A Mathematical Observation On Synthetic Aperture Radar

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ABSTRACT

A general synthetic aperture radar (SAR) signal model is derived based on the Maxwells equation, and three numerical simulations are analyzed and discussed. With this signal model, compressive sensing is applied to get a better image.

Keywords: compressed sensing, SAR, Maxwell's equations

1. INTRODUCTION

Synthetic aperture radar (SAR) has been developed in many different operations since the 1950s, The classic operations include strip-map operation, spotlight operations, ScanSAR operation, and so on. Under these operations the platform of SAR usually moves along a straight line while it illustrates the targets. These SAR systems are referred to as linear SAR. These linear SARs are used in many field, such as soil moisture, forestry, wetland, and agriculture by many researchers. Before we discuss the signal model and image reconstruction, the general signal model will derived from Maxwell's equations. As we know, the received signal of SAR is generated by the microwave around the receiving antenna, and all the fields of microwave satisfy Maxwell's equations. The image formation will reconstruct the scene from the received signals. So the image reconstruction is an inverse problem of electromagnetic wave propagation. The mathematical signal model is derived from the scattered field, and the approximations for the engineering application are presented including antenna pattern approximation, far-field approximation, and amplitude approximation for narrow band signal.

There are a number of different types of SAR imaging systems that are currently in use. Two of the most common SAR modes, which are used by modern systems, are stripmap-mode[1] and spotlight-mode[1]. However, it should be noted that these modes are not exclusive. That is, there are some SAR systems that can switch between imaging modes.

As was previously mentioned, the original SAR mode that was invented by Wiley was stripmap-mode SAR[1]. In the case of this SAR mode, the radar antenna is mounted at a fixed angle on a moving platform. stripmap-mode SAR systems are capable of producing high resolution images over a large region of the ground. This makes it useful for terrain mapping.

In the case of Spotlight-mode SAR[1], the radar beam is steered so that it remains focused on a single area of the target space. Spotlight-mode SAR systems can use either electronic or mechanical beam steering. The main advantage of this mode is that it has increased image resolution when compared to stripmap-mode SAR. However, this increase in resolution comes at the cost of decreased area coverage.

The best models retain as much physics as possible, though algorithms based on such models are often computationally intensive and difficult to apply to real-time radar environments or those involving large amounts of data. One such physics based on model on the geometrical theory of diffraction. This model gives an estimate of the geometry, location, and response to polarization for each scatterer. Another physics based model is based on Maxwell's equations for electromagnetics[2]. Though, this model does not use all of Maxwell's equations. An ideal model would use the full set of Maxwell's equations, but such a model only be applicable on systems that have antennas that take polarimetric measurements.

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Most model make a linearizing assumption known as the Born approximation, which is equivalent to assuming that the wave scatters only once before returning to the antenna. For example, when imaging underneath dense vegetation, most scattering is multiple scattering and the inability to account for this degrades the quality of the image formed. In addition, this approximation may introduce unwanted effects to the image such as shadowing and it ignores polarization changes as the waves scatter.

2. GENERAL SAR SIGNAL MODEL BASED ON MAXWELL'S EQUATION

2.1 Wave Propagation

The signals transmitted and measured by radar antenna are electromagnetic, therefore the appropriate model for radar is Maxwell's equation. For radar imaging, what is important is the wave nature of the transmitted and measured signals. Therefore, an appropriate model is given by[1]

$$(\nabla^2 - \frac{1}{c^2(x)}\partial_t^2)u(t,x) = -j(t,x) \tag{1}$$

Where $u(t,x)$ represents one component of the electromagnetic field due to some source $j(t,x)$ and $c(t,x)$ is local propagation speed of electromagnetic waves. Scattering causes singularities in the wave speed. In the absence of scatterers the speed of propagation is $c(t,x) = c_0$.

