1.9 Linear + Matrix Transformations

A transformation T : $\Re^n \to \Re^m$ is called linear if

(a) $T(\vec{U}+\vec{V}) = T(\vec{U}) + T(\vec{V})$

(b) $T(C\vec{U}) = CT(\vec{U})$

for all $\vec{U}, \vec{V}$ **in the domain of T and all scalars C.**

Note

(1)

$$\begin{aligned} T(\vec{0}) &= \vec{0} \\ T(\vec{0}) &= T(0.\vec{0}) = 0T(\vec{0}) = \vec{0} \end{aligned}$$

$$\uparrow$$

$$Scalars$$

(2)

$$\begin{aligned} T(C\vec{U}+D\vec{V}) &= T(C\vec{U}) + T(D\vec{V}) \\ &= CT(\vec{U}) + DT(\vec{V}) \end{aligned}$$

Representing Linear Transformations as $T(\vec{X}) = A\vec{X}$

EX **T :** $\Re^2 \rightarrow \Re^3$ **by**

$$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$$

(a) $T\left(\begin{bmatrix} 10 \\ -3 \end{bmatrix}\right) =$ **?**

$$
\begin{aligned}
T\left(\begin{bmatrix} 10 \\ -3 \end{bmatrix}\right) &= T\left(\underline{10}\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) + T\left(\underline{-3}\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) \\
&= 10T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) + (-3)T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) \\
&= 10\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + (-3)\begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \\
&= \begin{bmatrix} -2 \\ 5 \\ 12 \end{bmatrix}
\end{aligned}
$$

(b) $T\left(\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}\right) =$ **?**

$$
\begin{aligned}
T\left(\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}\right) &= T\left(X_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) + T\left(X_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) \\
&= X_1 T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) + X_2 T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) \\
&= X_1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + X_2 \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}
\end{aligned}
$$

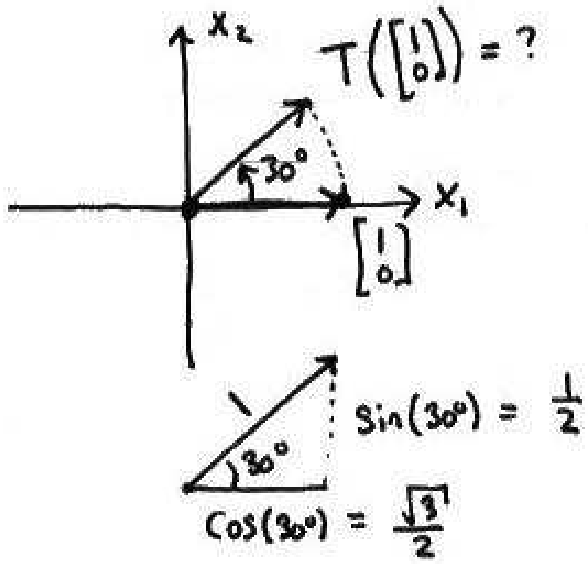
$$\nearrow \qquad \nwarrow$$

$$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) \quad T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$$

<u>EX</u> **T:**$\Re^2 \rightarrow \Re^2$ **that takes each** $\vec{X}$ **in** $\Re^2$ **and rotates it** 30^o **counter-clockwise around (0,0). Find a matrix A so that** $T(\vec{X}) = A\vec{X}$

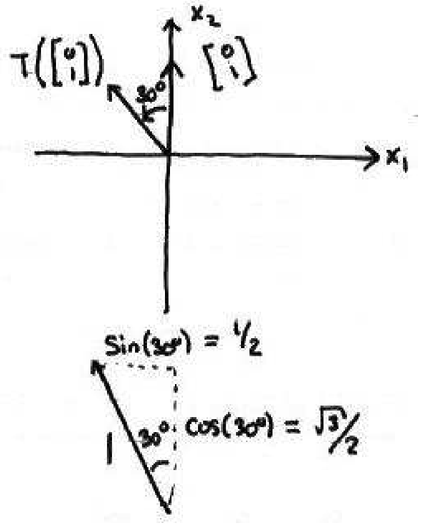
$$A = \left[T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) \; T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) \right]$$ **see last example**

