1.3 Vectors + Vector Equations

Vectors $\Re^n$

- **A vector is a matrix with one column**

$$\begin{bmatrix} 1 \\ 3 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \quad \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{bmatrix}$$

- **Vectors notations:** $\vec{a}$, $\vec{u}$, **...**

- $\Re$**: all real numbers**

$\Re^2$**: all real vectors** $\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$ **with 2 rows**

$\Re^3$**: all real vectors** $\begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix}$ **with 3 rows**

$\Re^n$**: all real vectors with n rows**

Vector Algebra

. $\vec{u} = \vec{v}$ $\vec{u}, \vec{v}$ **have the same number of rows**

$$\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}, \quad \vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}, \quad and \quad u_1 = v_1, u_2 = v_2, ..., u_n = v_n$$

. $\vec{u} + \vec{v}$ **:**

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \end{bmatrix}$$

. **c**$\vec{u}$ **:**

$$(-2) \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} = \begin{bmatrix} -2 \\ -6 \\ -10 \end{bmatrix}$$

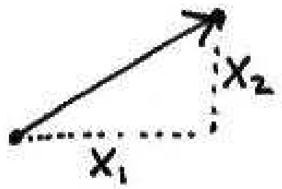
. $\vec{0}$ **(zero vector):**

$$\begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

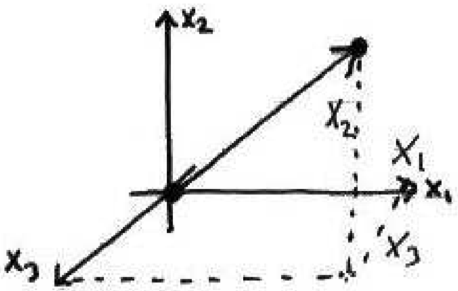
Geometric Interpretation

Vectors are represented by arrows

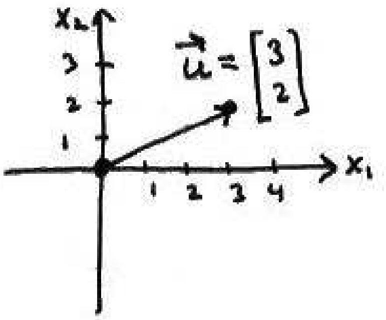
$$\vec{X} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} :$$

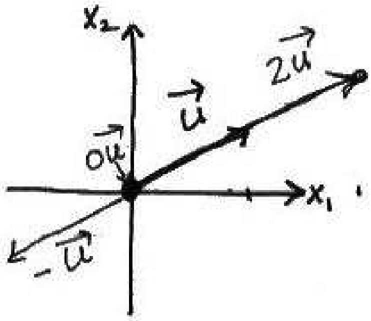


$$\vec{X} = \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} :$$



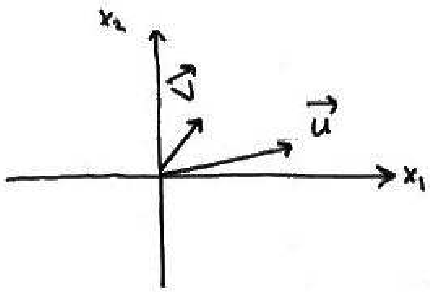
<u>Ex:</u> **In** R^2



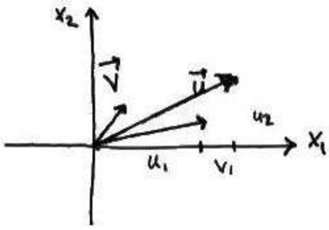


Scalar multiple of $\vec{u}$ lie along the same line as $\vec{u}$

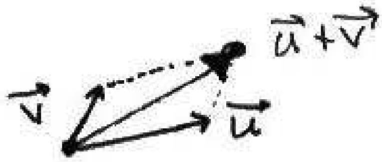
Adding two vectors $\vec{u}, \vec{v}$



$$\vec{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \qquad \vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \qquad \vec{u} + \vec{v} = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \end{bmatrix}$$



Parallel diagram Law:



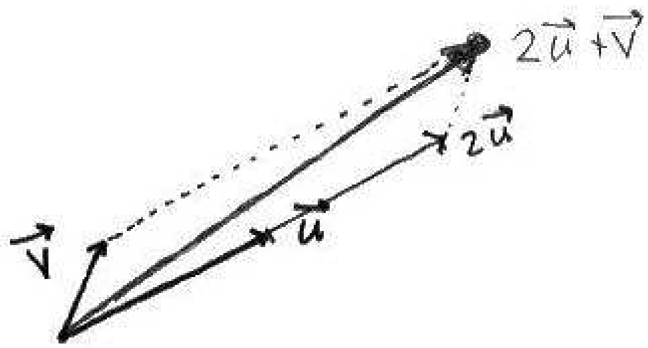
Create a parallel diagram using $\vec{u}, \vec{v}$ as sides.

