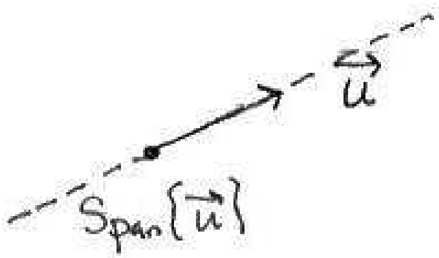
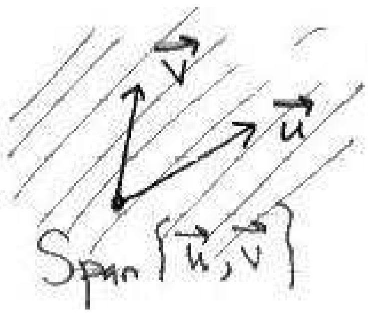
1.7 Linear Independence

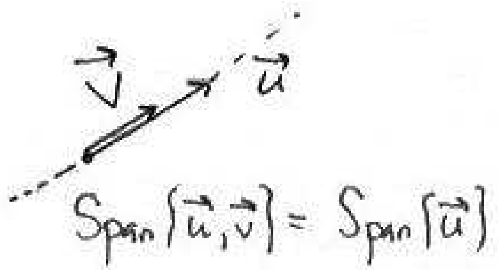
Preliminary Idea

Start with one nonzero vector in $\Re^2$



Add another vector:





$$\vec{U} \neq C\vec{V} \qquad \begin{aligned} \vec{U} &= C\vec{V} \\ 1 \times \vec{U} - C\vec{V} &= \vec{0} \\ \underset{A}{\begin{bmatrix} \vec{U} & \vec{V} \end{bmatrix}} \underset{\vec{X}}{\begin{bmatrix} 1 \\ -C \end{bmatrix}} &= \vec{0} \end{aligned}$$

$A\vec{X} = \vec{0}$ **has a nontrivial solution** $\vec{X}$**.**

A careful study of $A\vec{X} = \vec{0}$

$$A\vec{X} = \vec{0}$$

$$\begin{bmatrix} \vec{a_1} & \cdots & \vec{a_n} \end{bmatrix} \begin{bmatrix} X_1 \\ \vdots \\ X_n \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

$X_1\vec{a_1} + ... + X_n\vec{a_n} = \vec{0}$

One set of weights for this equation is :

$X_1 = 0, ..., X_n = 0$

Main issue: are there nontrivial weights ?

If so, one vector $\vec{a_1}, ...\vec{a_n}$ is a linear combination of the others.

Definition **A set of vectors** $\{\vec{V_1}, \vec{V_2}, ..., \vec{V_p}\}$ **in** $\Re^n$ **is called linearly independent if:**

$$X_1\vec{V_1} + X_2\vec{V_2} + ...X_p\vec{V_p} = \vec{0}$$

has only the trivial solutions $X_1 = 0, X_2 = 0, ..., X_p = 0$**.**

If there are nontrivial solutions, the vectors are called linearly dependent.

<u>EX</u> $\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \end{bmatrix} \right\}$

First, write the "important" starting equation :

$$X_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + X_2 \begin{bmatrix} 3 \\ 6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

solve to determine if $X_1 = 0, X_2 = 0$ is the only solution or not.

(a) $\begin{bmatrix} 1 & 3 & 0 \\ 2 & 6 & 0 \end{bmatrix} \backsim \begin{bmatrix} 1 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix} \; -2R_1 + R_2$

$$\begin{array}{ccccc} X_1 & + & 3X_2 & = & 0 \\ & & X_2 & & free \end{array}$$

Since X_2 is a free variable, it does not have to be 0.
So the vector equation will the solutions other than $(0, 0)$.
The vectors are <u>dependent</u>.

(b)OR, $X_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + X_2 \begin{bmatrix} 3 \\ 6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

By inspection

$$-3\begin{bmatrix}1\\2\end{bmatrix}+1\begin{bmatrix}3\\6\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$$

So the vectors are dependent as before.

(c)OR,

$$\begin{bmatrix}1\\2\end{bmatrix},\begin{bmatrix}3\\6\end{bmatrix}$$

$$\begin{bmatrix}3\\6\end{bmatrix}=3\begin{bmatrix}1\\2\end{bmatrix}$$

$$-3\begin{bmatrix}1\\2\end{bmatrix}+\begin{bmatrix}3\\6\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$$

So $\begin{bmatrix}1\\2\end{bmatrix},\begin{bmatrix}3\\6\end{bmatrix}$ **are dependent as before**

<u>**EX**</u> $\left\{\begin{bmatrix}1\\3\\-2\end{bmatrix},\begin{bmatrix}-3\\-5\\6\end{bmatrix},\begin{bmatrix}0\\5\\6\end{bmatrix}\right\}$

$$X_1\begin{bmatrix}1\\3\\-2\end{bmatrix}+X_2\begin{bmatrix}-3\\-5\\6\end{bmatrix}+X_3\begin{bmatrix}0\\5\\6\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\begin{bmatrix} 1 & -3 & 0 & 0 \\ 3 & -5 & 5 & 0 \\ -2 & 6 & 6 & 0 \end{bmatrix} \backsim \ldots \backsim \begin{bmatrix} (1) & -3 & 0 & 0 \\ 0 & (4) & 5 & 0 \\ 0 & 0 & (6) & 0 \end{bmatrix}$$ **Ech. Form**

No free variables,$X_1 = 0, X_2 = 0, X_3 = 0$ **is the only solution of the important vector eq.**

The vector are independent.