)}$ and $G^{(n)}$ stand for conservation density and associated flux, respectively. The conservation densities $F^{(n)}$, $n = -1, 0, 1, 2, \ldots$, are given by

$$\begin{aligned} &F^{(-1)} = -4\beta k^2 I^{(1)}, \qquad F^{(0)} = -4\beta k^2 I^{(2)} - 2\beta k^2 I_x^{(1)}, \\ &F^{(1)} = -4\beta k^2 I^{(3)} + (1-\beta)I^{(1)} - \beta k^2 I_{2x}^{(1)} - 2\beta k^2 I_x^{(2)}, \\ &F^{(2)} = -4\beta k^2 I^{(4)} + (1-\beta)I^{(2)} - \beta k^2 I_{2x}^{(2)} - 2\beta k^2\left[I_x^{(3)} + I^{(1)}I_x^{(1)}\right], \\ &\ldots \\ &F^{(n)} = (1-\beta)I^{(n)} - 4\beta k^2 I^{(n-2)} - \beta k^2 I_{2x}^{(n)} - 2\beta k^2 I_x^{(n-1)} \\ &\quad - 2\beta k^2 \sum_{i+j=n,\, i,j\geqslant 1} I^{(i)}I_x^{(j)}, \quad n \geqslant 3, \end{aligned} \tag{3.9}$$

and the associated fluxes $G^{(n)}$, $n = -1, 0, 1, 2, \ldots$, are given by

$$\begin{aligned} &G^{(-1)} = -2k^2\beta q_{x,t}, \qquad G^{(0)} = 0, \\ &G^{(1)} = \alpha I_{2x}^{(1)} - 6\alpha\left(\left(I^{(1)}\right)^2 + I^{(3)}\right) + 2\beta k^3 I^{(1)} \\ &\quad - 2k^2\beta q_{x,t}I^{(1)} + \alpha\left[6I^{(3)} + \frac{3}{2k^2}I^{(1)}\right], \\ &G^{(2)} = \alpha I_{2x}^{(2)} - 6\alpha\left(I^{(4)} + 2I^{(1)}I^{(2)}\right) + 6\alpha I^{(4)} + \frac{3}{2k^2}\alpha I^{(2)} \\ &\quad + 2\beta k^3 I^{(2)} - 2k^2\beta q_{x,t}I^{(2)}, \\ &\ldots \\ &G^{(n)} = \alpha I_{2x}^{(n)} - 2\alpha\Bigg[3I^{(n+2)} + 3\sum_{i+j=n+1,\, i,j\geqslant 1} I^{(i)}I^{(j)} \\ &\quad + \sum_{i+j+l=n,\, i,j,l\geqslant 1} I^{(i)}I^{(j)}I^{(l)}\Bigg] + 6\alpha I^{(n-2)} + \frac{3}{2k^2}\alpha I^{(n)} \\ &\quad + 2\beta k^3 I^{(n)} - 2k^2\beta q_{x,t}I^{(n)}. \end{aligned} \tag{3.10}$$

From (3.8)–(3.10), we observe that the first two equations of (3.8) are trivial. However, the third conservation law $F^{(1)} = G^{(1)}$ is exactly Eq. (2.2).

So far, we conclude that the ACH–KdV equation (1.6) is integrable with Lax pair, admits the bilinear Bäcklund transformation, and has infinitely many conservation laws.

4. Darboux covariant Lax pair

In this section, we shall construct the Darboux covariant Lax pair of (2.19) whose form is invariant under a gauge transformation.

Suppose that φ is a solution of the eigenvalue problem (2.19) corresponding to the parameter λ. Let us define the operator T as

$$T = \partial_x - \sigma, \quad \sigma = \partial_x \ln\varphi, \tag{4.1}$$

then we have a gauge transformation of the spectral problem (2.19)

$$\tilde{\varphi} = T\varphi. \tag{4.2}$$

This maps the operator $M(q)$ into a similar form

$$TM(q)T^{(-1)} = \tilde{M}(\tilde{q}) = M(\tilde{q}), \tag{4.3}$$

and the corresponding covariant condition is satisfied by

$$\tilde{q} = q + \Delta q, \quad \Delta q = 2\ln\varphi, \tag{4.4}$$

i.e.

