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Algebraic Traveling Wave Solutions of a Non-local Hydrodynamic-type Model

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Abstract In this paper we consider the algebraic traveling wave solutions of a non-local hydrodynamic-type model. It is shown that algebraic traveling wave solutions exist if and only if an associated first order ordinary differential system has invariant algebraic curve. The dynamical behavior of the associated ordinary differential system is analyzed. Phase portraits of the associated ordinary differential system is provided under various parameter conditions. Moreover, we classify algebraic traveling wave solutions of the model. Some explicit formulas of smooth solitary wave and cuspon solutions are obtained.

Keywords Algebraic traveling wave solution · Invariant algebraic curve · Solitary wave solution · Cuspon

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1 Introduction

Mathematical modeling of dynamical processes in a great variety of natural phenomena leads in general to nonlinear partial differential equations. There is a particular

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class of solutions for nonlinear equations that are of considerable interest, in particular, traveling wave solutions attract more attention. Such a wave is a special solution of governing equations, that may be localized or periodic, which does not change its shape and which propagates at a constant speed.

There are several classical models describing the motion of waves at the free surface of shallow water under the influence of gravity. Among these models, the best known is the Korteweg-de Vries (KdV) equation [1]

$$u_t + 6uu_x + u_{xxx} = 0. \tag{1.1}$$

The KdV equation admits solitary wave solutions, i.e. solutions of the form $u(x,t) = \varphi(x - ct)$ which travel at a fixed speed c, and vanish at infinity. The KdV solitary waves are smooth and retain their individuality after two solitary wave interaction and eventually emerge with their original shapes and speeds.

Another model, the Camassa-Holm (CH) equation [2]

$$u_t - u_{txx} + 3uu_x = 2u_x u_{xx} + uu_{xxx}. \tag{1.2}$$

arises as a model for the unidirectional propagation of shallow water waves over a flat bottom [2], as well as water waves moving over an underlying shear flow [3]. The CH equation has many remarkable properties that KdV does not have like solitary waves with singularities and breaking waves. The CH equation admits peaked solitary waves or "peakons" [2]: $u(x,t) = ce^{-|x-ct|}$, $c \neq 0$, which are smooth except at the crests, where they are continuous, but have a jump discontinuity in the first derivative. The CH equation also has algebro-geometric solutions associated with a Neumann system on a symplectic submanifold [4].

The CH model may be extended to multi-component generalizations. Chen and Liu [5] proposed the following generalized two-component CH system

$$\begin{cases} u_t - u_{txx} - Au_x + 3uu_x - \sigma(2u_x u_{xx} + uu_{xxx}) + \rho\rho_x = 0, \\ \rho_t + (\rho u)_x = 0, \end{cases} \tag{1.3}$$

which can be derived from shallow water theory with nonzero constant vorticity. By using dynamical system method, Li and Qiao [6] obtained solitary wave solutions, kink and anti-kink wave solutions, cusp wave solutions, breaking wave solutions, and smooth and non-smooth periodic wave solutions of this equation.

This paper deals with algebraic traveling wave solution, supported by the hydrodynamic-type system which takes into account the non-local effects [7–9]:

$$\begin{cases} u_t + \beta\rho^{n+1}\rho_x + \sigma[\rho^{n+1}\rho_{xxx} + 3(1+n)\rho^n\rho_x\rho_{xx} + n(1+n)\rho^{n-1}\rho_x^3] = 0, \\ \rho_t + \rho^2 u_x = 0, \end{cases} \tag{1.4}$$

where u is the mass velocity, ρ is the density, t is time, x is the mass (Lagrangian) coordinate related the commonly used (Eulerian) coordinate x_e in the following fashion:

$$x = \int^{x_e} \rho(t,s)ds,$$

$\beta > 0$ and $\sigma \neq 0$ are constant parameters. In this paper, by using bifurcation theory of dynamical system [10, 15], we study the smooth solitary wave and cuspon solutions of system (1.4). The idea is inspired by the study of the traveling waves of Camassa-Holm equation and Degasperis-Procesi equation [16, 19].