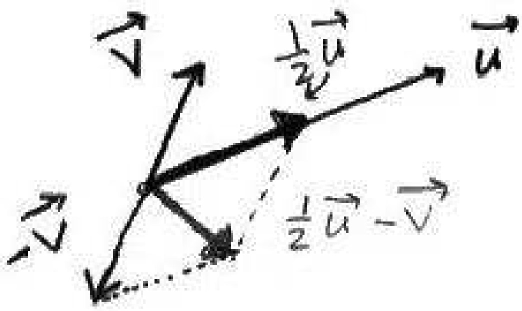
Tip-To-Tail:

Combinations of Vectors using Addition + Substraction

$2\vec{u} + \vec{v}$:

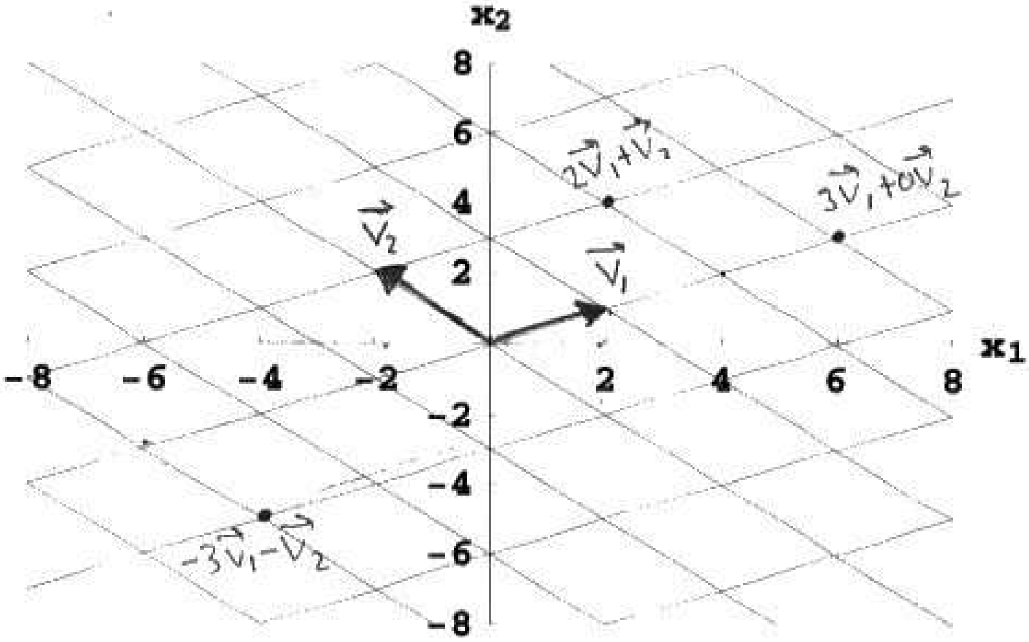
$\frac{1}{2}\vec{u} - \vec{v}$:





EXAMPLE: Let $\mathbf{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$. Express each of the following as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$:

$$\mathbf{a} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} -4 \\ 1 \end{bmatrix}, \quad \mathbf{c} = \begin{bmatrix} 6 \\ 6 \end{bmatrix}, \quad \mathbf{d} = \begin{bmatrix} 7 \\ -4 \end{bmatrix}$$



<u>Linear Combinations</u> (Creating new vectors from old vectors)

A <u>linear combination</u> of vectors $\vec{v_1}, \vec{v_2}, ...\vec{v_p}$ with scalar weights $c_1, c_2, ..., c_p$ is the vector:

$$c_1\vec{v_1} + c_2\vec{v_2} + ... + c_p\vec{v_p}$$

<u>Ex:</u>

$$\vec{V_1} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad \vec{V_2} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