$$\tilde{u} = u + \Delta u, \quad \Delta u = -2\sigma_x. \tag{4.5}$$

But this is not the case for the evolution equation of the spectral problem (2.19). Hence, it is necessary for us to find another covariant operator $N_{\mathrm{cov}}(q)$ with appropriate coefficients, such that the following equation holds by gauge transformation (4.2):

$$\tilde{N}_{\mathrm{cov}}(\tilde{q}) = N_{\mathrm{cov}}(\tilde{q}), \quad \tilde{q} = q + \Delta q, \quad \Delta q = 2\ln\varphi. \tag{4.6}$$

Assume φ is a solution of the following Lax pair

$$\begin{aligned} &M(q)\varphi = \lambda\varphi, \qquad N_{\mathrm{cov}}(q)\varphi = 0, \\ &N_{\mathrm{cov}}(q) = \left(1 - 3k^2\lambda\beta + b_1\right)\partial_t + 4\alpha\partial_x^3 - 4\beta k^2\partial_x^2\partial_t + b_2\partial_x + b_3 \\ &\qquad = 0, \end{aligned} \tag{4.7}$$

where b_1, b_2, b_3 are three functions to be determined. We require that it is sufficient for transformation T to map the operator $N_{\mathrm{cov}}(q)$ to a similar one, that is

$$\begin{aligned} &TN_{\mathrm{cov}}(q)T^{(-1)} = \tilde{N}_{\mathrm{cov}}(\tilde{q}), \\ &\tilde{N}_{\mathrm{cov}}(q) = \left(1 - 3k^2\lambda\beta + \tilde{b}_1\right)\partial_t + 4\alpha\partial_x^3 - 4\beta k^2\partial_x^2\partial_t + \tilde{b}_2\partial_x + \tilde{b}_3 \\ &\qquad = 0, \end{aligned} \tag{4.8}$$

such that $\tilde{b}_1, \tilde{b}_2, \tilde{b}_3$ satisfy the covariant condition

$$\tilde{b}_j = b_j(q + \Delta q) = b_j + \Delta b_j, \quad j = 1, 2, 3. \tag{4.9}$$

By an operator calculation from (4.8), we obtain

$$\begin{aligned} &\tilde{b}_1 - b_1 = \Delta b_1 = -8\beta k^2\sigma_x, \\ &\tilde{b}_2 - b_2 = \Delta b_2 = 12\alpha\sigma_x - 4\beta k^2\sigma_t, \\ &\tilde{b}_3 - b_3 = \Delta b_3 = 12\alpha\sigma_{2x} - 8\beta k^2\sigma_{x,t} + 12\alpha\sigma_x\sigma - 4\beta k^2\sigma_t\sigma \\ &\qquad\qquad + b_{2,x}, \end{aligned} \tag{4.10}$$

and σ satisfies

$$\begin{aligned} &\left(1 - 3\beta^2k^2\lambda + \tilde{b}_1\right)\sigma_t + 4\alpha\sigma_{3x} - 4\beta k^2\sigma_{2xt} + \tilde{b}_2\sigma_x + b_{3x} \\ &\quad + (\tilde{b}_3 - b_3)\sigma = 0. \end{aligned} \tag{4.11}$$

According to expression (4.10), it remains to determine the functions b_1, b_2, b_3 in the form of polynomial expressions in terms of q and its derivatives. We suppose

$$b_j = F_j(q, q_x, q_t, q_{x,t}, q_{xx}, \ldots), \quad j = 1, 2, 3, \tag{4.12}$$

and

$$\begin{aligned} \Delta b_j = \Delta F_j &= F_j(q + \Delta q, q_x + \Delta q_x, q_t + \Delta q_t, \ldots) \\ &\quad - F_j(q, q_x, q_t, \ldots), \end{aligned} \tag{4.13}$$

such that $\Delta q_{kx,lt} = 2(\ln\varphi)_{kx,lt}$, $k, l = 1, 2, \ldots$, and Δb_j given through (4.10).

Expanding the right-hand side of Eq. (4.13), we arrive at

$$\begin{aligned} \Delta b_1 = \Delta F_1 &= F_{1,q}\Delta q + F_{1,q_x}\Delta q_x + F_{1,q_t}\Delta q_t + \cdots \\ &= -8\beta k^2\sigma_x \\ &= -4\beta k^2\Delta q_{xx}, \end{aligned} \tag{4.14}$$

which indicates that b_1 can be determined up to an arbitrary constant c_1, that is

$$b_1 = F_1(q_{xx}) = -4\beta k^2 q_{xx} + c_1. \tag{4.15}$$

In the same way, we have

$$b_2 = F_2(q_{xx}) = 6\alpha q_{xx} - 2\beta k^2 q_{x,t} + c_2, \tag{4.16}$$

where c_2 is an arbitrary constant.