The whole paper is organized as follows. In Section 2, we discuss the algebraic traveling wave solutions and invariant algebraic curves. In Section 3, we study dynamical behavior of algebraic traveling wave solutions. In Section 4, we classify algebraic traveling wave solutions of the model. Some explicit formulas of algebraic traveling wave solutions are obtained.

2 Algebraic Traveling Wave Solutions and Invariant Algebraic Curves

We consider general n-th order partial differential equations with the following form

$$\frac{\partial^n u}{\partial x^n} = F\left(u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial t}, \cdots, \frac{\partial^{n-1} u}{\partial x^{n-1}}, \frac{\partial^{n-1} u}{\partial x^{n-2}\partial t}, \cdots, \frac{\partial^{n-1} u}{\partial x \partial t^{n-2}}, \frac{\partial^{n-1} u}{\partial t^{n-1}}\right), \tag{2.1}$$

where x and t are real variables and F is a smooth map. The traveling wave solutions of (2.1) are particular solutions of the form $u = u(x, t) = U(x - ct)$, where $U(\xi)$ satisfies the boundary conditions

$$\lim_{\xi \to -\infty} U(\xi) = a, \quad \lim_{\xi \to \infty} U(\xi) = b, \tag{2.2}$$

where a and b are constant root solutions, not necessarily different, of $F(u, 0, ..., 0) = 0$. Plugging $u(x, t) = U(x - ct)$ into (2.1) we get that $U(\xi)$ has to be a solution, defined for all $\xi \in R$, of the n-th order ordinary differential equation

$$U^{(n)} = F(U, U', -cU', \cdots, U^{(n-1)}, -cU^{(n-1)}, \cdots, (-c)^{(n-2)}U^{(n-1)}, (-c)^{(n-1)}U^{(n-1)}) \tag{2.3}$$

where $U = U(\xi)$ and its derivatives are taken with respect to ξ. The parameter c is called the speed of the traveling wave solution.

Definition 2.1 A function $u(x, t) = U(\xi) = U(x - ct)$ is called an algebraic traveling wave solution of (2.1) if $U(\xi)$ is a non constant function that satisfies (2.1) and (2.3) and there exists a polynomial $P \in R[z, w]$ such that $P(U(\xi), U'(\xi)) = 0$.

It is known that traveling wave solutions correspond to homoclinic ($a = b$) or heteroclinic ($a \neq b$) solutions of an associated n-dimensional system of ordinary differential equations. Recently, Gasull and Giacomini [20] proved the following theorem.

Theorem 2.1 [20] *The partial differential* (2.1) *has an algebraic traveling wave solution with wave speed c if and only if the first order differential system*

$$\begin{cases} y_1' = y_2, \\ y_2' = y_3, \\ \vdots \quad \vdots \\ y_{n-1}' = y_n, \\ y_n' = G(y_1, y_2, \cdots, y_n), \end{cases} \tag{2.4}$$

where

$$G(y_1, y_2, \cdots, y_n) = F(y_1, y_2, -cy_2, \cdots, y_n, -cy_n, \cdots, (-c)^{(n-2)}y_n, (-c)^{(n-1)}y_n), \tag{2.5}$$

has an invariant algebraic curve containing the critical points $(a, 0, \cdots, 0)$ *and* $(b, 0, \cdots, 0)$ *and no other critical points between them.*

Now, we need to seek a method to detect when a polynomial system of ordinary differential equations has algebraic invariant curves to determine whether some polynomial partial differential equation can have algebraic traveling wave solution. There are some works dealing with this problem in the n-dimensional setting [21, 23], the planar case is the most developed one. Consider a planar differential system,

$$\begin{cases} x' = P(x, y), \\ y' = Q(x, y), \end{cases} \tag{2.6}$$

where P and Q are polynomials of degree at most N, and assume that there is a polynomial $g(x, y)$ such that the set $g(x, y) = 0$ is non-empty and invariant by the flow of (2.6).