Scattering can be thought of as being due to perturbations in the wave speed, which we write as

$$\frac{1}{c^2(x)} = \frac{1}{c_0^2} - V(x) \tag{2}$$

Where $V(x)$ is the reflectivity function. Equation (1) and (2) do not provide an entirely accurate model for electromagnetic scattering from an object. Nevertheless, this is a commonly used model for radar scattering, with the understanding that $V(x)$ does not exactly correspond to the perturbation in the electromagnetic wave speed in the material.
In this model, the electric field can be divided into two component fields

$$u = u^{in} + u^{sc} \tag{3}$$

In the above identity, $u^{in}(t,x)$ is the incident field that is emitted by the antennas, and u^{sc} is the scattered field, which results from the interaction of the incident field with a target. Since the source of the incident field is a current on the antenna, u^{in} is modeled using the following non- homogeneous wave equation.

$$(\nabla^2 - \frac{1}{c_0^2(x)}\partial_t^2)u^{in}(t,x) = -j(t,x) \tag{4}$$

A wave equation that describes the propagation of the scattered field is derived from Eq(1) and Eq(4). This equation is given as follows

$$(\nabla^2 - \frac{1}{c_0^2(x)}\partial_t^2)u^{sc}(t,x) = -V(x)\partial_t^2(t,x) \tag{5}$$

Where $V(x)$ is the reflectivity function, which is given by

$$V(x) = \frac{1}{c_0^2} - \frac{1}{c^2(x)} \tag{6}$$

When the incident field comes into contact with a target, it induces a current, which causes the target to re-emit a weaker time shifted version of the same signal. However, $V(x)$ does not directly measure the intensity of the reflected signal. Instead, it indicates the level perturbation that occurs in the wave speed when the incident field comes in contact with the target plane. It will be assumed that the reflectivity function has compact support on the set of points on the target plane that have been illuminated by the antenna.

A fundamental solution of the wave equation is a generalized function satisfying

$$(\nabla^2 - \frac{1}{c_0^2(x)}\partial_t^2)g(t,x) = -\delta(t)\delta(x) \tag{7}$$

The solution of Eq(7) is

$$g(t,x) = \frac{\delta(t - \frac{|x|}{c_0})}{4\pi|x|} \tag{8}$$

The Green's function enables us to solve the constant-speed wave equation with any source term. The solution for the incident field is

$$u^{in}(t,x) = \int_{R^3} dy \int_R \frac{\delta(t - \tau - \frac{|x-y|}{c_0})}{4\pi|x-y|} j(\tau,y)d\tau \tag{9}$$

Since the antenna current density will be modeled such that $j(t,x) = p(t)(x - x_0)$. Subsisting this into Eq(9) yields the following expression

$$u^{in} = \frac{p(t - \frac{|x-x_0|}{c_0})}{4\pi|x-x_0|} \tag{10}$$

Furthermore, it is common to represent the current wave model in the frequency domain. Such a representation can be found by considering the following Helmholtz wave equation

$$(\nabla^2 + k^2)U^{in}(x,f) = -J(x,f) \tag{11}$$

In the above equation, U^{in} and J are the fourier transform of u^{in} and j, respectively, the equation can be solved by using the Helmholtz Green's function

$$G(x) = \frac{e^{-ik|x|}}{4\pi|x|} \tag{12}$$

Where $k = \frac{2\pi f}{c_0}$ denote the wavenumber. Then, the solution for the frequency domain representation of the incident field is given in terms of the following equation

$$U^{in}(x,f) = \int_{R^3} \frac{e^{-ik|x-y|}}{4\pi|x-y|} J(x,f)dy \tag{13}$$

more

$$U^{in}(x,f) = p(f)\frac{e^{-ik|x-x_0|}}{4\pi|x-x_0|} \tag{14}$$

2.2 The Lippmann- Schwinger Integral Equation

Since the scattered field is created as a result of the interaction of the incident field with the target scene, it would be useful if a formulation for the scattered field could be found directly in terms of the incident field. However, in general, this is not always possible. Consider that the scattered field can be described by the following equation