Expanding the coefficients of the operator ∂_t in Eqs. (4.8) and combining (4.15), one can readily obtain

$$q_{xx} = -\sigma^2 - \sigma_x + \gamma, \tag{4.17}$$

where γ is an arbitrary constant.

Regrouping (4.17), (4.13) and the third equation of (4.10), we get

$$\begin{aligned} \Delta b_3 = \Delta F_3 &= F_{3,q}\Delta q + F_{3,q_x}\Delta q_x + F_{3,q_t}\Delta q_t + \cdots \\ &= 3\alpha\Delta q_{3x} - 3\beta k^2\Delta q_{2x,t}, \end{aligned} \tag{4.18}$$

thus we have

$$b_3 = 3\alpha q_{3x} - 3\beta k^2 q_{2x,t} + c_3, \tag{4.19}$$

with an arbitrary constant c_3.

Choosing $c_1 = 3\beta\lambda k^2$, $c_2 = 2\beta k^3$, $c_3 = 0$, we get the covariant evolution equation

$$\begin{aligned} &N_{\mathrm{cov}}(q)\varphi = 0, \\ &N_{\mathrm{cov}}(q) = \left(1 - 4\beta k^2 q_{2x}\right)\partial_t + 4\alpha\partial_x^3 - 4\beta k^2\partial_x^2\partial_t \\ &\qquad + \left(6\alpha q_{2x} - 2\beta k^2 q_{x,t} + 2\beta k^3\right)\partial_x + 3\left(\alpha q_{3x} - \beta k^2 q_{2x,t}\right) \\ &\quad = 0. \end{aligned} \tag{4.20}$$

Thus, we obtain the following Darboux covariant Lax pair for Eq. (2.2)

$$\begin{aligned} &M(q)\varphi = \lambda\varphi, \qquad N_{\mathrm{cov}}(q)\varphi = 0, \\ &M(q) = \partial_x^2 + q_{2x}, \\ &N_{\mathrm{cov}}(q) = \left(1 - 4\beta k^2 q_{2x}\right)\partial_t + 4\alpha\partial_x^3 - 4\beta k^2\partial_x^2\partial_t \\ &\qquad + \left(6\alpha q_{2x} - 2\beta k^2 q_{x,t} + 2\beta k^3\right)\partial_x \\ &\qquad + 3\left(\alpha q_{3x} - \beta k^2 q_{2x,t}\right), \end{aligned} \tag{4.21}$$

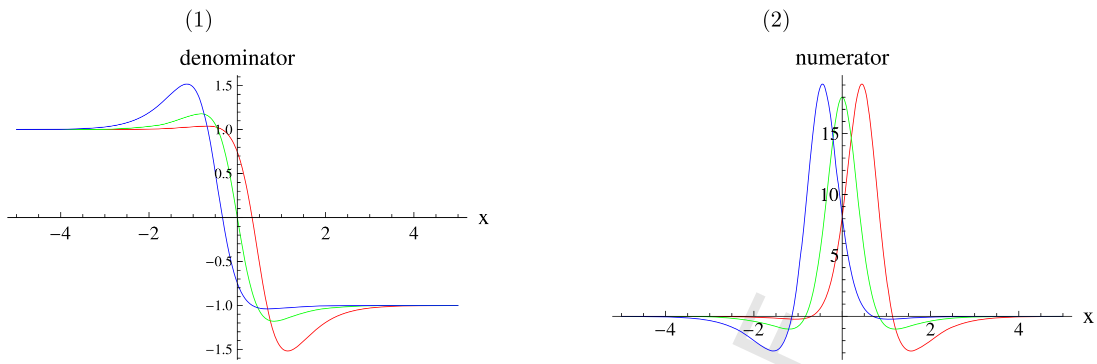

Fig. 1. (1) The denominator of two-soliton wave determined by (5.5) with the parameters $p_1 = 1, \alpha = 1, \beta = 3, k = 1$. Red line: $t = -1$; Green line: $t = 0$; Blue line: $t = 1$. (2) The numerator of two-soliton wave determined by (5.5) with the parameters $p_1 = 1, \alpha = 1, \beta = 3, k = 1$. Red line: $t = -1$; Green line: $t = 0$; Blue line: $t = 1$. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

i.e.

$$\begin{aligned}
&\varphi_{xx} - u\varphi = \lambda\varphi, \\
&\left(1 + 4\beta k^2 u\right)\varphi_t + 4\alpha\varphi_{xxx} - 4\beta k^2 \varphi_{xx,t} \\
&\quad + \left(-6\alpha u + 2\beta k^2 \partial_x^{-1} u_t + 2\beta k^3\right)\varphi_x \\
&\quad + 3\left(-\alpha u_x + \beta k^2 u_t\right)\varphi = 0.
\end{aligned} \tag{4.22}$$

Moreover, the two operators $N(q)$ and $N_{\text{cov}}(q)$ are related through the following formula

$$\begin{aligned}
N_{\text{cov}}(q) = {} & N(q) + \left(3\beta k^2\lambda - 3\beta k^2 q_{2x}\right)\partial_t + 3\alpha\partial_x^3 - 3\beta k^2 \partial_x^2 \partial_t \\
& + (3\alpha q_{2x} - 3\alpha\lambda)\partial_x + 3\alpha q_{3x} - 3\beta k^2 q_{2x,t}.
\end{aligned} \tag{4.23}$$

Hence, we arrive at the following proposition:

Proposition. *Under the gauge transformations (4.1) and (4.2), the ACH–KdV equation (1.6) possesses the following Darboux covariant Lax pair*

$$\begin{aligned}
& M\varphi = \lambda\varphi, \qquad N_{\text{cov}}\varphi = 0, \\
& N_{\text{cov}} = \left(1 - 4\beta k^2 q_{2x}\right)\partial_t + 4\alpha\partial_x^3 - 4\beta k^2 \partial_x^2 \partial_t \\
& \qquad + \left(6\alpha q_{2x} - 2\beta k^2 q_{x,t} + 2\beta k^3\right)\partial_x + 3\left(\alpha q_{3x} - \beta k^2 q_{2x,t}\right) \\
& \quad = 0,
\end{aligned} \tag{4.24}$$

and the Miura transformation between u and $\tilde{u}$ is

$$\tilde{u} = u - 2\sigma_x \tag{4.25}$$

where σ is given as in (4.1).

Moreover, one can easily check that the compatibility condition of the Darboux covariant Lax pair (4.21) does lead to the ACH–KdV equation (1.6) in Lax representation

$$\begin{aligned}
& \left[M(q), N_{cov}(q)\right] \\
& \quad = q_{2x,t} + \alpha(q_{5x} + 6q_{2x}q_{3x}) \\
& \qquad + \beta\left(2k^3 q_{3x} - 4k^2 q_{2x} q_{2x,t} - 2k^2 q_{3x} q_{x,t} - k^2 q_{4x,t}\right) \\
& \quad = u_t + \alpha(u_{xxx} - 6uu_x) \\
& \qquad + \beta\left(2k^3 u_x + 4k^2 u u_t + 2k^2 u_x \partial_x^{-1} u_t - k^2 u_{xxt}\right) \\
& \quad = 0.
\end{aligned} \tag{4.26}$$

5. Application of the Darboux transformation

In this section, we shall apply the above Darboux transformation (4.1) to give the explicit solutions of the ACH–KdV equation (1.6). To see this, we substitute the seed solution $u = 0$ corresponding eigenvalue $\lambda = p_i^2$, $p_i > 0$, $i = 1, 2$, into Lax pair (4.22), we get the basic solution for the Lax pair (4.22)

$$\begin{aligned}
& \varphi_i = 2\cosh\xi_i, \\
& \xi_i = p_i x + \frac{4\alpha p_i^3 + 2\beta k^2 p_i}{4\beta k^2 p_i^2 - 1} t + \nu_i, \quad i = 1, 2,
\end{aligned} \tag{5.1}$$

where ν_i, $i = 1, 2$, is arbitrary constant. From DT (4.2), we have

$$\sigma_1 = \partial_x \ln\varphi_1 = p_1 \tanh\xi_1. \tag{5.2}$$

So, we get one-soliton solution of the ACH–KdV equation

$$\tilde{u} = -2p_1^2 \operatorname{sech}^2 \xi_1. \tag{5.3}$$

According to the DT (4.1) and (4.2), we get

$$\tilde{\varphi}_2 = T\varphi_2 = (\partial_x - \sigma_1)\varphi_2 = 2p_2 \sinh\xi_2 - 2p_1 \tanh\xi_1 \cosh\xi_2,$$

