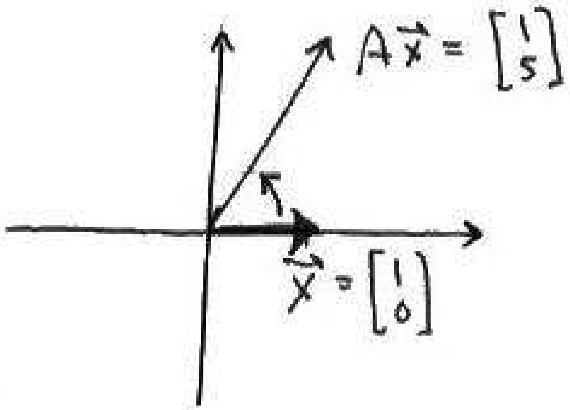
5.1 5.2 Eigenvalues + Eigenvectors

EX $\quad A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$

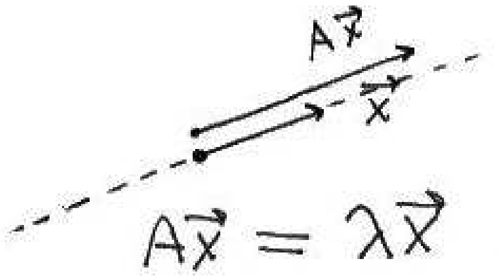
Matrix A will transform $\vec{X} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$ to $A\vec{X}$

$$\vec{X} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad A\vec{X} = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$$



We will look for particular vectors $\vec{X}$

where $A\vec{X}$ is parallel to $\vec{X}$.



Definition An eigenvector of an $n \times n$ matrix A is a nonzero vector $\vec{X}$ such that $A\vec{X} = \lambda\vec{X}$ ($A\vec{X}$ is parallel to $\vec{X}$) for some scalar λ . Such a scalar is called an eigenvalue .

(1) Why exclude $\vec{X} = \vec{0}$ in this definition ?

$$A\vec{0} = \vec{0} = \lambda\vec{0}$$

for all λ . There is no reasonalble way to associate $\vec{0}$ with one eigenvalue.

(2) If we know $\vec{X}$ is an eigenvector of A, we can find its eigenvalue :

$$A\vec{X} = \underline{?}\vec{X}$$

<u>EX</u> $A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$, Eigenvector $\begin{bmatrix} 6 \\ -5 \end{bmatrix}$

$$\begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} 6 \\ -5 \end{bmatrix} = \begin{bmatrix} -24 \\ 20 \end{bmatrix} = -4 \begin{bmatrix} 6 \\ -5 \end{bmatrix}$$

The eigenvalue is $\lambda = -4$

(3) If we know λ is an eigenvalue of A, we can find the associated eigenvectors :

$$A\vec{X} = \lambda\vec{X}$$
$$A\vec{X} - \lambda\vec{X} = \vec{0}$$
$$(A - \lambda I)\vec{X} = \vec{0}$$

<u>EX</u> $A = \begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix}$, 2 is an eigenvalue.

$$A\vec{X} = 2\vec{X}$$
$$A\vec{X} - 2\vec{X} = \vec{0}$$
$$(A - 2I)\vec{X} = \vec{0}$$

$$\begin{aligned} A - 2I &= \begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix} - 2\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 2 & -1 & 6 \\ 2 & -1 & 6 \\ 2 & -1 & 6 \end{bmatrix} \end{aligned}$$

$$\begin{bmatrix} 2 & -1 & 6 \\ 2 & -1 & 6 \\ 2 & -1 & 6 \end{bmatrix} \vec{X} = \vec{0}$$

$$\begin{bmatrix} 2 & -1 & 6 & 0 \\ 2 & -1 & 6 & 0 \\ 2 & -1 & 6 & 0 \end{bmatrix} \backsim \begin{bmatrix} 2 & -1 & 6 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \backsim \begin{bmatrix} 1 & -\frac{1}{2} & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{array}{ccccccc} X_1 & - & \frac{1}{2}X_2 & + & 3X_3 & = & 0 \\ & & & & X_2 & & \text{free} \\ & & & & X_3 & & \text{free} \end{array}$$

$$\vec{X} = \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}X_2 - 3X_3 \\ X_2 \\ X_3 \end{bmatrix} = X_2 \begin{bmatrix} \frac{1}{2} \\ 1 \\ 0 \end{bmatrix} + X_3 \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$$

