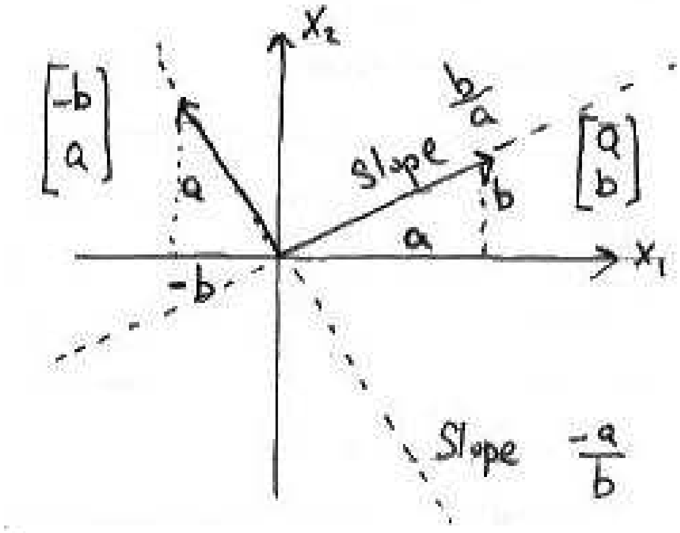
6.1 Inner Product, Length, Orthogonality



$$\begin{bmatrix} a \\ b \end{bmatrix}, \begin{bmatrix} -b \\ a \end{bmatrix}$$

$$\begin{bmatrix} a & b \end{bmatrix} \begin{bmatrix} -b \\ a \end{bmatrix} = a(-b) + ba = 0$$

EX $\vec{U} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad \vec{V} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$

$$\vec{U} \cdot \vec{V} = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

$$= (1)(1) + (2)(-1) + (3)(0)$$

$$= -1$$

Definition The inner product (dot product) of two vectors $\vec{U}, \vec{V}$ is

$$\begin{aligned}
\vec{U} \cdot \vec{V} &= \vec{U}^T \vec{V} \\
&= \begin{bmatrix} u_1 & u_2 & \cdots & u_n \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \\
&= u_1 v_1 + u_2 v_2 + \cdots + u_n v_n \quad \text{A Number!}
\end{aligned}$$

<u>Properties</u> $\vec{U}, \vec{V}, \vec{W}$ in $\Re^n$, c Scalar

(a) $\vec{U} \cdot \vec{V} = \vec{V} \cdot \vec{U}$

(b) $(\vec{U} + \vec{V}) \cdot \vec{W} = \cdots = (\vec{U} \cdot \vec{W}) + (\vec{V} \cdot \vec{W})$

(c)

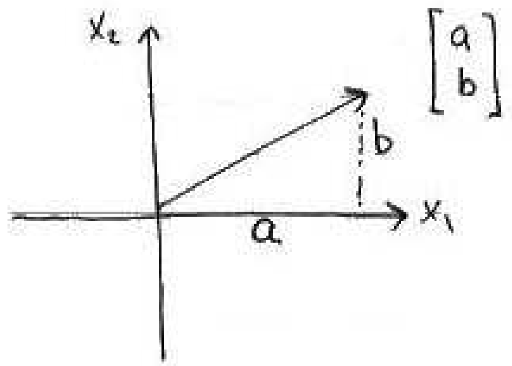
$$\begin{aligned} (c\vec{U}) \cdot \vec{V} &= \begin{bmatrix} cu_1 & cu_2 & \cdots & cu_n \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \\ &= cu_1 v_1 + cu_2 v_2 + \cdots + cu_n v_n \\ &= c(u_1 v_1 + u_2 v_2 + \cdots + u_n v_n) \\ &= c(\vec{U} \times \vec{V}) \end{aligned}$$

(d)

$$\begin{aligned} \vec{U} \cdot \vec{U} &= u_1 u_1 + u_2 u_2 + \cdots + u_n u_n \\ &= u_1^2 + u_2^2 + \cdots + u_n^2 \end{aligned}$$

$\vec{U} \cdot \vec{U} \geqslant 0$

$\vec{U} \cdot \vec{U} = 0 \Leftrightarrow \vec{U} = \vec{0}$



$$\textit{length of the vector} = \sqrt{a^2 + b^2}$$
$$= \sqrt{\begin{bmatrix} a \\ b \end{bmatrix} \cdot \begin{bmatrix} a \\ b \end{bmatrix}}$$

<u>Definition</u> The length or norm of

$\vec{V}$ is

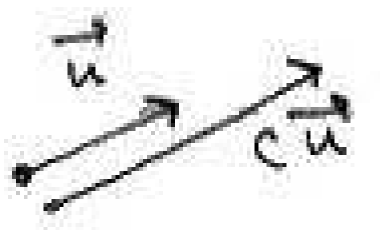
$$\|\vec{V}\| = \sqrt{\vec{V} \cdot \vec{V}}$$
$$= \sqrt{V_1^2 + V_2^2 + \cdots + V_n^2}$$

<u>EX</u>

$$\vec{X} = \begin{bmatrix} 1 \\ 2 \\ -1 \\ 1 \end{bmatrix} \qquad \|\vec{X}\| = \sqrt{1^2 + 2^2 + (-1)^2 + (1)^2} = \sqrt{7}$$

Properties of Length

$$
\begin{aligned}
(a)\ \ \|c\vec{U}\| &= \sqrt{(c\vec{U} \times (c\vec{U}))} \\
&= \sqrt{c^2(U \overset{\rightarrow}{\times} \vec{U})} \\
&= |c|\sqrt{\vec{U} \times \vec{U}} \\
&= |c|\ \ \|\vec{U}\|
\end{aligned}
$$



