6.5 Method of least Squares

Inconsistent system. But we want to find (x_1, x_2)that comes as close as passible to being a solution

$$\begin{aligned} x_1 &+ x_2 &= 0 \\ x_1 &+ 2x_2 &= 1 \\ x_1 &+ 3x_2 &= 3 \end{aligned}$$

$$\underset{A}{\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}} \underset{\vec{X}}{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}} = \underset{\vec{b}}{\begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}}$$

If we pick a $\vec{X} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and compute

$$A\vec{X} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix},$$

We measure how close this is to solving the system by

$A\vec{X} - \vec{b}$ (Residual Vector)

$\|A\vec{X} - \vec{b}\|$ (Residual error)

God is to find an $\vec{X}$ so that $\|A\vec{X} - \vec{b}\|$

is as small as possible.

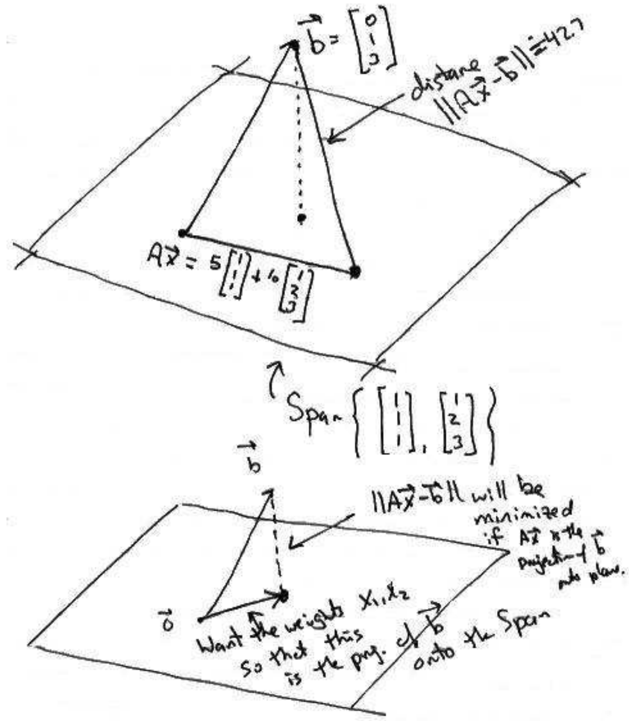
Finding(x_1, x_2) to minimize $\|A\vec{X} - \vec{b}\|$

$$\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}$$

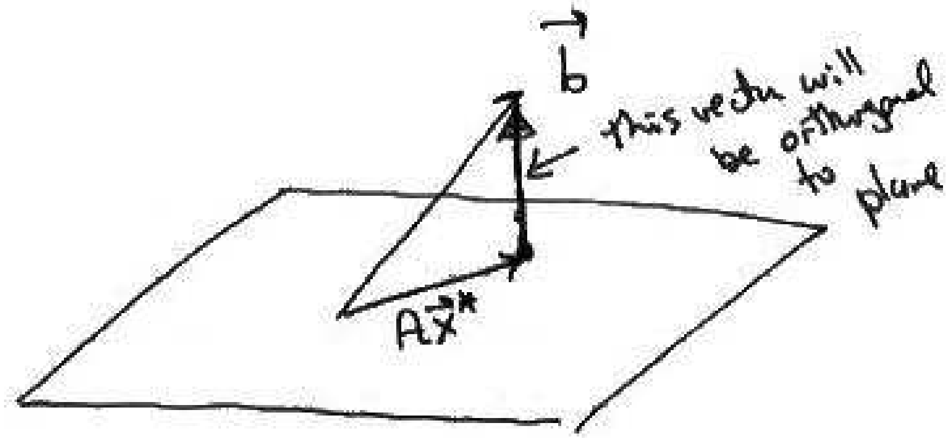
$$x_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}$$

No solution for $x_1, x_2 \Rightarrow \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}$ is not a linear combination of $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

So $\begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}$ is not in span $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right\}$



b = [0, 1, 3]
distance ||Ax - b|| ≐ 42.7
Ax = 5[1, 1, 1] + 10[1, 2, 3]
Span{[1, 1, 1], [1, 2, 3]}
b
||Ax - b|| will be minimized if Ax is the projection of b onto plane.
0
Want the weights x1, x2 so that this is the proj. of b onto the Span



$\vec{b} - A\vec{X}^*$ orthogonal to plane

$\Rightarrow \vec{b} - A\vec{X}^*$ orthogonal to every vector in plane

$\Rightarrow \vec{b} - A\vec{X}^*$ is orthogonal to $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

$$\Rightarrow \quad \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \cdot (\vec{b} - A\vec{X}^*) \quad = \quad 0$$

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot (\vec{b} - A\vec{X}^*) \quad = \quad 0 \qquad \vec{U} \cdot \vec{V} = 0$$

$$\vec{U}^T \vec{V} = 0$$

$$\Rightarrow \quad [\,1\ 1\ 1\,](\vec{b} - A\vec{X}^*) \quad = \quad 0$$

$$[\,1\ 2\ 3\,](\vec{b} - A\vec{X}^*) \quad = \quad 0$$

$$\Rightarrow \quad \underbrace{\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix}}_{A^T} (\vec{b} - A\vec{X}^*) \quad = \quad \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad A^T(\vec{b} - A\vec{X}^*) \quad = \quad \vec{0}$$

$$\Rightarrow \quad A^T\vec{b} - A^T A\vec{X}^* \quad = \quad \vec{0}$$

$$\Rightarrow \quad A^T A\vec{X}^* \quad = \quad A^T\vec{b}$$

The <u>normal equation</u> of a system

$$A\vec{X} = \vec{b}$$

is

$$A^T A\vec{X} = A^T\vec{b}.$$

A solution $\vec{X}^*$ of the normal eq is called a <u>least squares</u>

solution of $A\vec{X} = \vec{b}$. $\|A\vec{X}^* - \vec{b}\|$ is

minimized. Also $A\vec{X}^*$ is the projection

of $\vec{b}$ onto the span of the columns of A. (colA)

