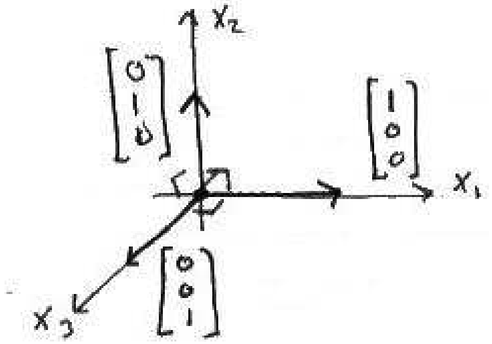
6.2 Orthogonal Sets

$$\vec{U} \cdot \vec{V} = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} \cdot \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} = u_1 v_1 + \cdots + u_n v_n$$

$\vec{U} \cdot \vec{V} = 0 \Rightarrow \vec{U}, \vec{V}$ are orthogonal

EX



$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = 0 + 0 + 0 = 0$$

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = 0 + 0 + 0 = 0$$

$$\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = 0 + 0 + 0 = 0$$

These 3 vectors are an orthogonal set.

Def A set of vectors $\{\vec{V}_1, \vec{V}_2, \cdots, \vec{V}_p\}$

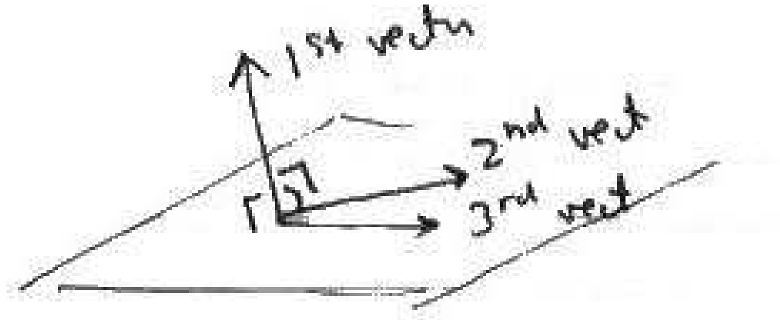
in$\Re^n$ is called an orthogonal set if

$\vec{V}_i \cdot \vec{V}_j = 0, \ \ i \neq j$

EX $\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \right\}$

This is not an orthogonal set because

$$\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} = 2 + 0 + 0 = 2 \neq 0$$



Theorem If $\{\vec{u_1}, \cdots, \vec{u_p}\}$ are nonzero vectors that form an orthogonal set then $\{\vec{u_1}, \cdots, \vec{u_p}\}$ is linearly independent.

Justification:

$c_1\vec{u_1} + c_2\vec{u_2} + \cdots + c_p\vec{u_p} = \vec{0}$

Take the innerprodud of both sides with $\vec{u_1}$

$$\begin{aligned} (c_1\vec{u_1} + c_2\vec{c_2} + \cdots + c_p\vec{u_p}) \cdot \vec{u_1} &= \vec{0} \cdot \vec{u_1} \\ c_1(\vec{u_1} \cdot \vec{u_1}) + c_2(\vec{u_2} \cdot \vec{u_1}) + \cdots + c_p(\vec{u_p} \cdot \vec{u_1}) &= 0 \\ c_1(\vec{u_1} \cdot \vec{u_1}) &= 0 \end{aligned}$$

$c_1 = 0$ or $\vec{u_1} \cdot \vec{u_1} = 0$. Since $\vec{u_1} \neq \vec{0}$,

$\vec{U} \cdot \vec{u_1} \neq 0$, so $c_1 = 0$. Similarly,by repeating

with $\vec{u_2}, \vec{u_3}, \cdots$,we see $c_2 = 0_1, c_2 = 0_2, \cdots$ So

$\{\vec{u_1}, \cdots, \vec{u_p}\}$ are independent.

Def An orthogonal basis for a subspace H.

in $\Re^n$ is a basis for H (indep. ,span H)

that is also an orthogonal set.

EX $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ is an orthog. basis for $\Re^2$

EX $\left\{ \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ -7 \end{bmatrix} \right\}$

(a) Verify this a basis for $\Re^3$

Let $A = \begin{bmatrix} 3 & -1 & 1 \\ 1 & 2 & 4 \\ 1 & 1 & -7 \end{bmatrix}$ $\quad detA = \cdots = -66 \neq 0$

A invertible $\Rightarrow$ column are independent

and Span $\Re^3$

$\Rightarrow$ the columns form a basis for$\Re^3$

(b) Verify the columns are orthogonal

$$\begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} = -3 + 2 + 1 = 0$$

$$\begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 4 \\ -7 \end{bmatrix} = 3 + 4 - 7 = 0$$

$$\begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 4 \\ -7 \end{bmatrix} = -1 + 8 - 7 = 0$$

(c) Use orthogonality to write linear (combinations).

