5.2,5.3 Similar Matrices and Diagonalization

$$A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix} \quad B = \begin{bmatrix} 3 & 14 \\ 2 & 0 \end{bmatrix} \quad C = \begin{bmatrix} 7 & 0 \\ 0 & -4 \end{bmatrix}$$

$$det A = 2 - 30 = -28 \quad det B = 0 - 28 = -28 \quad det C = -28$$

$$trace A = 1 + 2 = 3 \quad trace B = 3 + 0 = 3 \quad trac C = 7 - 4 = 3$$

$$\lambda = 7, -4 \qquad \lambda = 7, -4 \qquad \lambda = 7, -4$$

There is a connection between A, B, C

$$B = \underbrace{\begin{bmatrix} 3 & 4 \\ 1 & 1 \end{bmatrix}}_{P^{-1}} \overbrace{\begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}}^{A} \underbrace{\begin{bmatrix} -1 & 4 \\ 1 & -3 \end{bmatrix}}_{P}$$

Definition The $n \times n$ matrices A and B

are similar if there is an invertible matrix P so that

$$B = P^{-1}AP$$

OR

$$A = PBP^{-1}$$

Properties of Similar Matrices Suppose A,B are similar.

$$B = P^{-1}AP$$

$$
\begin{aligned}
(1)\quad det(B) &= det(P^{-1}AP) \\
&= detP^{-1} \quad detA \;\; detP \\
&= \tfrac{1}{detP} \quad detA \;\; detP \\
&= det(A)
\end{aligned}
$$

(2)A and B will have the same characteristic polynomial $(det(A - \lambda I)) = det(B - \lambda I)$, and so will have the same eigenvalues.

$$\begin{aligned} det(B - \lambda I) &= det(P^{-1}AP - \lambda I) \\ &= det(P^{-1}AP - \lambda P^{-1}P) \\ &= det(P^{-1}(A - \lambda I)P) \\ &= det(P^{-1})det(A - \lambda I)det(P) \\ &= det(A - \lambda I) \end{aligned}$$

Diagonalization

If A is similar to a diagonal matrix D

$$D = \begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & d_n \end{bmatrix}$$

then A is diagonalizable. In this case, we can convert A into a diagonal matrix D, by some marix P.

$$D = P^{-1}AP$$

A is $\quad 2 \times 2$

$$D = P^{-1}AP$$

$$\begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix} = \begin{bmatrix} \vec{P}_1 & \vec{P}_2 \end{bmatrix}^{-1} A \begin{bmatrix} \vec{P}_1 & \vec{P}_2 \end{bmatrix}$$

$$\begin{bmatrix} \vec{P}_1 & \vec{P}_2 \end{bmatrix} \begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix} = A \begin{bmatrix} \vec{P}_1 & \vec{P}_2 \end{bmatrix}$$

$$\begin{bmatrix} (d_1\vec{P}_1 + 0\vec{P}_2) & (0\vec{P}_1 + d_2\vec{P}_2) \end{bmatrix} = \begin{bmatrix} A\vec{P}_1 & A\vec{P}_2 \end{bmatrix}$$

1^{st} column : $\quad A\vec{P}_1 = d_1\vec{P}_1$

2^{st} column : $\quad A\vec{P}_2 = d_2\vec{P}_2$

d_1, d_2 are eigenvalues, $\vec{P}_1, \vec{P}_2$ are eigenvectors

$$D = P^{-1}AP$$

$$\begin{bmatrix} \text{Diagonal} \\ \text{matrix} \\ \text{of eigenvalues} \\ \text{of } A \end{bmatrix} = \begin{bmatrix} \quad \end{bmatrix}^{-1} A \begin{bmatrix} \text{matrix whose} \\ \text{columns are} \\ \text{eigenvectors of } A \\ P \end{bmatrix}$$

If P has indep. columns,

P will be invertible.

<u>EX:</u>

$$A = \begin{bmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{bmatrix}$$

Diagonalizable ? If so, P,D ?

(1) Find the eigenvalues of A

$$\Big[\, A \;-\; \lambda I \,\Big] = \begin{bmatrix} 4-\lambda & 2 & 2 \\ 2 & 4-\lambda & 2 \\ 2 & 2 & 4-\lambda \end{bmatrix} = \cdots$$

$$= (2-\lambda)^{(2)}(8-\lambda) = 0$$

Two eigenvalue :

$\lambda_1 = 2$ (algebraic multiplicity 2)

$\lambda_2 = 8$ (algebraic multiplicity 1)

(2) For each eigenvalue, find a basis for the eigenspace

(a) $\lambda_1 = 2$

$$(A - 2I)\vec{X} = \vec{0}$$

$$\begin{bmatrix} 2 & 2 & 2 & : & 0 \\ 2 & 2 & 2 & : & 0 \\ 2 & 2 & 2 & : & 0 \end{bmatrix} \backsim \begin{bmatrix} 1 & 1 & 1 & : & 0 \\ 0 & 0 & 0 & : & 0 \\ 0 & 0 & 0 & : & 0 \end{bmatrix}$$

$$\begin{array}{ccccccc} X_1 & + & X_2 & + & X_3 & = & 0 \\ & & & & X_2 & \text{free} & \\ & & & & X_3 & \text{free} & \end{array}$$

$$\vec{X} = \begin{bmatrix} -X_2 - X_3 \\ X_2 \\ X_3 \end{bmatrix} = X_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + X_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Eigenspace basis $\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$

$dim = 2$, the geometric multiplicity of $\lambda_1 = 2$.

(b) $\lambda = 8$

$$(A - 8I)\vec{X} = \vec{0} \Rightarrow \cdots \Rightarrow \vec{X} = X_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Eigenspace basis : $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$

$dim = 1$, geometric mult. of $\lambda_2 = 8$.

(3) Three independent eigenvectors

$$\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$\lambda_1 = 2$ $\qquad\qquad$ $\lambda_2 = 8$

So yes, A is diagondizable with

$$D = P^{-1}AP$$

$$\underset{E-vals}{\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 8 \end{bmatrix}} = \begin{bmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{bmatrix} \underset{E-vect.}{\overset{2 \quad 8}{\begin{bmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}}}$$