## 1.2 Row Reduction + Echelon Form

**The Three Basic Row Operations**

1. Interchange two rows
2. Multiply a row by a nonzero number
3. Add a multiple of one row to another row

**Goal:** Use the row operations to simplify a system of equations in order to easily see the solution.

**Ex**

$$\begin{array}{rcrcrcr} X_1 & - & 2X_2 & + & X_3 & = & 0 \\ & & 2X_2 & - & 8X_3 & = & 8 \\ -4X_1 & + & 5X_2 & + & 9X_3 & = & -9 \end{array}$$

**Leading Entries:** $1^{st}$ **nonzero number in a row**

$$\begin{bmatrix} (1) & -2 & 1 & | & 0 \\ 0 & (2) & -8 & | & 8 \\ (-4) & 5 & 9 & | & -9 \end{bmatrix}$$ **Augmented matrix for original system**

$$\begin{bmatrix} (1) & -2 & 1 & | & 0 \\ 0 & (1) & -9 & | & 4 \\ 0 & 0 & (2) & | & 6 \end{bmatrix}$$ **Represents a "simpler" system**

$$\begin{bmatrix} (1) & 0 & 0 & | & 29 \\ 0 & (1) & 0 & | & 16 \\ 0 & 0 & (1) & | & 3 \end{bmatrix}$$ **Represents the "simplest" system**

# Definition of Echelon Form of a Matrix

**A matrix is in echelon form if**

**1. All nonzero rows are above any row of all zeros.**

$$\begin{bmatrix} 0 & 0 \\ 1 & 2 \end{bmatrix} \quad \textbf{no} \qquad \begin{bmatrix} 3 & 0 \\ 0 & 0 \end{bmatrix} \quad \textbf{yes}$$

**2. Each leading entry of a row is in a column to the right of the leading entry of the row above it.**

$$\begin{bmatrix} 0 & (1) \\ (2) & 3 \end{bmatrix} \quad \textbf{no} \qquad \begin{bmatrix} (2) & 3 \\ 0 & (1) \end{bmatrix} \leftarrow \ \textbf{yes}$$

**3. All entries in a column below a leading entry are zero.**

$$\begin{matrix} & \textbf{No} & & \textbf{Yes} & \\ -3 & \begin{bmatrix} (1) & 2 \\ (3) & 4 \end{bmatrix} & \sim & \begin{bmatrix} (1) & 2 \\ 0 & (-2) \end{bmatrix} & -3R_1 + R_2 \end{matrix}$$

**A matrix is in reduced echelon form if it is in echelon form and**

**4. Each leading entry is 1 .**

**No** **Yes**

$$\begin{bmatrix} (2) & 1 & 1 \\ 0 & (3) & 9 \end{bmatrix} \sim \begin{bmatrix} (1) & 1/2 & 1/2 \\ 0 & (1) & 3 \end{bmatrix} \begin{matrix} 1/2R_1 \\ 1/3R_3 \end{matrix}$$

**5. All leading entries in a column above a leading entry are zero.**

$$\begin{matrix} \\ -2 \end{matrix} \begin{bmatrix} (1) & 2 & 1 \\ 0 & (1) & 4 \end{bmatrix} \sim \begin{bmatrix} (1) & 0 & -7 \\ 0 & (1) & 4 \end{bmatrix} \begin{matrix} -2R_2 + R_1 \\ \\ \end{matrix}$$

**Ex:**

$$-1\quad \begin{bmatrix} (2) & -2 & -2 & 12 \\ 0 & (1) & -4 & 8 \\ (2) & -1 & -6 & 20 \end{bmatrix}$$

$$\begin{bmatrix} (2) & -2 & -2 & 12 \\ 0 & (1) & -4 & 8 \\ 0 & (1) & -4 & 8 \end{bmatrix} \quad -R_1 + R_3$$

$$\begin{bmatrix} (2) & -2 & -2 & 12 \\ 0 & (1) & -4 & 8 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad -R_2 + R_3$$

**Echelon Form**

$$\begin{bmatrix} 2 & 0 & -10 & 28 \\ 0 & 1 & -4 & 8 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad 2R_2 + R_1$$

$$\begin{bmatrix} 1 & 0 & -5 & 14 \\ 0 & 1 & -4 & 8 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad 1/2\ R_1$$

**Reduced Echelon Form**

**If this was a system of equations the solution would be:**

$$
\begin{array}{rcrcl}
X_1 \qquad - 5X_3 &=& 14 \qquad X_1 &=& 14 + 5X_3 \\
X_2 - 4X_3 &=& 8 \qquad X_2 &=& 8 + 4X_3 \\
0X_3 &=& 0 \qquad X_3 &=& any \quad (free)
\end{array}
$$