1.1 Systems of Linear Equations

Linear equation in n variables $X_1, X_2, X_3, ..., X_n$

$$a_1X_1 + a_2X_2 + ... + a_nX_n = b$$

System of m linear equations

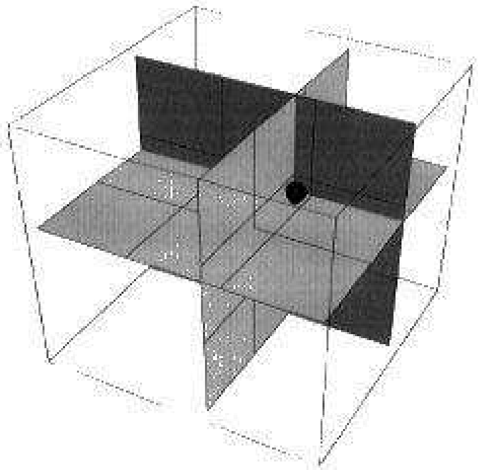
$$\begin{gathered}
a_{11}X_1 + a_{12}X_2 + ... + a_{1n}X_n = b_1 \\
a_{21}X_1 + a_{22}X_2 + ... + a_{2n}X_n = b_2 \\
. \\
. \\
. \\
a_{m1}X_1 + a_{m2}X_2 + ... + a_{mn}X_n = b_m
\end{gathered}$$

A solution is a set of numbers $(S_1, S_2, ..., S_n)$ **for which** $X_1 = S_1, X_2 = S_2, ..., X_n = S_n$ **satisfies all m equations.**

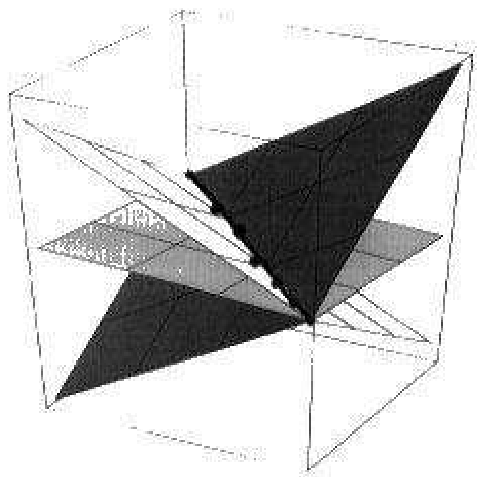
$$a_1X_1 + a_2X_2 + a_3X_3 = b$$

EXAMPLE: Three equations in three variables. Each equation determines a plane in 3-space.

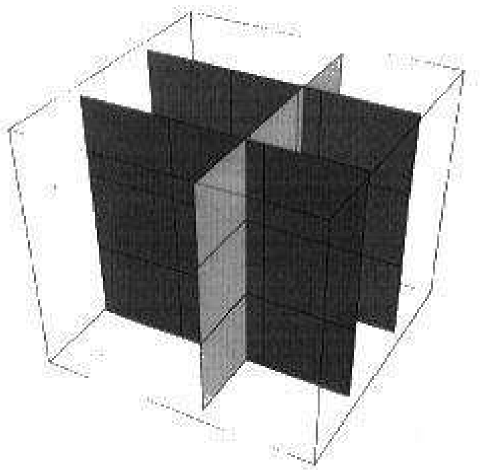
i) The planes intersect in one point. *(one solution)*



ii) The planes intersect in one line. *(infinitely many solutions)*



iii) There is not point in common to all three planes. *(no solution)*



Matrix Representations of Systems

An $m \times n$ matrix is array or table of m rows, n columns of numbers.

EX

$$\begin{aligned} X_1 - 2X_2 + X_3 &= 0 \\ 2X_2 - 8X_3 &= 8 \\ -4X_1 + 5X_2 + 9X_3 &= -9 \end{aligned}$$

Coefficient Matrix:

$$\begin{matrix} X_1 & X_2 & X_3 \end{matrix}$$

$$\begin{bmatrix} 1 & -2 & 1 \\ 0 & 2 & -8 \\ -4 & 5 & 9 \end{bmatrix} \begin{matrix} EQ1 \\ EQ2 \\ EQ3 \end{matrix}$$

