2.2 The Inverse of a matrix

Motivation

Reclnumbers: $aX = b \;\Rightarrow\; X = \frac{b}{a}$

Slow motion: $aX = b$

Suppose a^{-1} **is some number for which** $a^{-1}a = 1$**.**

$$\begin{aligned} aX &= b \\ a^{-1}(aX) &= a^{-1}(b) \\ (a^{-1}a)X &= a^{-1}b \\ 1X &= a^{-1}b \\ X &= a^{-1}b \;\longrightarrow\; \frac{b}{a} \end{aligned}$$

matrices: $AX = B$

Suppose A^{-1} **is another matrix where** $A^{-1}A = I$**.**

$$\begin{aligned} AX &= B \\ A^{-1}(AX) &= A^{-1}(B) \\ (A^{-1}A)X &= A^{-1}B \\ IX &= A^{-1}B \\ X &= A^{-1}B \end{aligned}$$

Definintion Suppose A is an $n \times n$ matrix (square).

If there is another $n \times n$ matrix A^{-1} such that

$$A^{-1}A = I \quad \textbf{AND} \quad AA^{-1} = I$$

then A is called invertible with inverse A^{-1}.

If such a matric A^{-1} does not exist, then A is called

not invertible or singular

How to Find A^{-1} (if it exists)

EX (Slow motion)

$$\mathbf{A}=\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

Is there a matrix A^{-1} where $AA^{-1} = A^{-1}A = \mathbf{I}$?

$$AA^{-1} = I$$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} \vec{C}_1 & \vec{C}_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \vec{C}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \vec{C}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 1 & 2 & 1 \\ 3 & 4 & 0 \end{array}\right] \quad \left[\begin{array}{cc|c} 1 & 2 & 0 \\ 3 & 4 & 1 \end{array}\right]$$

$$\searrow \quad \swarrow$$

$$-3 \quad \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 3 & 4 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & -2 & -3 & 1 \end{array}\right] \quad -3R_1 + R2$$

$$\left[\begin{array}{cc:cc} 1 & 0 & -2 & 1 \\ 0 & -2 & -3 & 1 \end{array}\right] \; R_2 + R_1$$

$$\left[\begin{array}{cc:cc} 1 & 0 & -2 & 1 \\ 0 & 1 & \frac{3}{2} & \frac{-1}{2} \end{array}\right] \; \frac{-1}{2} R_2$$

$$\vec{C}_1 = \begin{bmatrix} -2 \\ \frac{3}{2} \end{bmatrix} \quad \vec{C}_1 = \begin{bmatrix} 1 \\ \frac{-1}{2} \end{bmatrix}$$

So

$$A^{-1} = \begin{bmatrix} \vec{C}_1 & \vec{C}_2 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & \frac{-1}{2} \end{bmatrix}$$

Summary:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

$$\left[\begin{array}{cc:cc} 1 & 2 & 1 & 0 \\ 3 & 4 & 0 & 1 \end{array}\right] \quad \left[\begin{array}{c:c} A & I \end{array}\right]$$

$$\left[\begin{array}{cc:cc} 1 & 0 & -2 & 1 \\ 0 & 1 & \frac{3}{2} & \frac{-1}{2} \end{array}\right] \quad \left[\begin{array}{c:c} I & A^{-1} \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & \frac{-1}{2} \end{bmatrix}$$

<u>EX</u> $\quad \mathbf{A}=\begin{bmatrix} 1 & 2 & 0 \\ -2 & 1 & 1 \\ 0 & 5 & 1 \end{bmatrix}$

Find A^{-1} **if it exists.**

$$AA^{-1} = \mathbf{I}$$

$$2 \quad \begin{bmatrix} 1 & 2 & 0 & | & 1 & 0 & 0 \\ -2 & 1 & 1 & | & 0 & 1 & 0 \\ 0 & 5 & 1 & | & 0 & 0 & 1 \end{bmatrix}$$

$$-1 \quad \begin{bmatrix} 1 & 2 & 0 & | & 1 & 0 & 0 \\ 0 & 5 & 1 & | & 2 & 1 & 0 \\ 0 & 5 & 1 & | & 0 & 0 & 1 \end{bmatrix} \quad 2R_1 + R_2$$

$$\begin{bmatrix} 1 & 2 & 0 & | & 1 & 0 & 0 \\ 0 & 5 & 1 & | & 2 & 1 & 0 \\ 0 & 0 & 0 & | & -2 & -1 & 1 \end{bmatrix} \quad -R_2 + R_3$$