Let U be an open subset of R^2. We say that a non-constant function $H : U \to R$ is a first integral of the polynomial system (2.6) in U if H is constant on the trajectories of the polynomial system (2.6) contained in U; i.e. if

$$P(x, y)\frac{\partial H(x, y)}{\partial x} + Q(x, y)\frac{\partial H(x, y)}{\partial y} = 0. \tag{2.7}$$

For irreducible polynomials we have the following algebraic characterization of invariant algebraic curves. Given an irreducible polynomial of degree n, $f(x, y)$, then $f(x, y) = 0$ is an invariant algebraic curve for the system if there exists a polynomial of degree at most $N - 1$, $k(x, y)$, called the cofactor of f, such that

$$P(x, y)\frac{\partial f(x, y)}{\partial x} + Q(x, y)\frac{\partial f(x, y)}{\partial y} - k(x, y)f(x, y) = 0. \tag{2.8}$$

We are going to analyze a set of traveling wave solutions, having the form

$$u(x, t) = \phi(\xi), \quad \rho(x, t) = \varphi(\xi), \quad \xi = x - ct. \tag{2.9}$$

Inserting the ansatz (2.9) into the second equation of the system (1.4), we get, after one integration, the following quadrature:

$$\phi(\xi) = c_1 - \frac{c}{\varphi(\xi)}, \tag{2.10}$$

where c_1 is the constant of integration. In what follows, we assume that $c_1 = c/A_1$, where A_1 is a positive constant. Such a choice immediately leads to the following asymptotic behavior:

$$\lim_{|\xi|\to\infty} u(x,t) = 0, \qquad \lim_{|\xi|\to\infty} \rho(x,t) = A_1, \tag{2.11}$$

Inserting the ansatz (2.9) into the first equation of the system (1.4) and using the (2.10), we obtain the following second order ordinary differential equation

$$\frac{c^2}{\varphi} + \frac{\beta}{n+2}\varphi^{n+2} + \sigma[\varphi^{n+1}\varphi'' + (n+1)\varphi^n(\varphi')^2] = E, \tag{2.12}$$

where

$$E = \frac{c^2}{A_1} + \frac{\beta}{n+2}A_1^{n+2}, \tag{2.13}$$

is the integration constant determined from the conditions on infinity. Let us rewrite the (2.12) as the first order differential system:

$$\begin{cases} \frac{d\varphi}{d\xi} = y, \\ \frac{dy}{d\xi} = (\sigma\varphi^{n+2})^{-1}[E\varphi - c^2 - \frac{\beta}{n+2}\varphi^{n+3} - \sigma(n+1)\varphi^{n+1}y^2]. \end{cases} \tag{2.14}$$

Let $d\xi = (\sigma\varphi^{n+2})d\zeta$, then system (2.14) is equivalent to the following ordinary differential system

$$\begin{cases} \frac{d\varphi}{d\zeta} = \sigma\varphi^{n+2}y, \\ \frac{dy}{d\zeta} = E\varphi - c^2 - \frac{\beta}{n+2}\varphi^{n+3} - \sigma(n+1)\varphi^{n+1}y^2. \end{cases} \tag{2.15}$$

Theorem 2.2 [9] *System* (2.15) *has the following rational first integral*

$$H(x,y) = 2c^2\frac{\varphi^{n+1}}{n+1} + \frac{\beta}{(n+2)^2}\varphi^{2(n+2)} + \sigma y^2\varphi^{2(n+1)} - 2E\frac{\varphi^{n+2}}{n+2}, \tag{2.16}$$

and therefore gives invariant algebraic curves $h - H(x,y) = 0$ *for all* $h \in R$.

