5.5 Complex Eigenvalues

EX $A = \begin{bmatrix} 2 & 3 \\ -3 & 2 \end{bmatrix}$

$$\begin{aligned}
det(A - \lambda I) &= \begin{vmatrix} 2-\lambda & 3 \\ -3 & 2-\lambda \end{vmatrix} \\
&= (2-\lambda)^2 - (-9) \\
&= \lambda^2 - 4\lambda + 13 \\
0 &= \lambda^2 - 4\lambda + 13 \\
\lambda &= \frac{4 \pm \sqrt{16 - 4 \times 1 \times 13}}{2 \times 1} \\
&= \frac{4 \pm \sqrt{-36}}{2} \\
&= \frac{4 \pm 6i}{2} \qquad i^2 = -1 \\
&= 2 \pm 3i \\
&= 2 + 3i, \qquad 2 - 3i
\end{aligned}$$

Eigenvalue $\quad \lambda = 2 + 3i$

$$(A - (2+3i)I)\vec{X} = \vec{0}$$

$$\left[\begin{array}{cc:c} 2-(2+3i) & 3 & 0 \\ -3 & 2-(2+3i) & 0 \end{array}\right]$$

$$\left[\begin{array}{cc:c} -3i & 3 & 0 \\ -3 & -3i & 0 \end{array}\right]$$

$$\left[\begin{array}{cc:c} -i & 1 & 0 \\ -1 & -i & 0 \end{array}\right] \begin{array}{c} \frac{1}{3}R_1 \\ \frac{1}{3}R_2 \end{array}$$

$$\left[\begin{array}{cc:c} +1 & +i & 0 \\ -i & 1 & 0 \end{array}\right] \begin{array}{c} -R_2 \\ R_1 \end{array}$$

$$\begin{array}{ccccc} & & & -1 & \\ & & & \downarrow & \\ & iR_1 & i & i^2 & 0 \\ & R_2 & -i & 1 & 0 \\ & & \cdots & \cdots & \cdots \\ New & R_2 & 0 & 0 & 0 \end{array}$$

$$\left[\begin{array}{ccc} 1 & i & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$\begin{array}{ccccc} X_1 & + & iX_2 & = & 0 \\ & & X_2 & & \text{free} \end{array} \Rightarrow \begin{array}{ccc} X_1 & = & -iX_2 \\ X_2 & & \text{free} \end{array}$$

$$\vec{X} = \begin{bmatrix} -iX_2 \\ X_2 \end{bmatrix} = X_2 \begin{bmatrix} -i \\ 1 \end{bmatrix}$$

A basis for the eigenspace of $\lambda = 2 + 3i$ is $\begin{bmatrix} -i \\ 1 \end{bmatrix}$

Eigenvalue $\lambda = 2 - 3i$

$$(A - (2 - 3i)I)\vec{X} = \vec{0}$$

$$\left[\begin{array}{cc:c} 2-(2-3i) & 3 & 0 \\ -3 & 2-(2-3i) & 0 \end{array}\right]$$

$$\left[\begin{array}{cc:c} 3i & 3 & 0 \\ -3 & 3i & 0 \end{array}\right]$$

$$\left[\begin{array}{cc:c} 1 & -i & 0 \\ i & 1 & 0 \end{array}\right] \begin{array}{c} \frac{-1}{3}R_2 \\ \frac{1}{3}R_1 \end{array}$$

$$\begin{array}{lccc} & & -1 & \\ & & \downarrow & \\ -iR_1 & -i & i^2 & 0 \\ R_2 & i & 1 & 0 \\ & \cdots & \cdots & \cdots \\ Add & 0 & 0 & 0 \end{array}$$

$$\begin{bmatrix} 1 & -i & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{array}{lcl} X_1 & = & iX_2 \\ X_2 & & \text{free} \end{array}$$

$$\vec{X} = \begin{bmatrix} iX_2 \\ X_2 \end{bmatrix} = X_2 \begin{bmatrix} i \\ 1 \end{bmatrix}$$

A basis for the eigenspace of $\lambda = 2 - 3i$ is $\begin{bmatrix} i \\ 1 \end{bmatrix}$.

$$(1) \quad \lambda = 2 + 3i, \quad \vec{V} = \begin{bmatrix} -i \\ 1 \end{bmatrix}$$

$$\lambda = 2 - 3i, \quad \vec{V} = \begin{bmatrix} i \\ 1 \end{bmatrix}$$

The eigenvalues are conjugates of each other:

$\overline{a + bi} = a - bi$

$\overline{2 + 3i} = 2 - 3i.$

The eigenvectors are also conjugates of each other.