which implies

$$\begin{aligned}
& \sigma_2 = \frac{\tilde{\varphi}_{2,x}}{\tilde{\varphi}_2} = -p_1 \tanh\xi_1 + \frac{p_2^2 - p_1^2}{p_1 \tanh\xi_1 - p_2 \tanh\xi_2}, \\
& \xi_j = p_j x + \frac{4\alpha p_j^3 + 2\beta k^3 p_j}{4\beta k^2 p_j^2 - 1} t + \nu_j, \quad j = 1, 2.
\end{aligned} \tag{5.4}$$

Thus, we get a two-soliton solution

$$\tilde{\tilde{u}} = \frac{2(p_1^2 - p_2^2)(p_1^2 \operatorname{sech}^2 \xi_1 - p_2^2 \operatorname{sech}^2 \xi_2)}{p_1 \tanh\xi_1 - p_2 \tanh\xi_2}. \tag{5.5}$$

Next, we show special two-soliton solutions with singularities based on (5.5). Letting $p_1 = 1$, $p_2 = 2$, $\alpha = 1$, $\beta = 3$, $k = 1$, we can see that the two-soliton solution (5.5) has a singularity x_0 when $t = t_0$ is given, and these singularities satisfy (see Fig. 1(1)):

$$\begin{aligned}
& 0 < x_0 < 0.5, \quad \text{when } t = -1, \\
& x_0 = 0, \quad \text{when } t = 0, \\
& -0.5 < x_0 < 0, \quad \text{when } t = 1.
\end{aligned} \tag{5.6}$$

In the above cases, the numerator part of (5.5) is greater than 0 around the singularity x_0 (see Fig. 1(2)).

For our convenience, let us denote the denominator of (5.5) by

$$g(x, t) = p_1 \tanh\xi_1 - p_2 \tanh\xi_2,$$

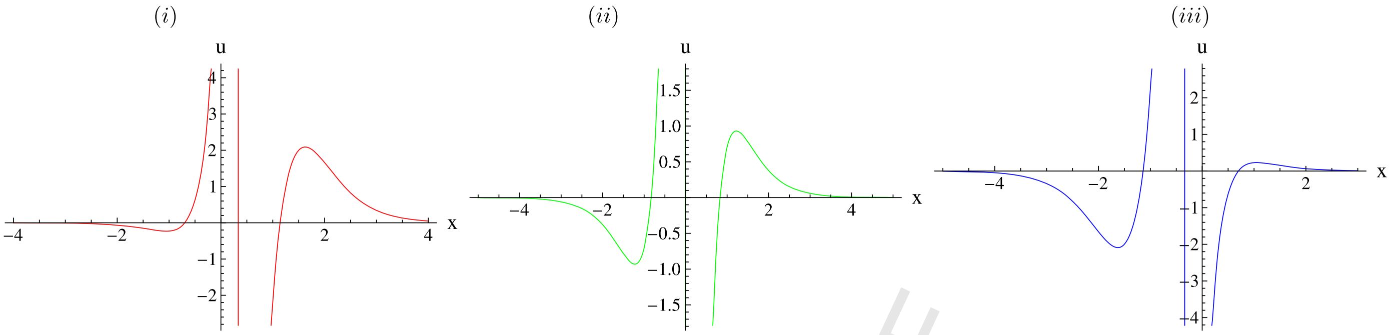

Fig. 2. The two-soliton wave determined by (5.5) with the parameters $p_1 = 1, \alpha = 1, \beta = 3, k = 1$; and the location of blowing up is different with different moment t. (i) Red line: $t = -1$ with the point of explosion $0 < x_0 < 0.5$; (ii) Green line: $t = 0$ with the point of explosion $x_0 = 0$; (iii) Blue line: $t = 1$ with the point of explosion $-0.5 < x_0 < 0$. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

then as shown in Fig. 1(1), the denominator of (5.5) is continuous and tends to zero as $x \to x_0$, and

$$\lim_{x\to x_0^-} g(x,t) = +0, \qquad \lim_{x\to x_0^-} g(x,t) = -0.$$

So we have

$$\lim_{x\to x_0^-} \tilde{\tilde{u}} = +\infty, \qquad \lim_{x\to x_0^+} \tilde{\tilde{u}} = -\infty,$$

and

$$\lim_{x\to -\infty} \tilde{\tilde{u}} = \lim_{x\to +\infty} \tilde{\tilde{u}} = 0.$$

According to Fig. 2, we may observe that the asymptotic behavior of the two-soliton solution (5.5): it tends to zero as $x \to \pm\infty$ while there exists a point of explosion for every different time t. The points of explosion vary with different time t, and the location of the blow up moves from left to right with increasing time t.