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = c_1 \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 4 \\ -7 \end{bmatrix}$$

$$\text{Method(1)} \quad \begin{bmatrix} 3 & -1 & 1 & 1 \\ 1 & 2 & 4 & 2 \\ 1 & 1 & -7 & 3 \end{bmatrix} \backsim \cdots \text{rref} \backsim \text{find } c_1, c_2, c_3$$

Method (2) Using orthogonality

$$c_1 \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 4 \\ -7 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

To find c_1 :take inner product of both sides with $\begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$

$$c_1 \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} +$$

$$c_3 \begin{bmatrix} 1 \\ 4 \\ -7 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$$

$$c_1(9 + 1 + 1) + c_2(0) + c_3(0) = (3 + 2 + 3)$$

$$c_1 = \frac{8}{11}$$

$$c_1 \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 4 \\ -7 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$c_2:$

$$\begin{aligned} 0 \ + \ c_2 \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} \ + \ 0 \ &= \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} \\ c_2(6) &= 6 \\ c_2 &= 1 \end{aligned}$$

$c_3:$

$$c_1 \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 4 \\ -7 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$c_3 = \frac{\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 4 \\ -7 \end{bmatrix}}{\begin{bmatrix} 1 \\ 4 \\ -7 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 4 \\ -7 \end{bmatrix}} = \frac{1+8-21}{1+16+49} = -\frac{2}{11}$$

Orthogonal Matrices

Def: An $n \cdot n$ matrix U is called an orthogonal matrix if $U^T U = I$

Note (a) This means U is invertible,

and $U^{-1} = U^T$

$$(b) \quad U = \begin{bmatrix} \vec{u_1} & \vec{u_2} & \vec{u_3} \end{bmatrix}$$

$$U^T = \begin{bmatrix} \vec{u_1}^T \\ \vec{u_2}^T \\ \vec{u_3}^T \end{bmatrix}$$

$$U^T U = \begin{bmatrix} \vec{u_1}^T \\ \vec{u_2}^T \\ \vec{u_3}^T \end{bmatrix} \begin{bmatrix} \vec{u_1} & \vec{u_2} & \vec{u_3} \end{bmatrix}$$

$$= \begin{bmatrix} \vec{u_1}^T \vec{u_1} & \vec{u_1}^T \vec{u_2} & \vec{u_1}^T \vec{u_3} \\ \vec{u_2}^T \vec{u_1} & \vec{u_2}^T \vec{u_2} & \vec{u_2}^T \vec{u_3} \\ \vec{u_3}^T \vec{u_1} & \vec{u_2}^T \vec{u_2} & \vec{u_3}^T \vec{u_3} \end{bmatrix}$$

$$U^TU = \begin{bmatrix} \vec{u_1}^T\vec{u_1} & \vec{u_1}^T\vec{u_2} & \vec{u_1}^T\vec{u_3} \\ \vec{u_2}^T\vec{u_1} & \vec{u_2}^T\vec{u_2} & \vec{u_2}^T\vec{u_3} \\ \vec{u_3}^T\vec{u_1} & \vec{u_2}^T\vec{u_2} & \vec{u_3}^T\vec{u_3} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\vec{u_1} \cdot \vec{u_1} = 1 \ \Rightarrow \quad \|\vec{u_1}\| = \sqrt{\vec{u_1} \cdot \vec{u_1}} = 1$$

$$\vec{u_2} \cdot \vec{u_2} = 1 \ \Rightarrow \quad \|\vec{u_2}\| = 1$$

$$\vec{u_3} \cdot \vec{u_3} = 1 \ \Rightarrow \quad \|\vec{u_3}\| = 1$$

$$\vec{u_1} \cdot \vec{u_2} = 0 \ \Rightarrow \ \vec{u_1}, \vec{u_2} \text{ are orthogonal.}$$

$$\vec{u_1} \cdot \vec{u_3} = 0 \ \Rightarrow \ \vec{u_1}, \vec{u_3} \text{ are orthogonal.}$$

$$\vec{u_2} \cdot \vec{u_3} = 0 \ \Rightarrow \ \vec{u_2}, \vec{u_3} \ are \ orthogonal.$$

So for U to be an orthogonal matrix

- the column of U are orthogonal,
- each column of U has length 1.