Proof In fact, the result can be seen from [9]. Here, for the convenience of the readers, we give proof. For the system (2.15), we have

$$\begin{aligned} & P(x,y)\tfrac{\partial H(x,y)}{\partial x} + Q(x,y)\tfrac{\partial H(x,y)}{\partial y} \\ = \ & \sigma\varphi^{n+2}y[2c^2\varphi^n + \tfrac{2\beta}{n+2}\varphi^{2n+3} + 2(n+1)\sigma y^2\varphi^{2n+1} - 2E\varphi^{n+1}] \\ & +[E\varphi - c^2 - \tfrac{\beta}{n+2}\varphi^{n+3} - \sigma(n+1)\varphi^{n+1}y^2](2\sigma y\varphi^{2(n+1)}) \\ = \ & 0. \end{aligned}$$

This completes the proof. □

3 Dynamical Behavior of the Differential System (2.15)

Let us now consider the dynamical behavior of the polynomial ordinary differential system (2.15). It is evident that all isolated singular points of system (2.15) are located on the horizontal axis $O\varphi$. They are determined by solutions of the algebraic equation

$$f(\varphi) = \frac{\beta}{n+2}\varphi^{n+3} - E\varphi + c^2 = 0. \tag{3.1}$$

Based on results of [7–9], next we show the existence of the singular points and the homoclinic orbits for the polynomial ordinary differential system (2.15).

Theorem 3.1 *For $\sigma \neq 0$, we have*

(1.) *Case I: $\sigma > 0$.*
(1) If $n = 2m-1, m = 1, 2, \cdots$, then

(a) if $\beta < \frac{c^2}{A_1^{n+3}}$, then system (2.15) has a saddle $(A_1, 0)$ and a center $(A_2, 0)$, where $A_1 < A_2$.
(b) if $\beta = \frac{c^2}{A_1^{n+3}}$, then system (2.15) has a cusp point $(A_1, 0)$.

(2) If $n = 2m, m = 0, 1, 2, \cdots$, then

(a)
(a) if $\beta < \frac{c^2}{A_1^{n+3}}$, then system (2.15) has a saddle $(A_1, 0)$ and two centers $(A_2, 0)$, $(A_3, 0)$, where $A_3 < A_1 < A_2$.
(b) if $\beta = \frac{c^2}{A_1^{n+3}}$, then system (2.15) has a cusp point $(A_1, 0)$ and a center $(A_3, 0)$, where $A_3 < A_1 = A_2$.

Moreover, if $\beta < \frac{c^2}{A_1^{n+3}}$, then there exists a homoclinic orbit bi-asymptotic to the saddle $(A_1, 0)$.

(2.) *Case II: $\sigma < 0$.*
(1) If $n = 2m-1, m = 1, 2, \cdots$, then

(a) if $\beta > \frac{c^2}{A_1^{n+3}}$, then system (2.15) has a saddle $(A_1, 0)$ and a center $(A_2, 0)$, where $A_1 > A_2$.
(b) if $\beta = \frac{c^2}{A_1^{n+3}}$, then system (2.15) has a cusp point $(A_1, 0)$.

(2) If $n = 2m, m = 0, 1, 2, \cdots$, then

(a) if $\beta > \frac{c^2}{A_1^{n+3}}$, then system (2.15) has two saddle $(A_1, 0)$, $(A_3, 0)$ and a centers $(A_2, 0)$, where $A_3 < A_2 < A_1$.
(b) if $\beta = \frac{c^2}{A_1^{n+3}}$, then system (2.15) has a cusp point $(A_1, 0)$ and a cusp point $(A_3, 0)$, where $A_3 < A_1 = A_2$.

Moreover, if $\frac{c^2}{A_1^{n+3}} < \beta < \frac{2(n+2)c^2}{(n+1)A_1^{n+3}}$, then there exists a homoclinic orbit bi-asymptotic to the saddle $(A_1, 0)$.

Proof It can be easily seen that one of the roots of (3.1) coincides with A_1. The location of the second real positive root depends on relations between the parameters.

Differentiating $f(\varphi)$ with respect to φ, we obtain

$$f'(\varphi) = \frac{(n+3)\beta}{n+2}\varphi^{n+2} - E. \tag{3.2}$$

□

Obviously, the function $f'(\varphi)$ has a zero φ_* if n is odd, and the function $f'(\varphi)$ has two zeros $\pm\varphi_*$ if n is even, where $\varphi_* > 0$. Let

$$f'(A_1) = \frac{(n+3)\beta}{n+2}A_1^{n+2} - E = 0. \tag{3.3}$$

Substituting (2.13) into (3.3) gives

$$\frac{(n+3)\beta}{n+2}A_1^{n+2} = \frac{c^2}{A_1} + \frac{\beta}{n+2}A_1^{n+2}. \tag{3.4}$$

Thus we obtain the critical parameters condition

$$c^2 = \beta A_1^{n+3}. \tag{3.5}$$

Furthermore analyzing we make the following conclusions.