$$(\nabla^2 + k^2)U^{sc}(x,f) = -V(x)U(x,f) \tag{15}$$

The solution to this equation can be found, in the same way as before, in the frequency domain to be

$$U^{sc}(x,f) = -\int_{R^3} \frac{e^{-ik|x-z|}}{4\pi|x-z|} V(Z)f^2U(x,f)dz \tag{16}$$

and in the time domain to be

$$u^{sc}(t,x) = \int_{R^3} dz \int_R \frac{\delta(t - \tau - \frac{|x-z|}{c_0})}{4\pi|x-z|} V(x)\partial_t^2 u(\tau,z)d\tau \tag{17}$$

It is clear that in both of the above expressions that the scattered field is dependent on the total field. Since in Eq(16) and Eq(17) the scattered field appears on both sides of the equation, it is not possible to exactly formulate the scattered field in terms of the incident field alone. In the following sections, a solution to this problem will be detailed.

2.3 The Born Approximation

For radar imaging, we measure u^{sc} at the antenna, and we would like to determine V. However, both V and u^{sc} in the neighborhood of the target V are unknown, and in Eq(17) these unknowns are multiplied together. This nonlinearity makes it difficult to solve for V. Consequently, almost all work on radar imaging involves making the Born approximation, which is also unknown as the weak-scattering or single-scattering approximation. This corresponds to replacing $u(t, x)$ on the right side of Eq(17) by u^{in}. The result in a formula of u^{sc} will be

$$u^{sc}(t,x) = \int_{R^3} dz \int_{R} \frac{\delta(t-\tau-\frac{|x-z|}{c_0})}{4\pi|x-z|} V(x)\partial_t^2 u^{in}(\tau,z)d\tau \tag{18}$$

In the frequency domain, the Born approximation is

$$U^{sc}(x,f) = -\int_{R^3} \frac{e^{-ik|x-z|}}{4\pi|x-z|} V(Z) f^2 U^{in}(z,f)dz \tag{19}$$

Since we know that

$$U^{in}(z,f) = p(f)\frac{e^{-ik|z-x_0|}}{4\pi|z-x_0|} \tag{20}$$

Thus,

$$U^{sc}(x,f) = -\int_{R^3} \frac{e^{-ik|x-z|}}{4\pi|x-z|} V(Z) f^2 p(f) \frac{e^{-ik|z-x_0|}}{4\pi|z-x_0|}dz \tag{21}$$

The Born approximation is very useful, because it makes the imaging problem linear.

2.4 Far-Field Wave Propagation Model

Often the flight path of the radar platform is designed in such a way that the maximum target distance from the origin, which is located at the target scene center, is much smaller than the distance of the antenna from the same origin. In this case, any computation of the received signal based on equation (21) can be simplified through the application of what is commonly referred to as the far-field approximation. This approximation can be understood by first noting that radar waves propagate as a spherical wave front. When the antenna is far from the target center, the curvature of this wavefront can be assumed to be negligible. Under this assumption, in the extreme far-field, radar wave propagation can be approximately represented by a plane wave. Actually, a good mathematical model is a rectangular distribution of point sources. We denote the length and width of the antenna by L and D, respectively, we denote the center of the antenna by x', thus a point on the antenna can be written $x_0 = x' + q$, where q is a vector from the center of the antenna to a point on the antenna. This assumption can be justified mathematically by applying a first order Taylor expansion to the range term $|z - x_0|$.

$$|z-x_0| = |z-x'-q| = |z-x'| - (\widehat{z-x'})\cdot q + O(\frac{L^2}{|z-x'|}) \tag{22}$$

Then, the phase term becomes

$$\frac{e^{-ik|z-x_0|}}{4\pi|z-x_0|} = \frac{e^{-ik|z-x'|}}{4\pi|z-x'|}\cdot e^{ik(\widehat{z-x'})\cdot q}[1+O(\frac{|L|^2}{|z-x'|})]\cdot[1+O(\frac{k|q|^2}{|z-x'|})] \tag{23}$$

