2.1 Matrix Algebra

$$A = \begin{bmatrix} \vec{a_1} & \vec{a_2} & ... & \vec{a_n} \end{bmatrix}$$

$\nwarrow\nearrow$

Each column is in $\Re^m$

$m \times n$ matrix

$$= \begin{bmatrix} a_{11} & a_{12} & ... & a_{1n} \\ a_{21} & a_{22} & ... & a_{2n} \\ \vdots & \vdots & ... & \vdots \\ a_{m1} & a_{m2} & ... & a_{mn} \end{bmatrix}$$

a_{ij} the $(i, j)^{th}$ entry in the matrix

$a_{11}, a_{22}, a_{33}, ...$ diagonal entries

(1) $A = B$ means A,B have the same size and each $a_{ij} = b_{ij}$

(2)

$$A + B = \begin{bmatrix} \vec{a_1} & ... & \vec{a_n} \end{bmatrix} + \begin{bmatrix} \vec{b_1} & ... & \vec{b_n} \end{bmatrix}$$

$$= \begin{bmatrix} (\vec{a_1} + \vec{b_1})(\vec{a_2} + \vec{b_2})...(\vec{a_n} + \vec{b_n}) \end{bmatrix}$$

$(A, B$ *must have the same size*$)$

(3)

$$\underset{\substack{\uparrow \\ number}}{cA} = c\begin{bmatrix} \vec{a_1} & \cdots & \vec{a_n} \end{bmatrix} = \begin{bmatrix} (c\vec{a_1}) & \cdots & (c\vec{a_n}) \end{bmatrix}$$

(4) $Q_{m\times n} = \begin{bmatrix} 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{bmatrix}$ **an** $m \times n$ **matrix of zeros**

(5) $I_n =$ **the** $n \times n$ **identity matrix with all entries 0 except for diagonal entries 1.**

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \ldots$$

<u>EX</u> $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, B = \begin{bmatrix} 3 & 5 \\ -1 & 2 \end{bmatrix}$

$$\begin{aligned} 3A - 2B + 4I &= 3\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} - 2\begin{bmatrix} 3 & 5 \\ -1 & 2 \end{bmatrix} + 4\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 3 & 6 \\ 9 & 12 \end{bmatrix} - \begin{bmatrix} 6 & 10 \\ -2 & 4 \end{bmatrix} + \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} \\ &= \begin{bmatrix} 1 & -4 \\ 11 & 12 \end{bmatrix} \end{aligned}$$

(6) Matrix Multiplication

$$A\vec{X} = \begin{bmatrix} \vec{a_1} & \vec{a_2} & \cdots & \vec{a_n} \end{bmatrix} \begin{bmatrix} X_1 \\ \vdots \\ X_n \end{bmatrix} = \begin{matrix} X_1\vec{a_1} + X_2\vec{a_2} + ... + X_n\vec{a_n} \\ (A\ vector) \end{matrix}$$

$$A \begin{bmatrix} \vec{X_1} & \vec{X_2} & ... & \vec{X_p} \end{bmatrix} = \begin{bmatrix} A\vec{X_1} & A\vec{X_2} & ... & A\vec{X_p} \end{bmatrix}$$

<u>EX</u> $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 0 \\ -1 & 2 \\ 0 & 3 \end{bmatrix}$

$$AB = A \begin{bmatrix} \vec{b_1} & \vec{b_2} \end{bmatrix} = \begin{bmatrix} A\vec{b_1} & A\vec{b_2} \end{bmatrix}$$

$$A\vec{b_1} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = 1 \begin{bmatrix} 1 \\ 4 \end{bmatrix} + (-1) \begin{bmatrix} 2 \\ 5 \end{bmatrix} + 0 \begin{bmatrix} 3 \\ 6 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$$

$$A\vec{b_2} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix} = \cdots = \begin{bmatrix} 13 \\ 28 \end{bmatrix}$$

$$AB = \begin{bmatrix} -1 & 13 \\ -1 & 28 \end{bmatrix}$$

$$\mathbf{BA=}\begin{bmatrix} 1 & 0 \\ -1 & 2 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

