2.8 Subspaces of $\Re^n$

Definition: A set of vectors H in $\Re^n$ is called a subspace of $\Re^n$ if

(a) The zero vector of $\Re^n$ is in H.

(b) For every $\vec{u}, \vec{v}$ in H, $\vec{u} + \vec{v}$ is also in H.

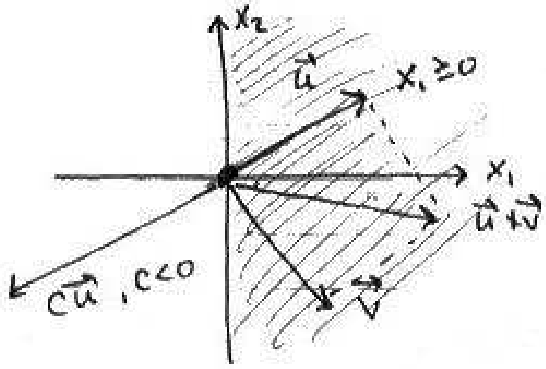
(c) For every $\vec{u}$ in H, scalar c, $c\vec{u}$ is also in H.

EX $\Re^2$

(a) $H = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} : x_1 \geqslant 0 \right\}$

$\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is in H; $\begin{bmatrix} -1 \\ 0 \end{bmatrix}$ is not in H.

$\vec{0}$ is in H $\quad \vec{u} + \vec{v}$ is in H?



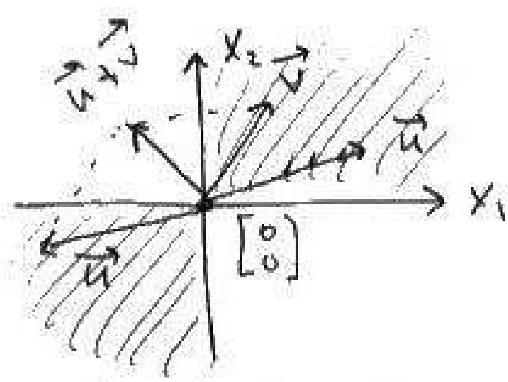
$$H \text{ is not a subspace; } \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ is in } H,$$

$$\text{but } -2 \begin{bmatrix} 1 & -2 \\ 1 & -2 \end{bmatrix} - 2 \not\geq \text{ is not in } H.$$

(b) $H = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} : x_1 x_2 \geqslant 0 \right\}$

$\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ -1 \end{bmatrix}$ is in H; $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$ is not in H.

H is not a closed under vector addition.



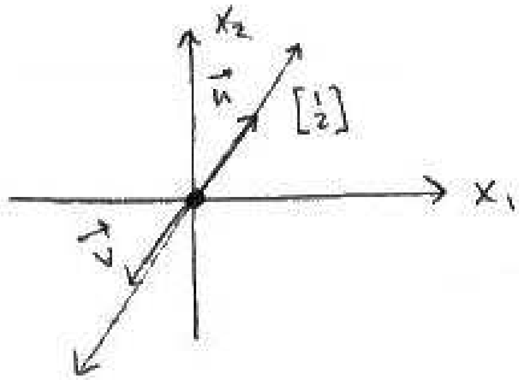
$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} -2 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$\nwarrow \quad \nearrow \qquad\qquad \uparrow$

in H Not in H

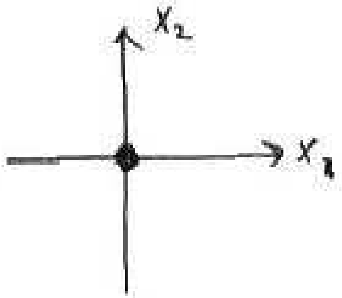
(c) $H = Span\left\{\begin{bmatrix}1\\2\end{bmatrix}\right\}$

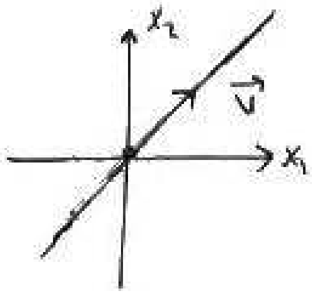
$\begin{bmatrix}2\\4\end{bmatrix}$ is in H; $\begin{bmatrix}-2\\4\end{bmatrix}$ is not in H.

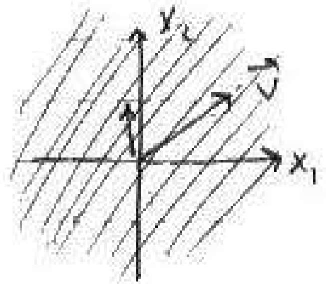


H is a Subspace of $\Re^2$.