1. When n is odd, then
   (1) if $\beta = \frac{c^2}{A_1^{n+3}}$, then $f(\varphi)$ has a zero $A_1 = \varphi_*$.
   (2) if $\beta < \frac{c^2}{A_1^{n+3}}$ and $\sigma > 0$, then $f(\varphi)$ has two zero A_1 and A_2 such that $A_2 > \varphi_* > A_1 > 0$.
   (3) if $\beta > \frac{c^2}{A_1^{n+3}}$ and $\sigma < 0$, then $f(\varphi)$ has two zero A_1 and A_2 such that $A_1 > \varphi_* > A_2 > 0$.
2. When n is even, then
   (1) if $\beta = \frac{c^2}{A_1^{n+3}}$, then $f(\varphi)$ has two zeros A_1 and A_3 such that $A_3 < -\varphi_* < 0 < \varphi_* = A_1$.
   (2) if $\beta < \frac{c^2}{A_1^{n+3}}$ and $\sigma > 0$, then $f(\varphi)$ has three zeros $A_i (i = 1, 2, 3)$ such that $A_3 < -\varphi_* < 0 < A_1 < \varphi_* < A_2$.
3. if $\beta > \frac{c^2}{A_1^{n+3}}$ and $\sigma < 0$, then $f(\varphi)$ has three zeros $A_i (i = 1, 2, 3)$ such that $A_3 < -\varphi_* < 0 < A_2 < \varphi_* < A_1$.

We next analyze the Jacobian matrix

$$\mathbf{M}(A_i, 0) = \begin{pmatrix} 0 & \sigma A_i^{n+2} \\ -f'(A_i) & 0 \end{pmatrix}.$$

A direct calculation shows that

$$J(A_i, 0) = det\mathbf{M}(A_i, 0) = \sigma f'(A_i)A_i^{n+2}. \tag{3.6}$$

For a singular point $(A_i, 0)$ of the planar system (2.15) the following classification holds true: if $J(A_i, 0) < 0$ then the singular point is a saddle; if $J(A_i, 0) > 0$ then it is a center; if $J(A_i, 0) = 0$ and the index of the singular point is zero then it is a cusp. By analyzing the formula (3.6), we easily determine the type of a singular point $(A_i, 0)$.

The detailed proof of the conditions assuring the existence of the homoclinic loops can be found in [9].

4 Classification and Explicit Formulas of Algebraic Traveling Wave Solutions

4.1 Weak Formulation

We consider the weak formulation of algebraic traveling wave solutions of system (1.4) for $n = 0$. For a traveling wave ansatz (2.9), (1.4) take the form

$$-c^2\varphi^{-2}\varphi_\xi + \beta\varphi\varphi_\xi + \sigma(\varphi\varphi_{\xi\xi\xi} + 3\varphi_\xi\varphi_{\xi\xi}) = 0. \tag{4.1}$$

By integrating with respect to ξ, combining (2.11) and (2.13), (4.1) is equivalent to the following integrated form

$$(\varphi^2)_{\xi\xi} = \frac{1}{\sigma}(-\beta\varphi^2 - \frac{2c^2}{\varphi} + 2E). \tag{4.2}$$

(4.2) makes sense for all $\varphi \in H^1_{loc}(R)$. The following definition is therefore natural.

Definition 4.1 *A function $\varphi \in H^1_{loc}(R)$ is a weak traveling wave solution to the equation (4.1) if φ satisfies (4.2) in distribution sense for some $E \in R$.*

By a similar approach just like Lemma 4 and 5 in [16], we give the following determinant theorem.