Then, we apply this approximation for the incident field.

$$U^{in}(z,f) = p(f)\frac{e^{-ik|z-x'|}}{4\pi|z-x'|}\cdot e^{(\widehat{z-x'})\cdot q} \tag{24}$$

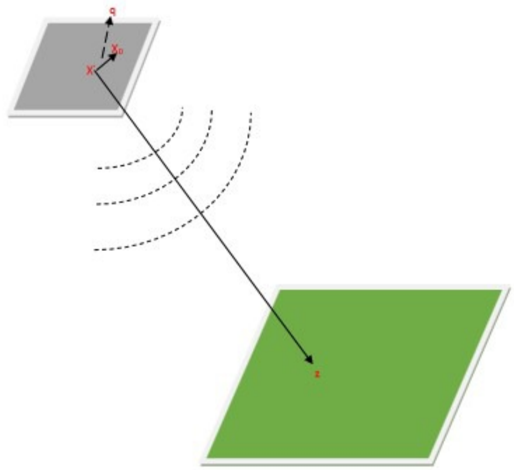

Figure 1.

2.5 Measured Data

We use a similar procedure to model how the receiving antenna measures the scattered field. In Eq(19), we substitute Eq(24) for the incident field to find the measured data.

$$U^{sc}(x,f) = -\int_{R^3} \frac{e^{-ik|x-z|}}{4\pi|x-z|} V(Z) f^2 p(f) \frac{e^{-ik|z-x'|}}{4\pi|z-x'|} \cdot e^{\widehat{(z-x')}\cdot q} dz \tag{25}$$

If we assume that the receiving antenna is at the same location as the transmitting antenna. we find that the scalar Born model for the received signal is

$$s(f) = \int e^{-2ik|z-x'|} \cdot A(f,x',z) \cdot V(z) dz \tag{26}$$

Where

$$A(f,x',z) = \frac{f^2}{(4\pi|z-x'|)^2} \cdot F(k,\widehat{z}) \tag{27}$$

and

$$F(k,\widehat{z}) = p(f) e^{ik\widehat{(z-x')}\cdot q} \tag{28}$$

Synthetic-aperture imaging involves a moving platform, and usually the antenna is pointed toward the earth. We denote by γ the antenna path. For a pulsed system, we assume that pulses are transmitted at times t_n and that the antenna position at time t_n is γ_n. Because the time scale on which the antenna moves is much slower than the time scale on which the electromagnetic waves propagate, we separate the time scales into a slow times, which corresponds to the n of t_n, and a fast time t. Using a continuum model for the slow time makes some of the analysis simpler but also leaves out some important effects that we will consider below. Using the continuum model for slow time, in Eq(26) we replace the antenna position x' by $\gamma(s)$

$$D(f,s) = \int e^{-2ik|\gamma(s)-z|} A(f,s,z) V(z) dz \tag{29}$$

With the additional assumption that the antennas are broadband and an appropriate symbol estimate of A. With this assumption, we can construct an approximate inverse operator, which we denote B. The reconstructed image I is formed by applying the inverse operator B to the data where B is of the form.

$$I(y) = B[D](y) := \int e^{-2ik|\gamma(s)-z_T|} Q(f,s,z) D(f,s) df ds \tag{30}$$

Where $z_T = (z, 0)$ and where Q is a filer to be determined below. The time-domain version is

$$I(y) = B[d](y) := \int e^{if(t - \frac{2|\gamma(s) - z_T|}{c})} Q(f, s, z) d(t, s) df ds dt \quad (31)$$