Subspaces of $\Re^2$:



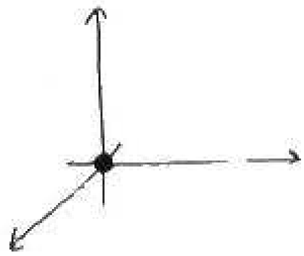


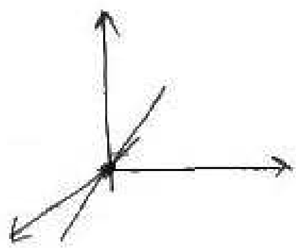


$$H = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\} \qquad H = Span\{\vec{V}\} \qquad H = \Re^2$$

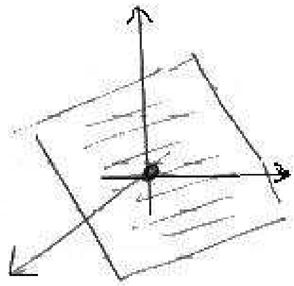
Subspaces of $\Re^3$:

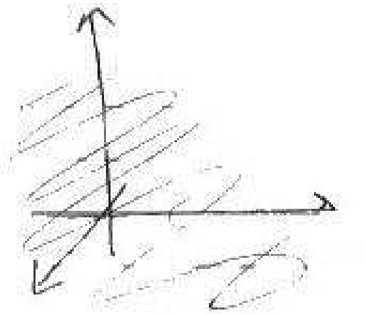
$$H = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$$





Line through origin

Plane through origin (not $\Re^2$) 

$H = \Re^3$ 

A set of vectors H in $\Re^n$ is called a subspace of $\Re^n$ if

(a) the zero vector of $\Re^n$ is in H

(b) for every $\vec{u}, \vec{v}$ in H, $\vec{u} + \vec{v}$ is also in H,

(c) for every $\vec{u}$ in H and scalar c,$c\vec{u}$ is also in H.

In $\Re^2$: $\left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$, lines through origin, $\Re^2$

In $\Re^3$: $\left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$, lines + planes through origin, $\Re^3$

In $\Re^4$:

Note If $\vec{v_1}, \vec{v_2}, \cdots, \vec{v_p}$ are vectors in $\Re^n$, and let

$$H = span\{\vec{V_1}, \cdots, \vec{V_p}\}$$

Then

(a) $\vec{0} = 0\vec{v_1} + 0\vec{v_2} + \cdots + 0\vec{v_p}$ shows $\vec{0}$ is in H

(b) If $\vec{U}, \vec{V}$ are in H,then

$\vec{U} = a_1\vec{v_1} + \cdots + a_p\vec{v_p}\vec{V} = b_1\vec{v_1} + \cdots + b_p\vec{v_p}\vec{U} + \vec{V} =$

$(a_1 + b_1)\vec{v_1} + \cdots + (a_p + b_p)\vec{v_p}$

So $\vec{U} + \vec{V}$ is also in H.

(c) If $\vec{U}$ is in H, and c is any scalar,

$$\begin{array}{ccccccc} \vec{U} & = & a_1\vec{v_1} & + & \cdots & + & a_p\vec{v_p} \\ c\vec{U} & = & (ca_1)\vec{v_1} & + & \cdots & + & (ca_p)\vec{v_p} \end{array}$$

so $c\vec{U}$ is also in H.

So $H = span\{\vec{v_1}, \cdots, \vec{v_p}\}$ is a subspace.

Two Subspaces Related to Matrices

Suppose $A = \begin{bmatrix} \vec{a_1} & \cdots & \vec{a_n} \end{bmatrix}$ is an $m \times n$ matrix.

(a) the column space of A is the set of all linear combinations of the columns of A.

$$Col(A) = Span\{\vec{a_1}, \cdots, \vec{a_n}\}$$

(b) the null space of A is the set of all solutions of $A\vec{X} = \vec{0}$

Nul(A)

<u>EX</u> $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ -1 & 1 \end{bmatrix}$

(a) Is $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ in the column space of A?

$$x_1 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \text{ (Important Eq.)}$$

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ -1 & 1 & 1 \end{bmatrix} \backsim \cdots \backsim \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix}$$

The important equation has no solution.

So $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is not in the $Col(A)$.

(b) Is $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ in the null space of A?

$$\begin{bmatrix} 1 & 2 \\ 0 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \text{ (Important Eq.)}$$

Check:

$$\begin{bmatrix} 1 & 2 \\ 0 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

So $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is not in $Nul(A)$.