$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$**:**



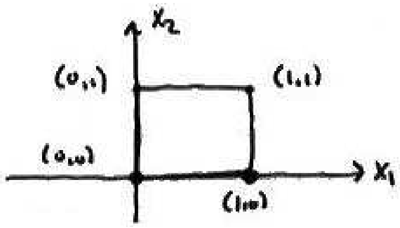
$$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} \frac{\sqrt{3}}{2} \\ \frac{1}{2} \end{bmatrix}$$

$T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$**:**



$$T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} \frac{-1}{2} \\ \frac{\sqrt{3}}{2} \end{bmatrix}$$

so $$T\left(\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}\right) = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{-1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$$



Rotate each corner 30^o

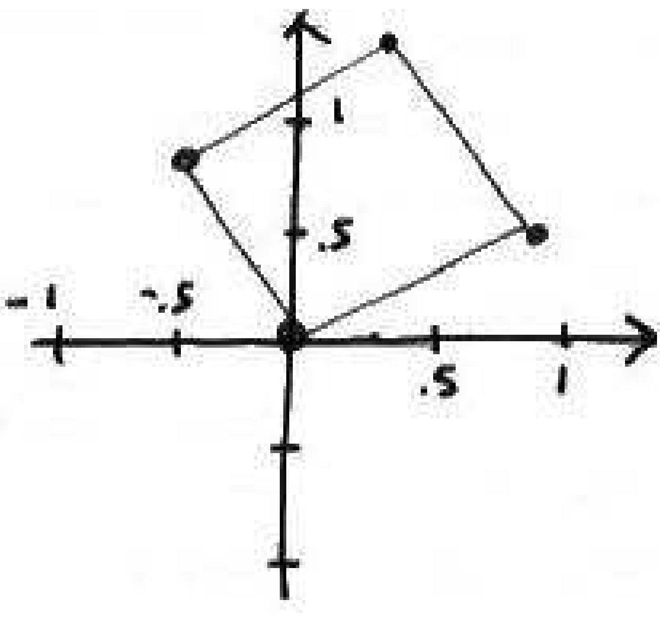
$$
\begin{aligned}
T\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right) &= A\begin{bmatrix} 0 \\ 0 \end{bmatrix} &=& \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\
T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) &= A\begin{bmatrix} 1 \\ 0 \end{bmatrix} &=& \begin{bmatrix} \frac{\sqrt{3}}{2} \\ \frac{-1}{2} \end{bmatrix} \\
T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) &= A\begin{bmatrix} 1 \\ 1 \end{bmatrix} &=& \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{-1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} \\
T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) &= A\begin{bmatrix} 0 \\ 1 \end{bmatrix} &=& \begin{bmatrix} \frac{-1}{2} \\ \frac{\sqrt{3}}{2} \end{bmatrix}
\end{aligned}
$$

$$T\left(\begin{bmatrix}0\\0\end{bmatrix}\right) = \left(\begin{bmatrix}0\\0\end{bmatrix}\right)$$

$$T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \left(\begin{bmatrix}.86\\.50\end{bmatrix}\right)$$

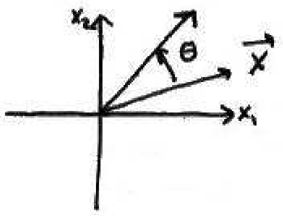
$$T\left(\begin{bmatrix}1\\1\end{bmatrix}\right) = \left(\begin{bmatrix}.37\\1.36\end{bmatrix}\right)$$

$$T\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = \left(\begin{bmatrix}-.50\\.86\end{bmatrix}\right)$$



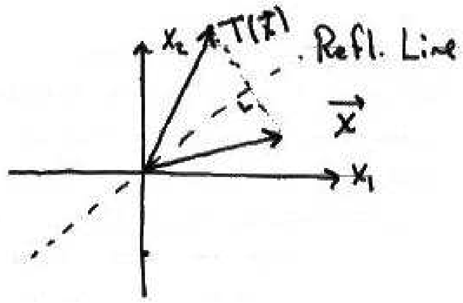
Geometric Transformations $T : \Re^2 \rightarrow \Re^2$

(1) Rotation counterclockwise θ



$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

(2) Reflection across a line through origin



EX Reflect across X_1-axis

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

(3) Vertical + Horizontal Scalings

(4) Projection onto a line

(5) Shear