- **(a) Find 3 different linear combinations of $\vec{V_1}$ and $\vec{V_2}$.**

$$C_1\vec{V_1} + C_2\vec{V_2}$$

$$0 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 0 \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 1 \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

$$2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 4 \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 8 \end{bmatrix}$$

- **(b) Is** $\begin{bmatrix} 4 \\ 5 \end{bmatrix}$ **a linear combination of** $\vec{V}_1, \vec{V}_2$?

$$C_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + C_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$$

$$\begin{bmatrix} C_1 \\ 2C_1 \end{bmatrix} + \begin{bmatrix} -C_2 \\ C_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$$

$$\begin{bmatrix} C_1 - C_2 \\ 2C_1 + C_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$$

$$\begin{array}{rcrcr} C_1 & - & C_2 & = & 4 \\ 2C_1 & + & C_2 & = & 5 \end{array}$$

$$-2 \begin{bmatrix} 1 & -1 & 4 \\ 2 & 1 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 4 \\ 0 & 3 & -3 \end{bmatrix} \; -2R_1 + R_2$$

$$\underline{(3C_2 = -3)}$$

This echelon form matrix represents a consistent system of equations. So C_1, C_2 **exist so that** $C_1\vec{V}_1 + C_2\vec{V}_2 = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$

So <u>yes</u>, $\begin{bmatrix} 4 \\ 5 \end{bmatrix}$ **is a combination of** $\vec{V}_1, \vec{V}_2$.

<u>Vector Equations + Linear Systems</u>

<u>Ex:</u>

$$\begin{array}{rcrcl} X_1 & + & 3X_2 & = & 7 \\ 2X_1 & - & 5X_2 & = & 0 \end{array} \qquad \textbf{Linear System}$$

$$\begin{bmatrix} X_1 + 3X_2 \\ 2X_1 - 5X_2 \end{bmatrix} = \begin{bmatrix} 7 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} X_1 \\ 2X_1 \end{bmatrix} + \begin{bmatrix} 3X_2 \\ -5X_2 \end{bmatrix} = \begin{bmatrix} 7 \\ 0 \end{bmatrix}$$

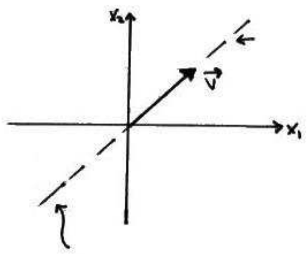
$$X_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + X_2 \begin{bmatrix} 3 \\ -5 \end{bmatrix} = \begin{bmatrix} 7 \\ 0 \end{bmatrix} \qquad \textbf{Vector Equation}$$

<u>Linear systems can be converted to a vector equation, and vise-versa.</u>

Generating Sets of Vectors (Span)

Ex: $\vec{V} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. **Using $\vec{V}$, we can generate other vectors by multiplying $\vec{V}$ by scalars:**

All multiples of $\vec{V}$ lie along this lines

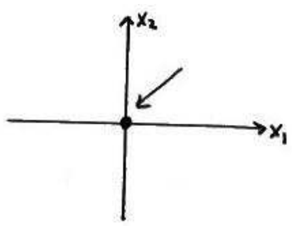


the multiples of $\vec{V}$ generate this line. called the span of $\vec{V}$, or span $\{\vec{V}\}$ for short.

Ex: $\vec{V} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. **But if $\vec{V}$ is the zero vector, the multiples of $\vec{V}$ are still $\vec{0}$.**

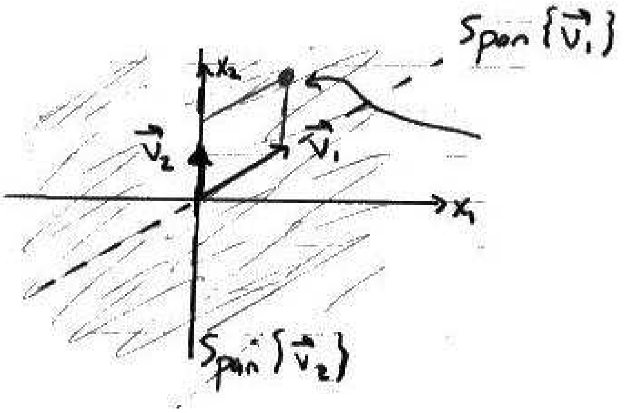
The multiples of $\vec{V}$ generate only this one point,

The span $\{\vec{V}\}$ is the origin.