(b) If $\vec{U} \neq \vec{0}$, then

$$\left(\frac{1}{\|\vec{U}\|}\right)\vec{U}$$

is a vector in the same direction as $\vec{U}$

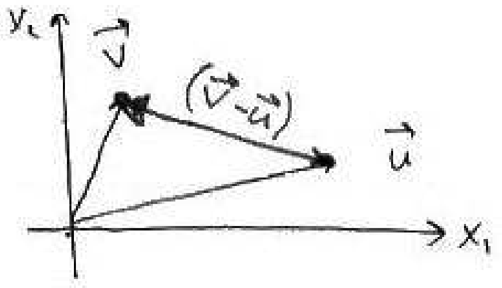
but has length 1.

<u>EX</u> $\vec{U} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad \|\vec{U}\| = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$

$$\vec{V} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix} \quad \|\vec{V}\| = \sqrt{\frac{1}{3} + \frac{1}{3} + \frac{1}{3}} = 1$$

(Normalizing vector $\vec{U}$).

(c) The <u>distance</u> between two vectors



$\vec{U} + (\vec{V} - \vec{U}) = \vec{V}$

distance from $\vec{U}$ *to* $\vec{V} = \|\vec{V} - \vec{U}\|$

Note: $\vec{V}$ *to* $\vec{U}$

$$
\begin{aligned}
\|\vec{U} - \vec{V}\| &= \sqrt{(\vec{U} - \vec{V}) \cdot (\vec{U} - \vec{V})} \\
&= \sqrt{(-1)(-1)(\vec{V} - \vec{U}) \cdot (\vec{V} - \vec{U})} \\
&= \sqrt{(\vec{V} - \vec{U}) \cdot (\vec{V} - \vec{U})} \\
&= \|\vec{V} - \vec{U}\|
\end{aligned}
$$

Definition Two vectors $\vec{U}$ and $\vec{V}$ in $\Re^n$ are orthogonal (perpendicular) if

$$\vec{U} \cdot \vec{V} = 0$$

EX $\vec{U} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ $\vec{V} = \begin{bmatrix} -2 \\ 4 \end{bmatrix}$

$\vec{U} \cdot \vec{V} = (2)(-2) + (1)(4) = 0$

$\vec{U}, \vec{V}$ are orthogonal.

Properties of Orthogonality

(a) In $\Re^n, \vec{0}$ is orthogonal to all other vectors

$$\begin{aligned}
\vec{U} \cdot \vec{0} &= \begin{bmatrix} u_1 & u_2 & \cdots & u_n \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \vdots \end{bmatrix} \\
&= u_1 \cdot 0 + u_2 \cdot 0 + \cdots + u_n \cdot 0 \\
&= 0
\end{aligned}$$

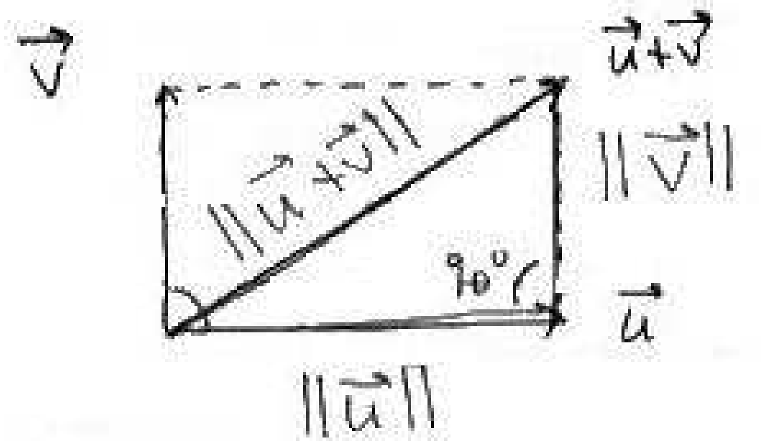
Properties of Orthogonality

(b) Suppose $\vec{U}, \vec{V}$ are orthogonal.

$$
\begin{aligned}
&\|\vec{U} + \vec{V}\|)^2 = (\|\vec{U} + \vec{V}\|)^2 \\
&= \left(\sqrt{(\vec{U} + \vec{V}) \cdot (\vec{U} + \vec{V})}\right)^2 \\
&= (\vec{U} + \vec{V}) \cdot (\vec{U} + \vec{V}) \\
&= \vec{U} \cdot (\vec{U} + \vec{V}) + \vec{V} \cdot (\vec{U} + \vec{V}) \\
&= (\vec{U} \cdot \vec{U}) + \underline{(\vec{U} \cdot \vec{V})} + (\vec{V} \cdot \vec{U}) + \underline{(\vec{V} \cdot \vec{V})} \\
&= \vec{U} \cdot \vec{U} + \vec{V} \cdot \vec{V} \\
&= (\sqrt{\vec{U} \cdot \vec{U}})^2 + (\sqrt{\vec{V} \cdot \vec{V}})^2 \\
&= \|\vec{U}\|^2 + \|\vec{V}\|^2
\end{aligned}
$$

$\|\vec{U} + \vec{V}\|^2 = \|\vec{U}\|^2 + \|\vec{V}\|^2$ is

a general Pythagorean Theorem in $\Re^n$



$$\|\vec{U}\|$$

$$\|\vec{U}\|^2 + \|\vec{V}\|^2 = \|\vec{U} + \vec{V}\|^2$$