$\underline{EX}$ $\quad U = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$

$$U^TU = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

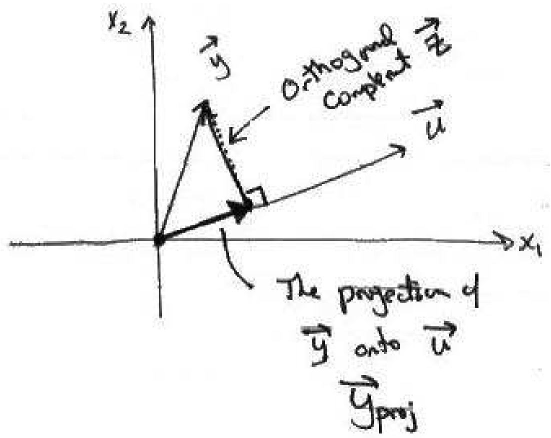
$$= \begin{bmatrix} \frac{1}{2} + \frac{1}{2} & \frac{-1}{2} + \frac{1}{2} \\ \frac{-1}{2} + \frac{1}{2} & \frac{1}{2} + \frac{1}{2} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

- U is an orthong. matrix
- Columns of U are orthong.,
- Each column of U has length 1.
- $U^{-T} = U^T = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$

Orthogonal Projections

Orthogonal Proj. of One Vector into another Vector



$$
\begin{array}{ccccc}
 & & \vec{y}_{proj} & = & c\vec{u} \\
\vec{y}_{proj} & + & \vec{z} & = & \vec{y} \\
(c\vec{u}) + \vec{z} & \cdot & \vec{u} & = & \vec{y} \cdot \vec{u} \\
c\vec{u} \cdot \vec{u} & + & \vec{z} \cdot \vec{u} & = & \vec{y} \cdot \vec{u} \\
 & & c & = & \frac{\vec{y} \cdot \vec{u}}{\vec{u} \cdot \vec{u}}
\end{array}
$$

The projection of $\vec{y}$ onto $\vec{u}$ is

$$\vec{y_{proj}} = \left(\frac{\vec{y}\cdot\vec{u}}{\vec{u}\cdot\vec{u}}\right)\vec{u}$$

<u>EX</u> $\vec{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \vec{y} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

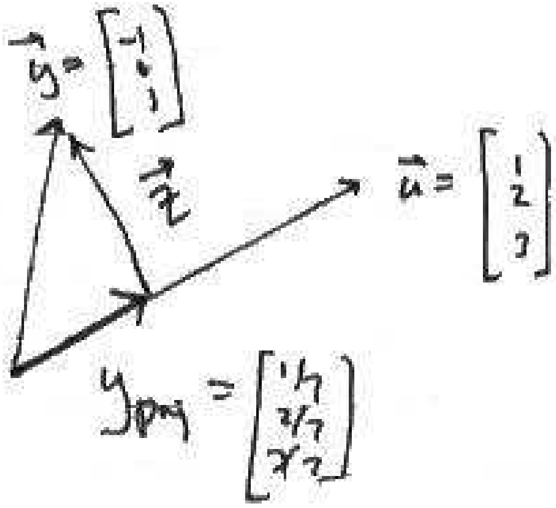
$$\vec{y_{Proj}} = \left(\frac{\vec{y}\cdot\vec{u}}{\vec{u}\cdot\vec{u}}\right)\vec{u} = \frac{\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}}{\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$= \frac{-1+0+3}{1+4+9} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \frac{2}{14} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{7} \\ \frac{2}{7} \\ \frac{3}{7} \end{bmatrix}$$

$$\vec{z} = \vec{y} - \vec{y_{proj}}$$

$$= \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} \frac{1}{7} \\ \frac{2}{7} \\ \frac{3}{7} \end{bmatrix} = \begin{bmatrix} \frac{-8}{7} \\ \frac{-2}{7} \\ \frac{4}{7} \end{bmatrix}$$



Orthogonality and Projections

If $\vec{u_1}, \vec{u_2}$ are orthogonal vectors and $\vec{X}$ is a linear combination of $\vec{u_1}, \vec{u_2}$.

