7.1 Symmetric Matrices

An $n \times n$ matrix A is symmetric if $A^T = A$

<u>*EX*</u> (*a*) $\begin{bmatrix} 1 & 2 \\ 2 & 7 \end{bmatrix}$ is symmetric.

$$\begin{bmatrix} 1 & 2 \\ 2 & 7 \end{bmatrix}^T = \begin{bmatrix} 1 & 2 \\ 2 & 7 \end{bmatrix}$$

(*b*) $\begin{bmatrix} 1 & -11 \\ 5 & 7 \end{bmatrix}$ is not symmetric.

(*c*) $\begin{bmatrix} 1 & 4 & 6 \\ 4 & 2 & 10 \\ 6 & 10 & 3 \end{bmatrix}$ is symmetric.

Properties of Transpose (Ch 2)

$$\begin{aligned}(A+B)^T &= A^T + B^T \\ (AB)^T &= B^T A^T \\ (CA)^T &= CA^T \\ (A^{-1})^T &= (A^T)^{-1}\end{aligned}$$

Orthogonal Matrices

Q is orthogonal matrix if $Q^TQ = I$

In this case

• $Q^{-1} = Q^T$(easy to find Q^{-1}: find Q^T)

• The columns of Q are an orthonormal set of vectors; or thogonal length 1

Properties of Symmetric Matrices

(1) If $A, \underline{B}$ are symmetric $n \times n$ matrices

$$(A+B)^T = A^T + B^T = (A+B)$$

So $A+B$ is symmetric.

(2) If A, B are symmetric $n \times n$ matrices,

$$(AB)^T = B^T A^T = BA \neq AB$$

AB is usually not symmetric.

(3) If C is <u>any</u> $n \times n$ matrix.

then $B = C^T C$ is symmetric:

$$\underline{(C^T C)}^T = C^T (C^T)^T = \underline{C^T C}$$

(4) If D is a diagonal matrix

$$\begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & \ddots & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & d_n \end{bmatrix}$$

D is symmetric

Eigenvalues + Eigenvectors of Symmetric Matrices

EX $A = \begin{bmatrix} 1 & 5 \\ 5 & 1 \end{bmatrix}$

$$\begin{aligned} det(A - \lambda I) &= \begin{vmatrix} 1-\lambda & 5 \\ 5 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 - 25 \\ &= \lambda^2 - 2\lambda - 24 \\ &= (\lambda - 6)(\lambda + 4) \end{aligned}$$

The eigenvalues are $\lambda = 6, -4$. Real, not complex

$$\lambda_1 = 6, \quad \vec{v_1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 5 \\ 5 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 6 \end{bmatrix} = 6 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\lambda_2 = -4, \vec{v_2} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 5 \\ 5 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ -4 \end{bmatrix} = -4 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$\vec{v_1} \cdot \vec{v_2} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix} = (-1) + (1) = 0$$

Eigenvectors are orthogonal.

We found 2 independent eigenvectors for the 2×2 matrix A. So matrix A is diagonalizable.

$$A = PDP^{-1}$$

Symmetric $n \times n$ matrix will have n independent eigenvectors.

$$A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 6 & 0 \\ 0 & -4 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}^{-1}$$

E-vectors E-values

$P = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ is <u>almost</u> an orthogonal matrix.

The columns are orthogonal, but may not have length $\|V\| = 1$

$\vec{v_1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\|\vec{v_1}\| = \sqrt{\begin{bmatrix} 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}} = \sqrt{2}$

$\vec{u_1} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is still an eigenvector for $\lambda = 6$, but $\|\vec{u_1}\| = 1$.

$\vec{v_2} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \qquad \|\vec{v_2}\| = \sqrt{2}$

$\vec{u_1} = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ is also an eigenvectors

for $\lambda = -4$, but $\|\vec{U_2}\| = 1$.

So redo the diagonalization:

$A = QDQ^{-1}$

$$= \underbrace{\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}} \begin{bmatrix} 6 & 0 \\ 0 & -4 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}^{-1}$$

This is an orthog. matrix $\qquad \downarrow Q^{-1} = Q^T$

$$A = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 6 & 0 \\ 0 & -4 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

Rotate 45^o $\qquad$ Rotate -45^o

Orthogonal Diagonalization of A.

$A^{100} = QD^{100}Q^T$

$$= \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 6^{100} & 0 \\ 0 & (-4)^{100} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$\underline{\text{EX}} \quad A = \begin{bmatrix} 7 & -4 & 4 \\ -4 & 5 & 0 \\ 4 & 0 & 9 \end{bmatrix}$$

$$\lambda_1 = 13, \quad \vec{v_1} = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} \rightarrow \|\vec{v_1}\| = \sqrt{4+1+4} = \sqrt{9} = 3$$

$$\lambda_2 = 7, \quad \vec{v_2} = \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix} \rightarrow \|\vec{v_2}\| = \sqrt{1+4+4} = 3$$

$$\lambda_3 = 1, \quad \vec{v_3} = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix} \rightarrow \|\vec{v_3}\| = 3$$

$$A = \underbrace{\begin{bmatrix} \frac{2}{3} & \frac{-1}{3} & \frac{2}{3} \\ \frac{-1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & \frac{-1}{3} \end{bmatrix}}_{Q} \underbrace{\begin{bmatrix} 13 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} \frac{2}{3} & \frac{-1}{3} & \frac{2}{3} \\ \frac{-1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & \frac{-1}{3} \end{bmatrix}}_{Q^T = Q^{-1}}$$