3. COMPRESSED SENSING

The primary interest in compressed sensing research is the inverse problem of recovering a signal $f \in C^N$ from noisy linear measurements$y = Af + n \in C^N$. The focus is on under determined problems where the forward operator $A \in C^{M \times N}$ has unit norm columns and forms an incomplete basis with $M \ll N$. The resulting ill-posed inverse problem is regularized assuming: (1) the unknown signal f is K-sparse or is compressible with K significant coefficients and (2) the noise process is bounded by $\| n \|_2 < \epsilon$. CS theory provides strong results which guarantee stable solution of the sparse signal recovery problem for a class of forward operators A that satisfies certain properties. One such class of operators is defined by bounding the singular values of the submatrices of A. Specifically, the restricted isometry constant (RIC) δ_K for forward operator A is the smallest $\delta \in (0, 1)$, such that[3]

$$(1 - \delta_K) \| x \|_2^2 \leq \| Ax \|_2^2 \leq (1 + \delta_K) \| x \|_2^2 \quad (32)$$

hold for all vectors x with at most K nonzero entries.

One of the key contributions of CS is that stable recovery of compressible, noisy signals can be achieved through the solution of the computationally tractable l_1 regularized inverse problem[3]

$$min_f \| f \|_1 \ \ subject \ \ to \| Af - y \|_2^2 \leq \epsilon^2 \quad (33)$$

At present, the least conservative available bound on the reconstruction performance guarantees that if $\delta_{2K} < \sqrt{2} - 1$ and $\| n \|_2 \leq \epsilon$, then the solution $\widehat{f}$ will satisfy[4]

$$\| f^* - \widehat{f} \|_2^2 \leq C_0 K^{-1/2} \| f^* - f_K \|_1 + C_1 \epsilon \quad (34)$$

where f_K is the best K-sparse approximation to the true solution f^*, C_0 and C_1 are small constants, and $\| \cdot \|_p$ represents the l_p norm. The optimization in (18) can be viewed as the convex relaxation of the NP-hard task of finding the sparsest feasible solution

$$min_f \| f \|_0 \ \ subject \ to \| Af - y \|_2^2 \leq \epsilon \quad (35)$$

where $\| \cdot \|_0$ is the l_0 norm, i.e., the number of nonzero entries in the vector. In radar and other array processing applications, imperfect calibration implies that precise knowledge of A is not available. Recent work has shown that a bounded unknown additive disturbance to the matrix A still permits a RIC-based guarantee on reconstruction performance that reduces to the result as the disturbance bound approaches zero.

Consider an unknown matrix $A \in C^{M \times N}$ and an orthonormal basis $(A_i)_i$ for $C^{M \times N}$. Then there exist coefficient $(s_i)_i$, such that

$$A = \sum_{i=0}^{MN-1} s_i A_i \quad (36)$$

Our goal is to identify the coefficients $(s_i)_i$. Since the basis elements are fixed, identifying $(s_i)_i$ is tantamount to discovering A. We will do this by designing a test function $f = (f_0, \cdots, f_{N-1})^T \in C^N$ and observing $Af \in C^M$. Here, $(\cdot)^T$ denotes the transpose of a vector or a matrix. For instance, A may represent an unknown communication channel which needs to be identified for equalization purposes.

For simplicity, from now on assume that $N = M$. The observation vector can be reformulated as

$$y = \sum_{i=0}^{N^2-1} s_i A_i f = \sum_{i=0}^{N^2-1} s_i \varphi_i = \Phi s \quad (37)$$

where the i-th atom $\varphi_i = A_i f$ is a column vector of length N, the concatenation of the atoms $\Phi = (\varphi_0 | \varphi_1 | \cdots | \varphi_{N^2-1}$ is an $N \times N^2)$ matrix, and $s = (s_0, s_1, \cdots, s_{N^2-1})^T$ is a column vector of length N^2. The system of equation in (37) is clearly highly under determined. If s is sufficiently sparse, then there is hope of recovering s from y.

4. CONCLUSION

In this paper, we derived the SAR signal model from the scalar form of Maxwells equations. Then in the simulation, compressed sensing will be used to construct point targets. In particular, classical radar detection techniques may fail to resolve the two targets whose regions are intersecting. In contrast, CS radar will be able to distinguish them as long as the total number of targets is enough.

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