$$\vec{X} = c_1\vec{u_1} + c_2\vec{u_2}$$

where

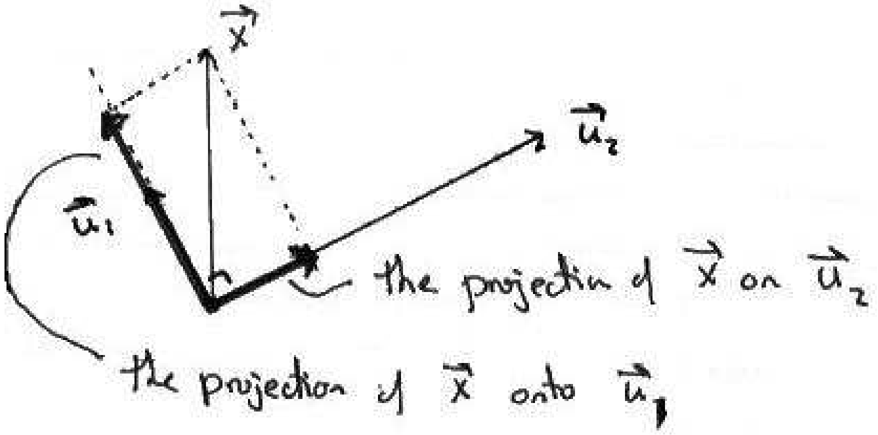
$$\vec{X} = (\tfrac{\vec{X}\cdot\vec{u_1}}{\vec{u_1}\cdot\vec{u_1}})\vec{u_1} + (\tfrac{\vec{X}\cdot\vec{u_2}}{\vec{u_2}\cdot\vec{u_2}})\vec{u_2}$$

Note

$(\frac{\vec{X}\cdot\vec{u_1}}{\vec{u_1}\cdot\vec{u_1}})\vec{u_1}$ is the Projection of $\vec{X}$ onto $\vec{u_1}$

$(\frac{\vec{X}\cdot\vec{u_2}}{\vec{u_2}\cdot\vec{u_2}})\vec{u_2}$ is the projection of $\vec{X}$ onto $\vec{u_2}$

Geometrically



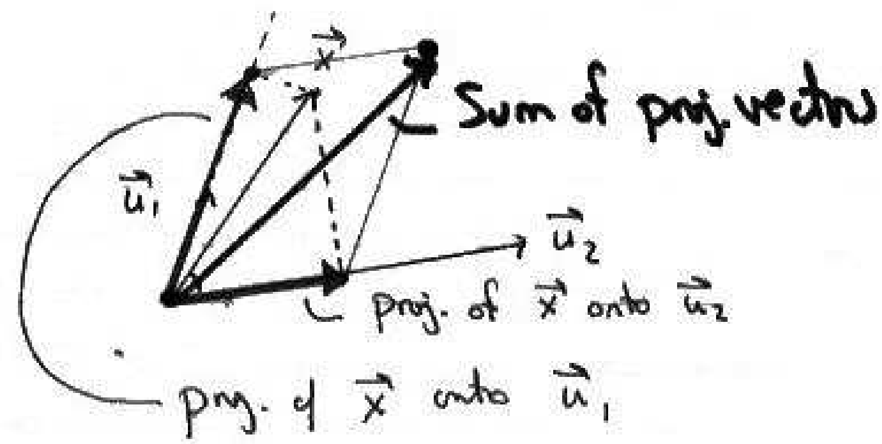
the projection of $\vec{X}$ on $\vec{u_2}$

the projection of $\vec{X}$ onto $\vec{u_1}$

$\vec{X} = (\text{proj. of } \vec{X} \text{ onto } \vec{u_1}) + (\text{proj. of } \vec{X} \text{ onto } \vec{u_2})$

$\vec{X} = (\frac{\vec{X}\cdot\vec{u_1}}{\vec{u_1}\cdot\vec{u_1}})\vec{u_1} + (\frac{\vec{X}\cdot\vec{u_2}}{\vec{u_2}\cdot\vec{u_2}})\vec{u_2}$

If $\vec{u_1}$ and $\vec{u_2}$ are <u>not</u> orthogonal:

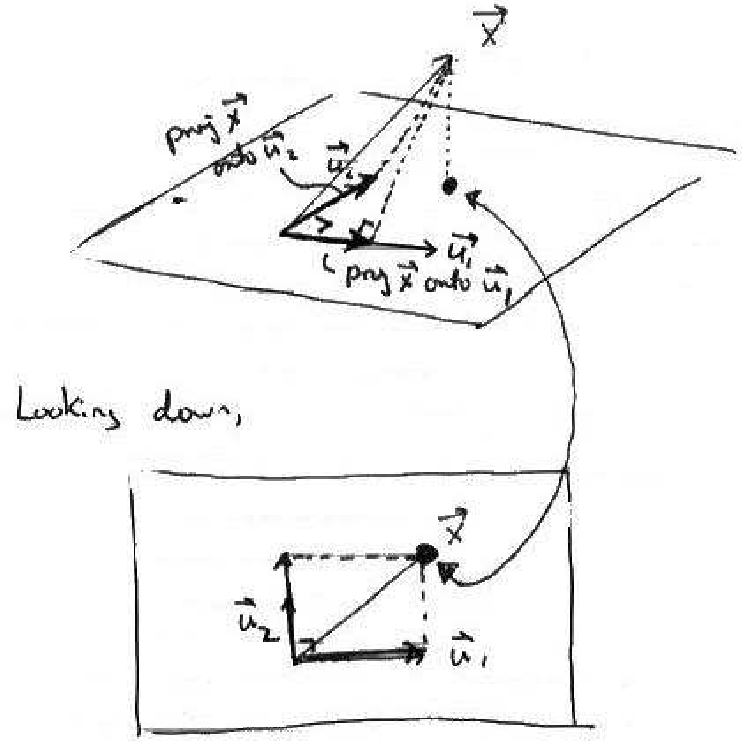


proj. of $\vec{X}$ onto $\vec{u_2}$

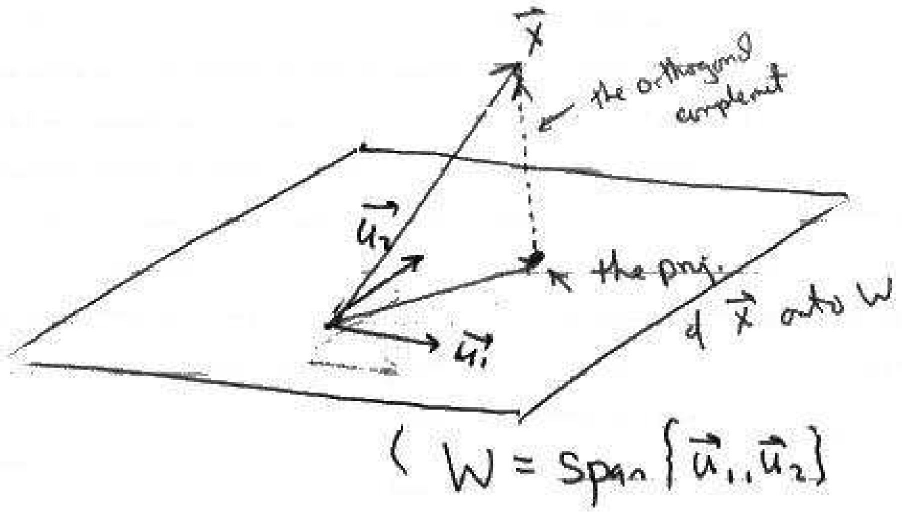
proj. of $\vec{X}$ onto $\vec{u_1}$

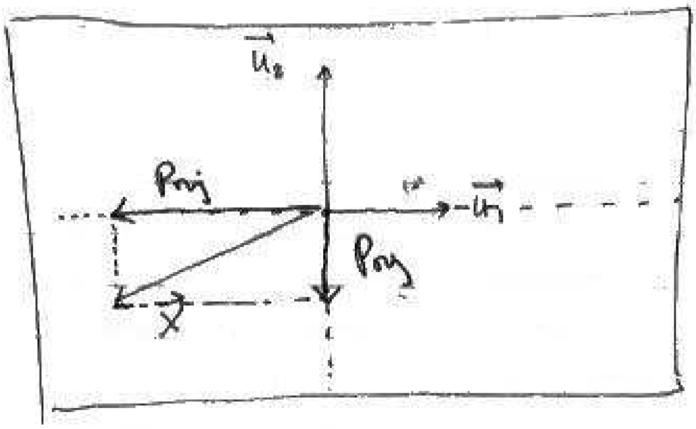
$\vec{X} \neq (\text{proj. of } \vec{X} \text{ onto } \vec{u_1}) + (\text{proj. of } \vec{X} \text{ onto } \vec{u_2})$

If $\vec{u}_1$, $\vec{u}_2$ are orthogonal but $\vec{X}$ is <u>not</u> in span $\{\vec{u}_1, \vec{u}_2\}$



$$\vec{X}_{\text{proj onto plane}} = (\text{proj. of } \vec{X} \text{ onto } \vec{u}_1) + (\text{proj. of } \vec{X} \text{ onto } \vec{u}_2)$$
$$= \left(\frac{\vec{X}\cdot\vec{u}_1}{\vec{u}_1\cdot\vec{u}_1}\right)\vec{u}_1 + \left(\frac{\vec{X}\cdot\vec{u}_2}{\vec{u}_2\cdot\vec{u}_2}\right)\vec{u}_2$$


$\vec{x}$
the orthogonal complement
$\vec{u}_2$
the proj. of $\vec{x}$ onto W
$\vec{u}_1$
(W = Span{$\vec{u}_1$, $\vec{u}_2$}


$\vec{u}_2$
Proj
$\vec{u}_1$
Proj
$\vec{x}$