Theorem 4.1 *Any bounded function φ belongs to $H^1_{loc}(R)$ and is a weak traveling wave solution to (4.1) with the speed c if and only if satisfying the following two statements:*

(A). There are disjoint open intervals $J_i, i \geqslant 1$, and a closed set C such that $R \setminus C = \bigcup_{i=1}^{\infty} J_i$, $\varphi \in C^\infty(J_i)$ for $i \geqslant 1$, $\varphi(\xi) \neq 0$ for $\xi \in \bigcup_{i=1}^{\infty} J_i$ and $\varphi(\xi) = 0$ for $\xi \in C$.

(B). For each $E \in R$, there exists $h \in R$ such that

$$\varphi_\xi^2 = G(\varphi), \quad \xi \in J_i, \tag{4.3}$$

where

$$G(\varphi) = \frac{-\beta\varphi^4 + 4E\varphi^2 - 8c^2\varphi + 4h}{4\sigma\varphi^2}, \tag{4.4}$$

and $\varphi \to 0$, at any finite endpoint of J_i.

We next introduce two notations in [16]. We say that a continuous function φ has a *peak* at ξ_0 if φ is smooth locally on either side of ξ_0 and

$$\lim_{\xi\uparrow\xi_0}\varphi_\xi(\xi)=-\lim_{\xi\downarrow\xi_0}\varphi_\xi(\xi)=a,\quad a\neq 0,\quad a\neq\pm\infty.$$

Wave profiles with peaks are called *peaked waves* or *peakons*.

Similarly, a continuous function φ is said to have a cusp at ξ_0 if φ is smooth locally on both sides of ξ_0 and

$$\lim_{\xi\uparrow\xi_0}\varphi_\xi(\xi)=-\lim_{\xi\downarrow\xi_0}\varphi_\xi(\xi)=\pm\infty,$$

We will call waves with cusps *cusped waves* or *cuspons*.

4.2 Classification of Algebraic Traveling Wave Solutions

The hydrodynamic-type system (1.4) admits smooth solitary wave and cuspon solutions. It is notable that system (1.4) has not peakon solution.

From invariant algebraic curves $h-H(x,y)=0$ or (4.3) we obtain

$$y^2=\frac{\beta(\frac{4h}{\beta}-\frac{8c^2}{\beta}\varphi+\frac{4E}{\beta}\varphi^2-\varphi^4)}{4\sigma\varphi^2}, \tag{4.5}$$

where

$$h=A_1(c^2-\frac{\beta}{4}A_1^3). \tag{4.6}$$

Combining (2.12),(2.16) and (4.6), then (4.1) reduces to

$$y^2=\frac{(\varphi-A_1)^2(B_1-\varphi)(\varphi-B_2)}{4\sigma\varphi^2}=F(\varphi), \tag{4.7}$$

where

$$B_1=\sqrt{\frac{4c^2}{\beta A_1}}-A_1,\quad B_2=-\sqrt{\frac{4c^2}{\beta A_1}}-A_1. \tag{4.8}$$

Obviously, $B_1\geqslant B_2$. From (4.8) we know that $A_1<B_1$ if $\beta<\frac{c^2}{A_1^3}$ and $A_1>B_1$ if $\beta>\frac{c^2}{A_1^3}$. Any algebraic traveling wave solutions φ must satisfy the following initial and boundary values problem

$$\begin{cases}(\varphi_\xi)^2=\frac{(\varphi-A_1)^2(B_1-\varphi)(\varphi-B_2)}{4\sigma\varphi^2},\\ \lim_{\xi\to\pm\infty}\varphi(\xi)=A_1,\\ \varphi(0)\in\{0,B_1\},\end{cases} \tag{4.9}$$

(4.7) implies

(1) if $\sigma>0$, then $B_2\leqslant\varphi\leqslant B_1$;
(2) if $\sigma<0$, then $\varphi\geqslant B_1$ or $\varphi\leqslant B_2$.

From (4.7), we know that $F(\varphi)$ has a simple zero at B_1 and a double zero at A_1. Depending on whether the zero is double or simple, φ has different behavior.