The eigenvectors associated with eigenvalue 2 are

$$\vec{X} = X_2 \begin{bmatrix} \frac{1}{2} \\ 1 \\ 0 \end{bmatrix} + X_3 \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$$

any X_2, X_3 except for $X_2 = 0 = X_3$.

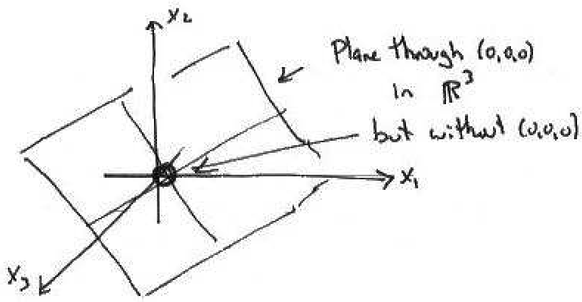
So for example, if $X_2 = 2, X_3 = 4$

$$\vec{X} = 2 \begin{bmatrix} \frac{1}{2} \\ 1 \\ 0 \end{bmatrix} + 4 \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -11 \\ 2 \\ 4 \end{bmatrix}$$

$$\begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix} \begin{bmatrix} -11 \\ 2 \\ 4 \end{bmatrix} = \begin{bmatrix} -22 \\ 4 \\ 8 \end{bmatrix} = 2 \begin{bmatrix} -11 \\ 2 \\ 4 \end{bmatrix}$$

Graphically,

$$\vec{X} = X_2 \begin{bmatrix} \frac{1}{2} \\ 1 \\ 0 \end{bmatrix} + X_3 \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$$



The eigenvectors and add $\vec{0}$ back in , we generate a subspace called the <u>eigenspace</u> of A associated with $\lambda = 2$

$$\text{Eigenspace} = \text{Span} \left\{ \begin{bmatrix} \frac{1}{2} \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \right\}$$

$$\text{Basis} = \left\{ \begin{bmatrix} \frac{1}{2} \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \right\}$$

(4) The null space of $(A - \lambda I)$ when λ is an eigenvalue is called the <u>eigenspace</u> of A associated with λ . It is the solutions of

$$(A - \lambda I)\ \vec{X} = \vec{0}$$

and consisted of

(a) eigenvectors for λ, and

(b) $\vec{0}$

(5) The eigenvalues of A can be found by using the characteristic polynomial of A:

$$A\vec{X} = \lambda\vec{X} \text{ for some } \vec{X} \neq \vec{0}$$

$\Leftrightarrow$ $(A - \lambda I)\vec{X} = \vec{0}$ for some $\vec{X} \neq \vec{0}$

$\Leftrightarrow$the columns of $(A - \lambda I)$ are dependent

$\Leftrightarrow$$(A - \lambda I)$ is not invertible (singular)

$\Leftrightarrow$ $det(A - \lambda I) = 0$

- $det(A - \lambda I)$ is called the characteristic polynomial of A.

- $det(A - \lambda I)= 0$ is called the characteristic equation of A.
- Solutions of $det(A - \lambda I) = 0$ are the exigences of A.