This matrix tells us that there is <u>no</u> matrix A^{-1} where $AA^{-1} = \mathbf{I}$ **.**

A is not invertible (singular)

A Formula for 2×2 Matrix Inverses

$$\mathbf{A}=\begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \begin{bmatrix} a & b & \vdots & 1 & 0 \\ c & d & \vdots & 0 & 1 \end{bmatrix}$$

$-cR_1$	$-ca$	$-cb$	$-c$	0
aR_2	ac	ad	0	9
	$---$	$---$	$---$	$--$
Add	0	$ad-bc$	$-c$	9

$$\begin{bmatrix} a & b & \vdots & 1 & 0 \\ 0 & ad-bc & \vdots & -c & a \end{bmatrix}$$

For A^{-1} to exist, we need $ad-bc \neq 0$

$$\begin{bmatrix} a & b & \vdots & 1 & 0 \\ 0 & 1 & \vdots & \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{bmatrix}$$

$\Downarrow$ **One more step to row reduce**

$$\begin{bmatrix} 1 & 0 & \vdots & \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ 0 & 1 & \vdots & \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{bmatrix}$$

OR

$$A^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \quad \overset{A}{\begin{bmatrix} a & b \\ c & d \end{bmatrix}}$$

(1) Reverse diagonal entries

(2) Change sign of the nondiagonal entries

(3) Multiple by $\frac{1}{ad-bc}$

OR $A = \begin{bmatrix} 1 & 2 \\ 2 & 6 \end{bmatrix}$

$det(A) = ad - bc = (1)(6) - (2)(2) = 2$

$$A^{-1} = \frac{1}{2} \begin{bmatrix} 6 & -2 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ -1 & \frac{1}{2} \end{bmatrix}$$

Using A^{-1} to Solve $A\vec{X} = \vec{b}$

EX

$$\begin{array}{rcrcl} X_1 & + & 3X_2 & = & 1 \\ 2X_1 & + & 4X_2 & = & 6 \end{array}$$

$$\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \end{bmatrix}$$

$$(\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix})^{-1} \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = (\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix})^{-1} \begin{bmatrix} 1 \\ 6 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \frac{1}{(1)(4)-(2)(3)} \begin{bmatrix} 4 & -3 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 6 \end{bmatrix}$$

$$\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \frac{1}{-2} \begin{bmatrix} -14 \\ 4 \end{bmatrix} = \begin{bmatrix} 7 \\ -2 \end{bmatrix})$$

The soln. $X_1 = 7, X_2 = -2$

Properties of Matrix Inverses

(1) If A is invertible , then $A\vec{X} = \vec{b}$ is consistent with exactly one solution .

$$\vec{X} = A^{-1}\vec{b}$$

(2) If A is invertible with inverse A^{-1}, then A^{-1} is also invertible $(A^{-1})^{-1} = A$

$$AA^{-1} = I$$

A^{-1} is the inverse of A

A is the inverse of A^{-1}

(3) If A,B are two invertible $n \times n$ matrices, then AB is another invertible matrix with

$$(AB)^{-1} = B^{-1}A^{-1}$$

$$(AB)(?) = I$$

$$\begin{aligned} (AB)(B^{-1}A^{-1}) &= A(BB^{-1})A^{-1} \\ &= A(I)A^{-1} \\ &= AA^{-1} \\ &= I \end{aligned}$$

$$\begin{aligned}
(?)(AB) &= I \\
(B^{-1}A^{-1})(AB) &= B^{-1}(A^{-1}A)B \\
&= B^{-1}(I)B \\
&= B^{-1}B \\
&= I
\end{aligned}$$

So:

$$(AB)^{-1} = B^{-1}A^{-1}$$

(4) If A is invertible, then so is A^T <u>and</u> $(A^T)^{-1} = (A^{-1})^T$

Check:

$$\begin{aligned}
A^T(A^{-1})^T &= (A^{-1}A)^T \\
&= (I)^T \\
&= I \\
(A^{-1})^T A^T &= (AA^{-1})^T = (I)^T = I
\end{aligned}$$

<u>**Page. of transpose $(AB)^T = B^T A^T$**</u>