Theorem 4.2 *1. When φ approaches the double zero A_1 of $F(\varphi)$ so that $F'(A_1)=0$, $F''(A_1)\neq 0$, then the solution φ satisfies*

$$\varphi(\xi)-A_1 \sim a\exp(-|\xi|\sqrt{|F''(A_1)|}), \quad \xi\to\pm\infty, \tag{4.10}$$

for some constant a, thus $\varphi\to A_1$ exponentially as $x\to\pm\infty$.

2. If φ approaches the simple zero B_1 of $F(\varphi)$ so that $F(B_1)=0$, $F'(B_1)\neq 0$, then the solution φ satisfies

$$\varphi(\xi)=B_1+\frac{1}{4}\xi^2F'(B_1)+O(\xi^4), \quad \xi\to 0, \tag{4.11}$$

where $\varphi(0)=B_1$ and $\varphi'(0)=0$.

3. If $\sigma<0$, $\beta=\frac{4c^2}{A_1^3}$, then $B_1=0$ is a simple pole of $F(\varphi)$, and the solution φ satisfies

$$\varphi(\xi)=b\xi^{2/3}+O(\xi^{4/3}), \quad \xi\to 0 \tag{4.12}$$

and

$$\varphi_\xi=\begin{cases}\frac{2}{3}b|\xi|^{-1/3}+O(\xi^{1/3}), & \xi\downarrow 0,\\ -\frac{2}{3}b|\xi|^{-1/3}+O(\xi^{1/3}), & \xi\uparrow 0,\end{cases} \tag{4.13}$$

for some constant b, thus φ has a cusp.

Proof Because A_1 is a double zero of $F(\varphi)$, we have

$$\varphi_\xi^2=(\varphi-A_1)^2F''(A_1)+O((\varphi-A_1)^3), \quad \varphi\to A_1. \tag{4.14}$$

Furthermore, we get

$$\frac{d\xi}{d\varphi}=\frac{1}{\sqrt{(\varphi-A_1)^2F''(A_1)+O((\varphi-A_1)^3)}}. \tag{4.15}$$

Since

$$\sqrt{(\varphi-A_1)^2F''(A_1)+O((\varphi-A_1)^3)}=|\varphi-A_1|(\sqrt{|F''(A_1)|}+O(\varphi-A_1)), \tag{4.16}$$

and

$$\frac{1}{\sqrt{|F''(A_1)|}+O(\varphi-A_1)}=\frac{1}{\sqrt{|F''(A_1)|}}+O(\varphi-A_1) \tag{4.17}$$

we get

$$\frac{d\xi}{d\varphi}=\frac{1}{|\varphi-A_1|\sqrt{F''(A_1)}}+O(1). \tag{4.18}$$

Integration gives Eq. (4.10). □

We know that $F(\varphi)$ has a simple zero B_1 so that $F(B_1)=0$, $F'(B_1)\neq 0$, then

$$\varphi_\xi^2=(\varphi-B_1)F'(B_1)+O((\varphi-B_1)^2), \quad \varphi\to B_1. \tag{4.19}$$

We get

$$\frac{d\xi}{d\varphi}=\frac{1}{\sqrt{(\varphi-B_1)F'(B_1)+O((\varphi-B_1)^2)}}. \tag{4.20}$$

Since

$$\sqrt{(\varphi-B_1)F'(B_1)+O((\varphi-B_1)^2)}=\sqrt{|\varphi-B_1|}(\sqrt{|F'(B_1)|}+O(\varphi-B_1)) \tag{4.21}$$

and

$$\frac{1}{\sqrt{|F'(B_1)| + O(\varphi - B_1)}} = \frac{1}{\sqrt{|F'(B_1)|}} + O(\varphi - B_1), \tag{4.22}$$

we get

$$\frac{d\xi}{d\varphi} = \frac{1}{\sqrt{|(\varphi - B_1)F'(B_1)|}} + O((\varphi - B_1)^{1/2}). \tag{4.23}$$