EX $A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$

$$\begin{aligned} A - \lambda I &= \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} (1-\lambda) & 6 \\ 5 & (2-\lambda) \end{bmatrix} \\ det(A - \lambda I) &= \begin{vmatrix} (1-\lambda) & 6 \\ 5 & (2-\lambda) \end{vmatrix} \\ &= (1-\lambda)(2-\lambda) - 30 \\ &= \lambda^2 - 3\lambda - 28 \text{ Char. Poly} \end{aligned}$$

$$\begin{aligned} det(A - \lambda I) = 0 &\Rightarrow \lambda^2 - 3\lambda - 28 = 0 \quad \text{Char. Eq.} \\ &\Rightarrow (\lambda - 7)(\lambda + 4) = 0 \\ &\Rightarrow \lambda = 7, \quad \lambda = -4 \end{aligned}$$

The eigenvalues of $\begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$ are $7, -4$.

Note:
$$\begin{aligned} 7 + (-4) &= 3 = 1 + 2 = traceA. \\ (7)(-4) &= -28 = 1.2 - 5.6 = detA \end{aligned}$$

The eigenvectors for $\lambda = 7$:

$$A\vec{X} = 7\vec{X}$$

$$(A - 7I)\vec{X} = \vec{0}$$

$$\left(\begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix} - 7 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \vec{X} = \vec{0}$$

$$\begin{bmatrix} -6 & 6 \\ 5 & -5 \end{bmatrix} \vec{X} = \vec{0}$$

$$\begin{bmatrix} -6 & 6 & 0 \\ 5 & -5 & 0 \end{bmatrix} \backsim \begin{bmatrix} 1 & -1 & 0 \\ 1 & -1 & 0 \end{bmatrix} \backsim \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{array}{rcll} X_1 & - & X_2 & = 0 \\ & & X_2 & free \end{array} \Rightarrow \begin{array}{rcl} X_1 & = & X_2 \\ X_2 & = & X_2 \end{array}$$

$$\vec{X} = \begin{bmatrix} X_2 \\ X_2 \end{bmatrix} = X_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad X_2 \neq 0$$

A basis for the eigenspace with λ is $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$

The eigenvectors for $\lambda = -4$:

$$A\vec{X} = -4\vec{X}$$

$$(A + 4I)\vec{X} = \vec{0}$$

$$\left(\begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix} + 4 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \vec{X} = \vec{0}$$

$$\begin{bmatrix} 5 & 6 \\ 5 & 6 \end{bmatrix} \vec{X} = \vec{0}$$

$$\begin{bmatrix} 5 & 6 & 0 \\ 5 & 6 & 0 \end{bmatrix} \backsim \begin{bmatrix} 1 & \frac{6}{5} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{array}{rcll} X_1 & + & \frac{6}{5}X_2 & = 0 \\ & & X_2 & \text{free} \end{array} \Rightarrow \begin{array}{rcl} X_1 & = & \frac{-6}{5}X_2 \\ X_2 & = & \text{free} \end{array}$$

$$\vec{X} = \begin{bmatrix} -\frac{6}{5}X_2 \\ X_2 \end{bmatrix} = X_2 \begin{bmatrix} -\frac{6}{5} \\ 1 \end{bmatrix}$$

A basis for the eigenspace for $\lambda = -4$ is $\left\{ \begin{bmatrix} -6 \\ 5 \end{bmatrix} \right\}$

(6) If A is triangular, say

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & \cdots & a_{nn} \end{bmatrix}$$

$$\begin{aligned} det(A - \lambda I) &= \begin{vmatrix} (a_{11-\lambda}) & a_{12} & \cdots & a_{1n} \\ 0 & (a_{22-\lambda}) & \cdots & a_{2n} \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & \cdots & (a_{nn-\lambda}) \end{vmatrix} \\ &= (a_{11} - \lambda)(a_{11} - \lambda) \cdots (a_{nn} - \lambda) \\ 0 &= (a_{11} - \lambda)(a_{22} - \lambda) \cdots (a_{11} - \lambda) \\ & \lambda = a_{11}, \lambda = a_{22}, \cdots, \lambda = a_{nn} \end{aligned}$$

The eigenvalues are $a_{11}, a_{22}, \cdots, a_{nn}$, the diagonal entries.