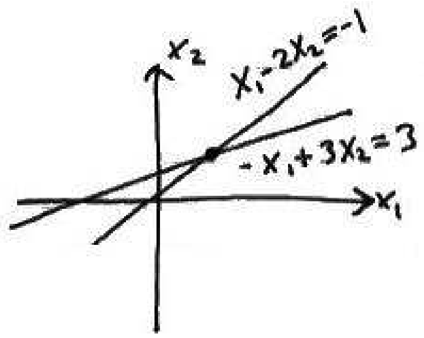
Augmented Matrix

$$\begin{bmatrix} 1 & -2 & 1 & | & 0 \\ 0 & 2 & -8 & | & 8 \\ -4 & 5 & 9 & | & -9 \end{bmatrix}$$

Solving Systems by Elimination

EX:

$$\begin{aligned} X_1 - 2X_2 &= -1 \\ -X_1 + 3X_2 &= 3 \end{aligned}$$

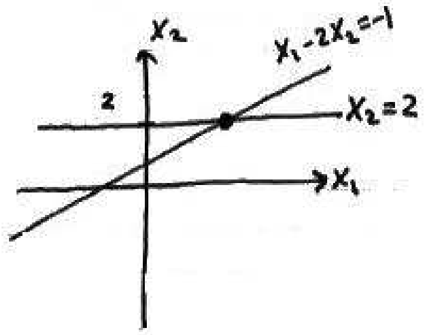


Adding $EQ1$ **to** $EQ2$**:**

$$\begin{array}{rrcr} X_1 & -2X_2 & = & -1 \\ -X_1 & +3X_2 & = & 3 \\ -- & --- & - & -- \\ 0 & +X_2 & = & 2 \end{array}$$

New System:

$$\begin{aligned} X_1 - 2X_2 &= -1 \\ X_2 &= 2 \end{aligned}$$

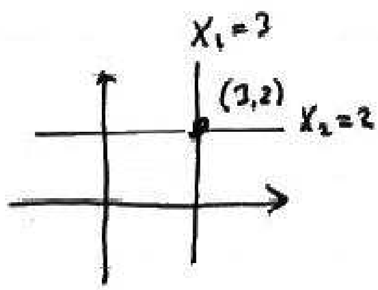


<u>$2 * EQ2$ **add:**</u>

$$\begin{aligned} X_1 - 2X_2 &= -1 \\ 2X_2 &= 4 \\ ------ &- -- \\ X_1 + 0 &= 3 \end{aligned}$$

New System:

$$\begin{aligned} X_1 &= 3 \\ X_2 &= 2 \end{aligned}$$



<u>**One solution** $(3, 2)$</u>

EX:

$$\begin{aligned} X_1 - 2X_2 + X_3 &= 0 \\ 2X_2 - 8X_3 &= 8 \\ -4X_1 + 5X_2 + 9X_3 &= -9 \end{aligned}$$

$$\begin{array}{rrrrcr} 4 * EQ1 & 4X_1 & -8X_2 & +4X_3 & = & 0 \\ EQ3 & -4X_1 & +5X_2 & +9X_3 & = & -9 \\ & --- & --- & --- & - & -- \\ Add & & -3X_2 & +13X_3 & = & -9 \end{array}$$

New System:

$$\begin{aligned} X_1 - 2X_2 + X_3 &= 0 \\ 2X_2 - 8X_3 &= 8 \\ -3X_2 + 13X_3 &= -9 \end{aligned}$$

$$\begin{array}{rrrcr} 3 * EQ1 & 6X_2 & -24X_3 & = & 24 \\ 2 * EQ3 & -6X_2 & +26X_3 & = & -18 \\ & --- & --- & - & -- \\ Add & & 2X_3 & = & 6 \end{array}$$

New System:

$$
\begin{aligned}
X_1 - 2X_2 + X_3 &= 0 \\
2X_2 - 8X_3 &= 8 \\
2X_3 &= 6
\end{aligned}
$$

$$\begin{array}{rrrcr} EQ2 & 2X_2 & -8X_3 & = & 8 \\ 4*EQ3 & & 8X_3 & = & 24 \\ & --- & --- & - & - \\ & 2X_2 & & = & 32 \end{array}$$

New System:

$$\begin{aligned} X_1 - 2X_2 + X_3 &= 0 \\ 2X_2 \qquad &= 32 \\ 2X_3 &= 6 \end{aligned}$$