Ex: $\vec{V}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \vec{V}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

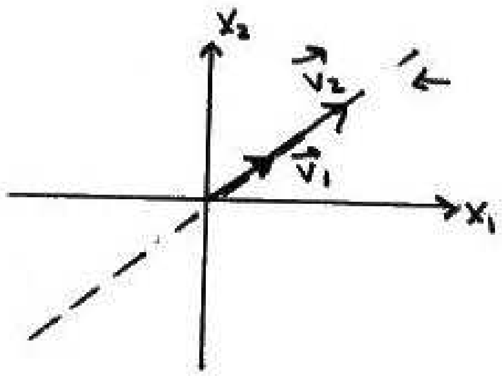
Every vector in $\Re^2$ is the sum of a vector $C_1\vec{V}_1$ and a vector $C_2\vec{V}_2$



The combinations $C_1\vec{V}_1 + C_2\vec{V}_2$ now generate all points in the plane. Here, Span $\{\vec{V}_1, \vec{V}_2\} = \Re^2$

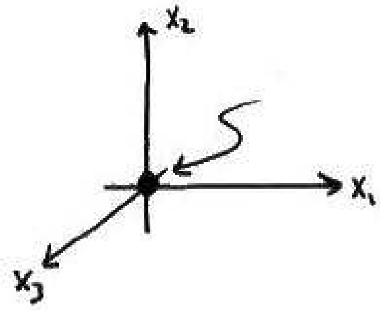
Ex: $\vec{V}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \vec{V}_2 = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$. **But here, $\vec{V}_1$ and $\vec{V}_2$ lie along the same line. Combinations of $\vec{V}_1$ and $\vec{V}_2$ just generale move vectors along the same line.**

Span$\{\vec{V}_1, \vec{V}_2\}$ is a line through origin.

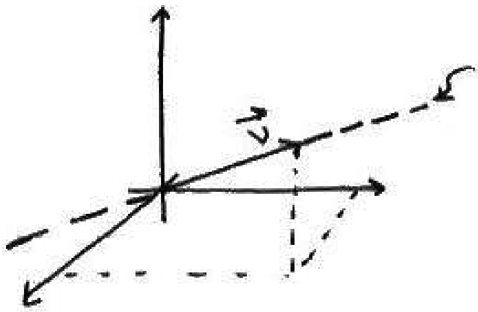


<u>Ex:</u> In $\Re^2$, we saw that the span of vectors may generate a point, a line through the origin, or all of the plane $\Re^2$.

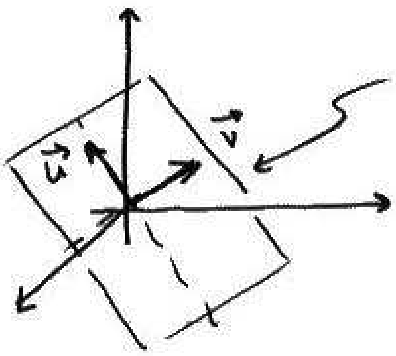
In $\Re^3$, the multiples $\vec{0}$, Span $\{\vec{0}\}$ is the origin (a point):



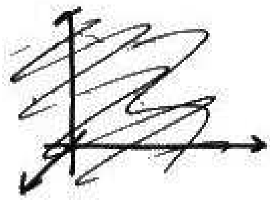
If $\vec{V}$ is not $\vec{0}$, the multiples $\vec{V}$, span$\{\vec{V}\}$, form a line through the origin:



If $\vec{U}, \vec{V}$ are nonzero vectors that don't lie along the same line, the combinations $\vec{U}, \vec{V}$, Span$\{\vec{U}, \vec{V}\}$ form a plane through the origin:



If $\vec{U}, \vec{V}, \vec{W}$, are 3 vectors in different directions, span$\{\vec{U}, \vec{V}, \vec{W}\}$ is all of $\Re^3$:



<u>Ex:</u>

$$\textbf{Is} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \textbf{ in Span} \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} \right\} \textbf{?}$$

$$\textbf{Is} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \textbf{ a combination of} \left(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} \right) \textbf{?}$$

$$C_1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + C_2 \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} \stackrel{?}{=} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

(Check to see if weights C_1, C_2 exist by reducing an appropriate augmented matrix to echelon form.)