(2) $\begin{bmatrix} 2 & 3 \\ -3 & 2 \end{bmatrix}$ is diagonalizable

$$A \quad = \quad P \; D \; P^{-1}$$

$$\downarrow \text{vect.}$$

$$\begin{bmatrix} 2 & 3 \\ -3 & 2 \end{bmatrix} = \begin{bmatrix} -i & i \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2+3i & 0 \\ 0 & 2-3i \end{bmatrix} \begin{bmatrix} -i & i \\ 1 & 1 \end{bmatrix}^{-1}$$

Matrices of the form $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$

$$\begin{bmatrix} 2 & 3 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} -5 & -3 \\ 3 & -5 \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

- Eigenvalues

$$A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

$$det(A - \lambda I) = \begin{vmatrix} a-\lambda & -b \\ b & a-\lambda \end{vmatrix} = (a-\lambda)^2 + b^2$$

$$\begin{aligned} 0 &= (a-\lambda)^2 + b^2 \\ (a-\lambda)^2 &= -b^2 \\ a-\lambda &= \pm bi \\ -\lambda &= -a \pm bi \\ \lambda &= a \pm bi \end{aligned}$$

EX $\quad A = \begin{bmatrix} 5 & 1 \\ -1 & 5 \end{bmatrix}$

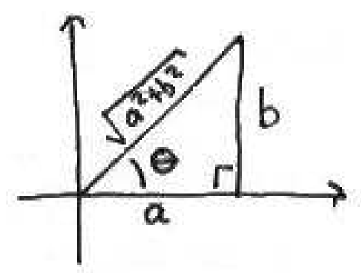
Eigenvalues $\quad \lambda = 5 \pm 1 \times i$

- Geometric interpretation

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix} = \sqrt{a^2+b^2} \begin{bmatrix} \frac{a}{\sqrt{a^2+b^2}} & \frac{-b}{\sqrt{a^2+b^2}} \\ \frac{b}{\sqrt{a^2+b^2}} & \frac{a}{\sqrt{a^2+b^2}} \end{bmatrix}$$

$\frac{a}{\sqrt{a^2+b^2}} = \cos\theta$

$\frac{b}{\sqrt{a^2+b^2}} = \sin\theta$



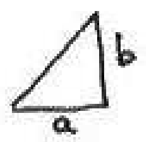
$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix} \quad = \quad \sqrt{a^2+b^2} \qquad \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$\nearrow$ (2)Scale the result $\qquad$ $\uparrow$ (1)Rotate vectors by θ

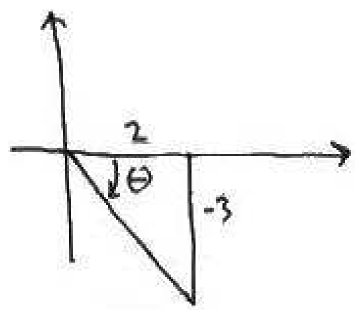
$\underline{\text{EX}}$ $A = \begin{bmatrix} 2 & 3 \\ -3 & 2 \end{bmatrix}$

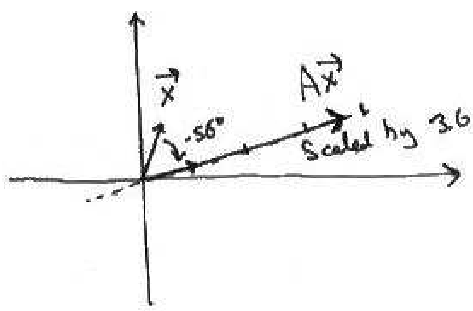
Scaling factor $= \sqrt{(2)^2 + (-3)^2} = \sqrt{13} = 3.6$

Rotation: $a = 2$, $b = -3$



$tan\theta = -3/2, \quad \theta = tan^{-1}(-3/2) \approx -56^o$:





$A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$. From last line, we saw.

- Eigenvalues are $\lambda = a \pm bi \;\; (i^2 = -1)$,
- Rotates vectors + scales by a factor

We will see that all other 2×2 matrices of real numbers are <u>similar to</u> a matrix of this form.