If we choose the seed solution $u = 0$ corresponding to eigenvalue $\lambda = -p_1^2$, $p_1 > 0$, we have the basic solution

$$\varphi_1 = \cos\xi_1 + \sin\xi_1, \quad \xi_1 = p_1 x + \frac{4\alpha p_1^3 + 2\beta k^3 p_1}{4\beta k^2 p_1^2 - 1} t + \nu_1, \tag{5.7}$$

and

$$\begin{aligned}\sigma_1 = \partial_x \ln\varphi_1 &= p_1 \frac{\cos\xi_1 - \sin\xi_1}{\cos\xi_1 + \sin\xi_1} \\ &= p_1\big(\sec(2\xi_1) - \tan(2\xi_1)\big).\end{aligned} \tag{5.8}$$

We get a periodic solution of Eq. (1.6)

$$\tilde{u} = \frac{4p_1^2}{1+\sin(2\xi_1)} = 4p_1^2\big(\sec^2(2\xi_1) - \sec(2\xi_1)\tan(2\xi_1)\big). \tag{5.9}$$

Substituting (5.9) into the Lax pair (4.22), and assuming $\lambda = -p_2^2 > 0$, then the eigenfunction satisfies

$$\begin{aligned}\sigma_2 = \frac{\varphi_{2,x}}{\varphi_2} &= -p_1 \frac{\cos\xi_1 - \sin\xi_1}{\cos\xi_1 + \sin\xi_1} \\ &\quad - \frac{p_1^2 - p_2^2}{p_1 \frac{\cos\xi_1 - \sin\xi_1}{\cos\xi_1 + \sin\xi_1} - p_2 \frac{\cos\xi_2 - \sin\xi_2}{\cos\xi_2 + \sin\xi_2}}, \\ \xi_j = p_j x + \frac{4\alpha p_j^3 + 2\beta k^3 p_j}{4\beta k^2 p_j^2 - 1} t + \nu_j, &\quad j = 1, 2,\end{aligned} \tag{5.10}$$

where $\sigma_{2,x} + \sigma_2^2 = -p_2^2 + \frac{4p_1^2}{(\cos\xi_1 + \sin\xi_1)^2}$.

Thus, we get another soliton solution of Eq. (1.6)

$$\begin{aligned}\tilde{\tilde{u}} &= \frac{2p_1^2}{(\cos\xi_1 + \sin\xi_1)^2} \\ &\quad - \frac{2(p_1^2 - p_2^2)(p_1^2 - p_2^2 - p_2^2\sin(2\xi_1) + p_1^2\sin(2\xi_2))}{[(p_1 - p_2)\cos(\xi_1 + \xi_2) - (p_1 + p_2)\cos(\xi_1 - \xi_2)]^2}.\end{aligned} \tag{5.11}$$

In this case, both the one-periodic and two-periodic solutions are singular and their singularities appear periodically.

6. Conclusions

In this paper, we study a new generalization of the associated Camassa–Holm equation. This equation is shown to be integrable in the sense of Lax pair. The bilinear Bäcklund transformations are presented by virtue of the Bell polynomial theory. At the same time, its infinite conservation laws are constructed, and conserved densities and fluxes are given with explicit recursion formulas. Furthermore, a Darboux transformation for this equation is derived with the help of the gauge transformation between two Lax pairs. As an application, soliton and periodic wave solutions are given through the Darboux transformation. Simultaneously, we analyze the asymptotic behavior of the two-soliton solution, and we observe it tends to zero as $x \to \pm\infty$ while there exists a point of explosion for every different time t. The points of explosion vary with different time t, and the location of the blow up moves from left to right with increasing time t along with x-axis.

Acknowledgements

This work was supported by grants from the National Science Foundation of China (No. 11371244) and the Texas Norman Hackerman Advanced Research Program (No. 003599-0001-2009) and the key discipline “Applied Mathematics” of Shanghai Second Polytechnic University (No. A30XK1322100).

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