Least Squares + Projections

$$\begin{bmatrix} 1 & 2 \\ 2 & 1 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 7 \\ 3 \\ 1 \end{bmatrix}$$

- Find the least squares solution by solving the normal equation.

$$A = \underset{2 \times 3}{\begin{bmatrix} 1 & 2 \\ 2 & 1 \\ -1 & 4 \end{bmatrix}} \quad \vec{b} = \underset{3 \times 2}{\begin{bmatrix} 7 \\ 3 \\ 1 \end{bmatrix}}$$

$$A^T A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 1 \\ -1 & 4 \end{bmatrix} = \begin{bmatrix} 6 & 0 \\ 0 & 21 \end{bmatrix}$$

$$A^T \vec{b} = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & 4 \end{bmatrix} \begin{bmatrix} 7 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 12 \\ 21 \end{bmatrix}$$

Normal equation $A^T A \vec{X} = A^T \vec{b}$

$$\begin{bmatrix} 6 & 0 \\ 0 & 21 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} 12 \\ 21 \end{bmatrix}$$

$$\begin{bmatrix} 6 & 0 & 12 \\ 0 & 21 & 21 \end{bmatrix} \backsim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \end{bmatrix}$$

The least square solution is $x_1 = 2, x_2 = 1$.

How close is $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ to being a solution of

$$\begin{bmatrix} 1 & 2 \\ 2 & 1 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 7 \\ 3 \\ 1 \end{bmatrix}?$$

with $\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$$A\vec{X} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ 2 \end{bmatrix}$$

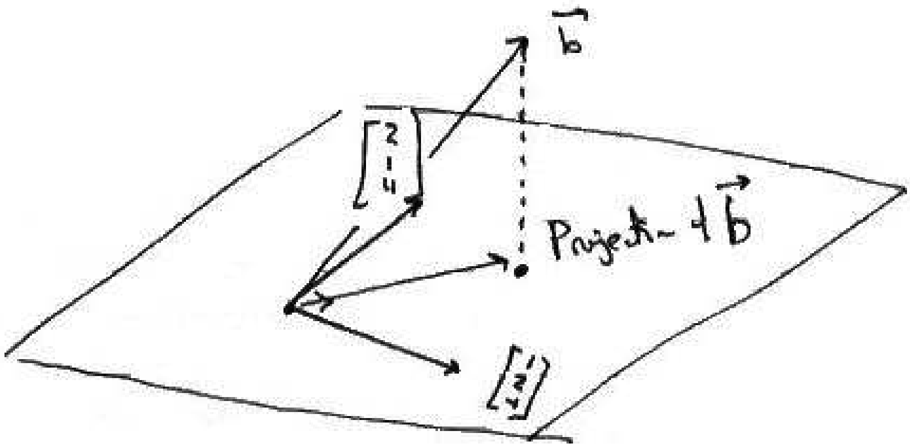
$$\begin{aligned} \|A\vec{X} - \vec{b}\| &= \left\| \begin{bmatrix} 4 \\ 5 \\ 2 \end{bmatrix} - \begin{bmatrix} 7 \\ 3 \\ 1 \end{bmatrix} \right\| \\ &= \left\| \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix} \right\| = \sqrt{9+4+1} \\ &= \sqrt{14} = 3.74 \end{aligned}$$

Finally, $\begin{bmatrix} 4 \\ 5 \\ 2 \end{bmatrix}$ is the projection of $\begin{bmatrix} 7 \\ 3 \\ 1 \end{bmatrix}$ onto the span

$$\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix}.$$

• A second approach based on the fact that the columns of A are orthogonal:

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \\ -1 & 4 \end{bmatrix} \qquad \vec{b} = \begin{bmatrix} 7 \\ 3 \\ 1 \end{bmatrix}$$



Since $\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix}$ are orthogonal, the proj.

of $\vec{b}$ onto the plane

$$\begin{aligned}
\vec{b}_{\text{proj}} &= \frac{\vec{b}\cdot\vec{U}}{\vec{U}\cdot\vec{U}}\vec{U} + \frac{\vec{b}\cdot\vec{V}}{\vec{V}\cdot\vec{V}}\vec{V} \\
&= \frac{\begin{bmatrix} 7 \\ 3 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}}{\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}} \cdot \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + \frac{\begin{bmatrix} 7 \\ 3 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix}}{\begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix}} \cdot \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix}
\end{aligned}$$

$$\vec{b}_{\text{proj}} = \frac{12}{6}\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + \frac{21}{21}\begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix}$$

$$= 2\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + 1\begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix}$$

$$\nwarrow \quad \nearrow$$

The least squares solution of $A\vec{X} = \vec{b}$ is $\vec{X} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

The projection $\vec{b}_{\text{proj}} = \begin{bmatrix} 4 \\ 5 \\ 2 \end{bmatrix}$