2×2 Matrices with Complex Eigenvalues

Suppose A has an eigenvalue

$$\lambda = a + ib$$

with an associated eigenvector

$$\vec{V} \;=\; \begin{bmatrix} X_1 + Y_1 i \\ X_2 + Y_2 i \end{bmatrix} \;=\; \underset{\vec{X}}{\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}} \;+\; i \underset{\vec{Y}}{\begin{bmatrix} y_1 \\ Y_2 \end{bmatrix}}$$

Then

$$A\vec{V} = \lambda\vec{V}$$

$$\begin{aligned}
A(\vec{X} + iA\vec{Y}) &= (a + bi) \quad (\vec{X} + i\vec{Y}) \\
& \qquad\qquad\qquad\qquad \downarrow \qquad \textit{Multiply out} \\
A\vec{X} + iA\vec{Y} &= a\vec{X} + ai\vec{Y} \quad + \quad bi\vec{X} + bi^2\vec{Y} \\
& \qquad\qquad\qquad\qquad \uparrow \qquad (-1) \\
A\vec{X} + iA\vec{Y} &= (a\vec{X} - b\vec{Y}) \quad + \quad i(a\vec{Y} + b\vec{X})
\end{aligned}$$

Compare the real part and imaginary part

$$A\vec{X} = a\vec{X} - b\vec{Y}$$

$$A\vec{Y} = a\vec{Y} + b\vec{X}$$

Combine into one matrix equation by

$$\begin{aligned} A[\vec{X}\ \vec{Y}] &= \quad [A\vec{X} \qquad\qquad A\vec{Y}] \\ & \qquad\ \ \downarrow \qquad\qquad\quad \downarrow \qquad \curvearrowleft \textit{From last page} \\ &= [(a\vec{X}-b\vec{Y})\ \ (b\vec{X}+a\vec{Y})] \\ &= \quad [\vec{X}\ \ \vec{Y}] \quad \begin{bmatrix} a & b \\ -b & a \end{bmatrix} \end{aligned}$$

So we can write

$$A = \begin{bmatrix} \vec{X} & \vec{Y} \end{bmatrix} \begin{bmatrix} a & b \\ -b & a \end{bmatrix} \begin{bmatrix} \vec{X} & \vec{Y} \end{bmatrix}^{-1}$$

Let $P = \begin{bmatrix} \vec{X} & \vec{Y} \end{bmatrix}$, the matrix whose columns come from the eigenvalue $\vec{V} = \vec{X} + i\vec{Y}$ and let $R = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$, a matrix from the eigenvalue $\lambda = a + ib$.

$$A = PRP^{-1}$$

A factorization of matrix A different from diagonalization.

$\underline{\text{EX}}$ $A = \begin{bmatrix} 5 & -5 \\ 1 & 1 \end{bmatrix}$

$$\begin{aligned} det(A - \lambda I) &= \begin{vmatrix} 5-\lambda & -5 \\ 1 & 1-\lambda \end{vmatrix} \\ &= (5-\lambda)(1-\lambda) \ - \ (-5) \\ &= \lambda^2 - 6\lambda + 10 \\ \lambda &= \tfrac{6 \pm \sqrt{36 - 4\times 1 \times 10}}{2\times 1} \\ &= \tfrac{6 \pm \sqrt{-4}}{2} \\ &= \tfrac{6 \pm 2i}{2} \\ &= 3 \pm i \end{aligned}$$

$\lambda = 3 + i$

$$(A - (3+i)I)\vec{V} = \vec{0} \Rightarrow \cdots \text{ Solve for } \vec{V}$$

$$\vec{V} = V_2 \begin{bmatrix} 2+i \\ 1 \end{bmatrix}$$

$$\lambda = 3+i \quad \vec{V} = \begin{bmatrix} 2+i \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} + i \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$a = 3, b = 1 \quad \vec{X} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \quad \vec{Y} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$A = \begin{bmatrix} \vec{X} & \vec{Y} \end{bmatrix} \begin{bmatrix} a & b \\ -b & a \end{bmatrix} \begin{bmatrix} \vec{X} & \vec{Y} \end{bmatrix}^{-1}$$

↗ See last page

$$\begin{bmatrix} 5 & -5 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}^{-1}$$