$$B = \begin{bmatrix} \vec{a_1} & \vec{a_2} & \vec{a_3} \end{bmatrix} = \begin{bmatrix} B\vec{a_1} & B\vec{a_2} & B\vec{a_3} \end{bmatrix}$$

$$B\vec{a_1} = \begin{bmatrix} 1 & 0 \\ -1 & 2 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 4 \end{bmatrix} = 1\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + 4\begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 7 \\ 12 \end{bmatrix}$$

$$B\vec{a_2} = \begin{bmatrix} 1 & 0 \\ -1 & 2 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 2 \\ 5 \end{bmatrix} = \cdots = \begin{bmatrix} 2 \\ 8 \\ 15 \end{bmatrix}$$

$$B\vec{a_3} = \begin{bmatrix} 1 & 0 \\ -1 & 2 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 3 \\ 6 \end{bmatrix} = \cdots = \begin{bmatrix} 3 \\ 9 \\ 18 \end{bmatrix}$$

$$BA = \begin{bmatrix} 1 & 2 & 3 \\ 7 & 8 & 9 \\ 12 & 15 & 18 \end{bmatrix} \qquad (Note: BA \neq AB)$$

(7) Powers of a Square Matrix

A - $n \times n$ **(square)**

$A^0 = I_n$

A^1**=A**

$A^2 = A \cdot A$

$A^3 = A \cdot A \cdot A = (A^2) \quad A$

$\vdots$

$A^k = A \cdots A (K - times)$

<u>EX</u> A= $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$

$$A^2 = AA = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}$$

$$A^3 = A^2 A = \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 6 \\ 0 & 1 \end{bmatrix}$$

Properties of Matrix Multiplication

(a) AB may <u>not</u> be BA

(b)

$$\underset{m\times n}{\mathbf{A}}\ \underset{n\times p}{\mathbf{B}} = \underset{m\times p}{\mathbf{C}}$$

(c)

$$AI = \cdots = A$$
$$IA = \cdots = A$$

(d)AB=AC does not guarantee B=C.

(e)AB=0 does not guarantee A=0 or B=0

Transpose

$$\vec{X} = \begin{bmatrix} X_1 \\ \vdots \\ X_n \end{bmatrix} \qquad \vec{X}^T = \begin{bmatrix} X_1 & \cdots & X_n \end{bmatrix}$$

$$\vec{Y} = \begin{bmatrix} Y_1 & \cdots & Y_n \end{bmatrix} \qquad \vec{Y}^T = \begin{bmatrix} Y_1 \\ \vdots \\ Y_n \end{bmatrix}$$

$$A = \begin{bmatrix} \vec{a_1} & \cdots & \vec{a_n} \end{bmatrix} \qquad A^T = \begin{bmatrix} \vec{a_1}^T \\ \vec{a_2}^T \\ \vdots \\ \vec{a_n}^T \end{bmatrix}$$

EX $A = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix} \qquad A^T = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$

$3 \times 2 \qquad\qquad 2 \times 3$

EX $\vec{U} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \qquad \vec{V} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$

$$
\begin{aligned}
\vec{U}^T\vec{V} &= \begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \end{bmatrix} \\
&= 3[1] + 4[2] \\
&= [3] + [8] \\
&= [11] \\
&= 11
\end{aligned}
$$

Basic Transpose Properties

(a) $(A^T)^T = A$

(b) $(A + B)^T = A^T + B^T$

(c) $(AB)^T \neq A^T B^T$

EX $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad B = \begin{bmatrix} -1 & 2 \\ 1 & 1 \end{bmatrix}$

$$(AB)^T = \left(\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 1 & 1 \end{bmatrix} \right)^T = \begin{bmatrix} 1 & 4 \\ 1 & 10 \end{bmatrix}^T = \begin{bmatrix} 1 & 1 \\ 4 & 10 \end{bmatrix}$$

$$A^T B^T = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 4 \\ 6 & 6 \end{bmatrix}$$

$$B^T A^T = \begin{bmatrix} -1 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 4 & 10 \end{bmatrix}$$

$(AB)^T = B^T A^T$