<u>EX</u> $\begin{bmatrix} 3 & 0 & 0 \\ 2 & 0 & 0 \\ 1 & -1 & 3 \end{bmatrix}$ Triangular

$$Eigenvalues \;\; are \;\; 3, \qquad \underset{\substack{\uparrow \\ Allowed.}}{0,} \qquad 3$$

(7) $\lambda = 0$ can be an eigenvalue of A.

$$\begin{aligned} A\vec{X} &= 0\vec{X} \\ or \quad A\vec{X} &= \vec{0} \end{aligned}$$

The eigenspace is then Nul(A). This only happens when A is not invertile.

EX $\quad A = \begin{bmatrix} 1 & 3 \\ 1 & 3 \end{bmatrix}$ is not invertible ($detA = 0$)

$\lambda = 0$ is an eigenvalue. The other is $\lambda = 4$.

$\vec{X}$ eigenvector, λ eigenvalue $\quad A\vec{X} = \lambda\vec{X}$

Geometric :

$det(A - \lambda I) = 0$ (Chan. Eq) $\Rightarrow$ eigenvalue λ

$(A - \lambda I)\vec{X} = \vec{0} \quad \Rightarrow \quad$ eigenvectors $\vec{X}$

triangular matrix $\Rightarrow \lambda = a_{ij}$.

Linear Independence of Eigenvectors

Suppose A is an $n \times n$ matrix.

Suppose $\vec{V}_1, \vec{V}_2, \cdots, \vec{V}_p$ are eigenvectors,

corresponding to distinct eigenvalues $\lambda_1, \cdots, \lambda_p$.

Then $\{\vec{V}_1, \vec{V}_2, \cdots, \vec{V}_p\}$ are linearly independent.

For the case of two eigenvectors :

Suppose $\vec{V}_1$ and $\vec{V}_2$ are two eigenvectors of A.

$$\vec{V}_1 \neq \vec{0}, \quad A\vec{V}_1 = \lambda_1 \vec{V}_1$$

$$\vec{V}_2 \neq \vec{0}, \quad A\vec{V}_2 = \lambda_2 \vec{V}_2$$

Suppose $\lambda_1 \neq \lambda_2$.

(1) $c_1\vec{V}_1 + c_2\vec{V}_2 = \vec{0}$

Multiply (1) by A

$$\begin{aligned} A(c_1\vec{V}_1 \;&+\; c_2\vec{V}_2) \;=\; A\vec{0} \\ c_1\underline{A\vec{V}_1} \;&+\; c_2\underline{A\vec{V}_2} \;=\; \vec{0} \end{aligned}$$

(2) $c_1\lambda_1\vec{V}_1 + c_2\lambda_2\vec{V}_2 = \vec{0}$

(1) $c_1\vec{V_1} + c_2\vec{V_2} = \vec{0}$

(2)$(c_1\lambda_1)\vec{V_1} + (c_2\lambda_2)\vec{V_2} = \vec{0}$

Multiply (1) by $(-\lambda_2)$ and add to (2)

$$
\begin{array}{llcccccc}
-\lambda_2 & (1) & -c_1\lambda_2\vec{V_1} & - & c_2\lambda_2\vec{V_2} & = & \vec{0} & \\
 & (2) & c_1\lambda_1\vec{V_1} & + & c_2\lambda_2\vec{V_2} & = & \vec{0} & \\
 & & \cdots & \cdots & \cdots & \cdots & \cdots &
\end{array}
$$

Add $\quad c_1\lambda_1\vec{V_1} \ - \ c_1\lambda_2\vec{V_1} \ = \ \vec{0}$

$$c_1 \ - \ (\lambda_1 - \lambda_2) \ \vec{V_1} \ = \ \vec{0}$$

So

(1) $c_1\vec{V_1} + c_2\vec{V_2} = \vec{0}$

$c_1 = 0, c_2 = 0$ is the only solution.

So $\vec{V_1}, \vec{V_2}$ are linearly independent.