...Continue, eliminate X_3 term in EQ1, then eliminate X_2 term in EQ1

Final System:

$$\begin{aligned} X_1 \qquad &= 29 \\ X_2 \quad &= 16 \\ X_3 &= 3 \end{aligned}$$

<u>**One solution (29,16,3)**</u>

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$$2 \quad \begin{bmatrix} 1 & h & -3 \\ -2 & 4 & 6 \end{bmatrix}$$

$$\begin{bmatrix} 1 & h & -3 \\ 0 & (2h+4) & 0 \end{bmatrix} \quad 2R_1 + R_2$$

$h = -2$

$$\begin{bmatrix} 1 & -2 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{aligned} X_1 - 2X_2 &= -2 \quad \Rightarrow X_1 = -2 + 2X_2 \\ 0 &= 0 \quad \leftarrow X_2 \ is\ arbitrary \end{aligned}$$

$(-2 + 2X_2, X_2)$ $\underline{Many\ solutions\ if\ h = -2}$

$h \neq -2$

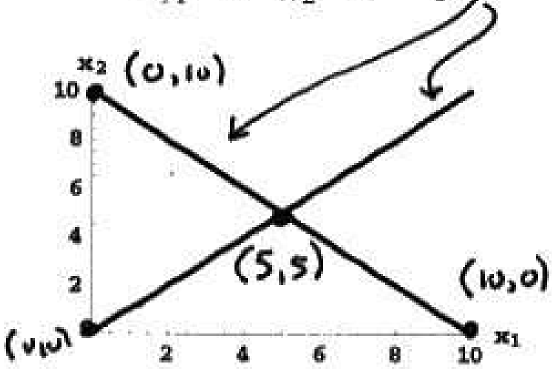
$$\begin{bmatrix} 1 & h & -2 \\ 0 & (2h+4) & 0 \end{bmatrix}$$

$$\begin{aligned} X_1 + hX_2 &= -2 \quad \rightarrow \quad X_1 = -2 \\ (2h+4)X_2 &= 0 \quad \Rightarrow \quad X_2 = 0 \end{aligned}$$

$(-2, 0):$ **One unique solution** **if** $h \neq -2$

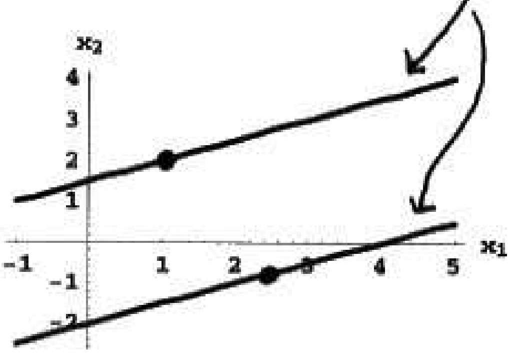
EXAMPLE Two equations in two variables:

$$\begin{aligned} x_1 + x_2 &= 10 \\ -x_1 + x_2 &= 0 \end{aligned}$$



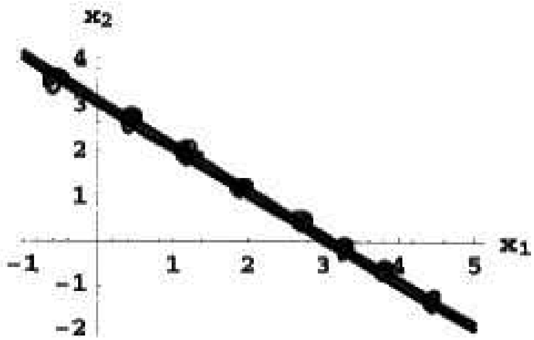
one unique solution

$$\begin{aligned} x_1 - 2x_2 &= -3 \\ 2x_1 - 4x_2 &= 8 \end{aligned}$$



no solution

$$\begin{aligned} x_1 + x_2 &= 3 \\ -2x_1 - 2x_2 &= -6 \end{aligned}$$



infinitely many solutions

BASIC FACT: A system of linear equations has either

(i) exactly one solution *(consistent)* or

(ii) infinitely many solutions *(consistent)* or

(iii) no solution *(inconsistent)*.

Figure 1: