3.2 Properties of Determinants

(1) If A is an $n \times n$ triangular matrix
(all entries are o either above or below the diagonal),
then

$$det(A) = a_{11}a_{22}\cdots a_{nn}$$

EX $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}$ $det(A) = 1 \times 4 \times 6 = 24$

(2) Suppose two rows of a matrix are interchanged.
The determinant changes sign.

$$\underbrace{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}_{A} = - \underbrace{\begin{vmatrix} a_{21} & a_{22} \\ a_{11} & a_{12} \end{vmatrix}}_{B}$$

$det(A) = a_{11}a_{22} - a_{12}a_{21}$
$det(B) = a_{21}a_{12} - a_{11}a_{22} = -det(A)$

(3) Suppose one row of a matrix is multiplied by K. The determinant changes by a factor of K.

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

$$\begin{vmatrix} ka_{11} & ka_{12} \\ ka_{21} & ka_{22} \end{vmatrix} = \quad ka_{11}a_{22} \quad - \; ka_{12}a_{21}$$

$$= \quad k(a_{11}a_{22} \quad - \; a_{21}a_{12})$$

$$= \; k\begin{vmatrix} ka_{11} & ka_{12} \\ ka_{21} & ka_{22} \end{vmatrix}$$

(4) If a multiple of one row of a matrix is *ADDED* to another row, the determinant does not change.

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

Add c times R_1 *to* R_2

$$\begin{vmatrix} a_{11} & a_{12} \\ ca_{11} & ca_{12} + a_{22} \end{vmatrix} = a_{11}(ca_{12} + a_{22}) - a_{12}(ca_{11} + a_{21})$$

$$= ca_{11}a_{12} + a_{11}a_{22} - ca_{12}a_{21} - a_{12}a_{21}$$

$$= \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

<u>EX</u> Use row operations to find the determinant of $\begin{bmatrix} 0 & 2 & 4 \\ 4 & 2 & 3 \\ 1 & 3 & 6 \end{bmatrix}$

Apply row operations to reduce the matrix to echelon form, keeping track of any changes to the determinant.

$$\begin{vmatrix} 0 & 2 & 4 \\ 4 & 2 & 3 \\ 1 & 3 & 6 \end{vmatrix} = (-1)\begin{vmatrix} 1 & 3 & 6 \\ 4 & 2 & 3 \\ 0 & 2 & 4 \end{vmatrix} \quad \text{Rows 1+3 Exchanged}$$

$$= (-1)\begin{vmatrix} 1 & 3 & 6 \\ 0 & -10 & -21 \\ 0 & 2 & 4 \end{vmatrix} \quad \text{Add} - 4 \times R_1 \ \ to \ \ R_2$$

$$(No \ \ change)$$

$$2(-1)^2\begin{vmatrix} 1 & 3 & 6 \\ 0 & 1 & 2 \\ 0 & -10 & -21 \end{vmatrix} \longleftarrow = (-1)^2\begin{vmatrix} 1 & 3 & 6 \\ 0 & 2 & 4 \\ 0 & -10 & -21 \end{vmatrix} \quad \text{Rows 2,3 Exchanged}$$

$$2(-1)^2\begin{vmatrix} 1 & 3 & 6 \\ 0 & 1 & 2 \\ 0 & 0 & -1 \end{vmatrix} = (-1)^2\begin{vmatrix} 1 & 3 & 6 \\ 0 & 2 & 4 \\ 0 & 0 & -1 \end{vmatrix} \quad \text{Add } 5 \times R_2 \ to \ R_3$$

$$2(-1)^2(1)(1)(-1) = (-1)^2 \times (1)(2)(-1)$$

$$-2 = -2$$

(5) Suppose $A \backsim U$ with no row scalings
(row interchanges,row replacement)

where U is in echelon form. Then

$det(A) = (-1)^r det(U) \quad r =^\sharp$ row interchanges

(6) $det(A) = 0 \Leftrightarrow det(U) = 0 \Leftrightarrow$ there

is a zero on the diagonal of $U \Leftrightarrow$

A will not have n pivot columns $\Leftrightarrow$

A is not invertible.

Determinant Properties (Continued)

(7) $\quad det(AB) = det(BA)$

EX $\quad A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad B = \begin{bmatrix} -1 & 0 \\ 3 & 1 \end{bmatrix}$

$$AB = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 2 \\ 9 & 4 \end{bmatrix}$$

$$det(AB) = 5 \times 4 - 2 \times 9 = 2$$

$$BA = \begin{bmatrix} -1 & 0 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} -1 & -2 \\ 6 & 10 \end{bmatrix}$$

$$det(BA) = (-1)(10) - (6)(-2) = 2$$

(8) $det(AB) = det(A)det(B)$

<u>EX</u> $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ $B = \begin{bmatrix} -1 & 0 \\ 3 & 1 \end{bmatrix}$

$detA = -2 \qquad detB = -1$

$det(A)det(B) = (-2)(-1) = 2 = det(AB)$

Note:

$$\begin{aligned} det(AB) &= det(A)det(B) \\ & \quad\ \ \sharp \quad \sharp \\ &= det(B)det(A) \\ &= det(BA) \end{aligned}$$

(9) $det(A^T) = det(A)$

(10) $det(A^{-1}) = 1/det(A)$