Integration gives

$$-\xi = \frac{2}{\sqrt{|F'(B_1)|}}\sqrt{|\varphi - B_1|} + O((\varphi - B_1)^{3/2}), \tag{4.24}$$

where we use the fact $\varphi(0) = B_1$. Therefore

$$\xi^2 = \frac{4}{|F'(B_1)|}|\varphi - B_1|(1 + O(\varphi - B_1))^2. \tag{4.25}$$

Using that $(1 + O(\varphi - B_1))^2 = 1 + O(\varphi - B_1)$, we obtain

$$\xi^2 = \frac{4}{|F'(B_1)|}|\varphi - B_1| + O((\varphi - B_1)^2). \tag{4.26}$$

This equation shows that $O(\xi^4) = O((\varphi - B_1)^2)$. From (4.26) we get (4.11).

If $\sigma < 0$, $\beta = \frac{4c^2}{A_1^3}$, then $B_1 = 0$ is a simple pole of $F(\varphi)$, i.e. $1/F(\varphi)$ has a simple zero. There exists a smooth function $h(\varphi)$ defined in a neighborhood of $\varphi = 0$, such that

$$\frac{1}{\sqrt{F(\varphi)}} = h(\varphi)\sqrt{\varphi}, \quad h(0) > 0, \quad h(\varphi) = h(0) + O(\varphi),$$

where

$$h(\varphi) = \frac{2\sqrt{-\sigma}}{\sqrt{(\varphi - A_1)^2(\varphi - B_2)}}, \quad h(0) = \frac{2}{A_1}\sqrt{\frac{\sigma}{B_2}}.$$

This implies, in view of (4.7), that

$$\frac{d\xi}{d\varphi} = h(0)\sqrt{\varphi} + O(\varphi^{3/2}). \tag{4.27}$$

Integration gives

$$\xi = \frac{2h(0)}{3}\varphi^{3/2} + O(\varphi^{5/2}), \tag{4.28}$$

where $\varphi(0) = 0$. Hence

$$\xi^{2/3} = (\frac{2h(0)}{3})^{2/3}\varphi(1 + O(\varphi))^{2/3}. \tag{4.29}$$

Since $(1 + O(\varphi))^{2/3} = 1 + O(\varphi)$, we get

$$\xi^{2/3} = (\frac{2h(0)}{3})^{2/3}\varphi + O(\varphi^2). \tag{4.30}$$

This equation shows that $O(\varphi^2) = O(\xi^{4/3})$. We arrive at

$$\varphi(\xi) = b|\xi|^{2/3} + O(\xi^{4/3}), \quad \xi \to 0, \tag{4.31}$$

where $b = (\frac{3}{2h(0)})^{2/3}$.

As for φ_ξ, we get

$$\varphi_\xi = \frac{1}{h(\varphi)\sqrt{\varphi}} = \frac{1}{(h(0) + O(\varphi))\sqrt{b|\xi|^{2/3} + O(\xi^{4/3})}}. \tag{4.32}$$

Observing that

$$\sqrt{b|\xi|^{2/3} + O(\xi^{4/3})} = \sqrt{b}\xi^{1/3} + O(\xi), \tag{4.33}$$

we obtain

$$\varphi_\xi = \frac{1}{h(0)\sqrt{b}\xi^{1/3}(1 + O(\xi^{2/3}))}. \tag{4.34}$$

Using that $\frac{1}{1+O(\xi^{2/3}))} = 1 + O(\xi^{2/3}))$, we deduce

$$\varphi_\xi = \frac{1}{h(0)\sqrt{b}}\xi^{-1/3} + O(\xi^{1/3}). \tag{4.35}$$

Therefore we obtain (4.13). This completes the proof of Theorem 4.2.

4.3 Explicit Formulas of Algebraic Traveling Wave Solutions

Using the standard phase portrait analytical technique (see Figs. 1, 2, 3, 4 and 5) and above conclusions, we consider the following three case.

Case I :$\sigma > 0$, $\beta < \frac{c^2}{A_1^3}$.

In this case, we have

$$B_2 < 0 < A_1 < B_1, \quad \varphi(0) = B_1, \quad A_1 < \varphi \leqslant B_1.$$

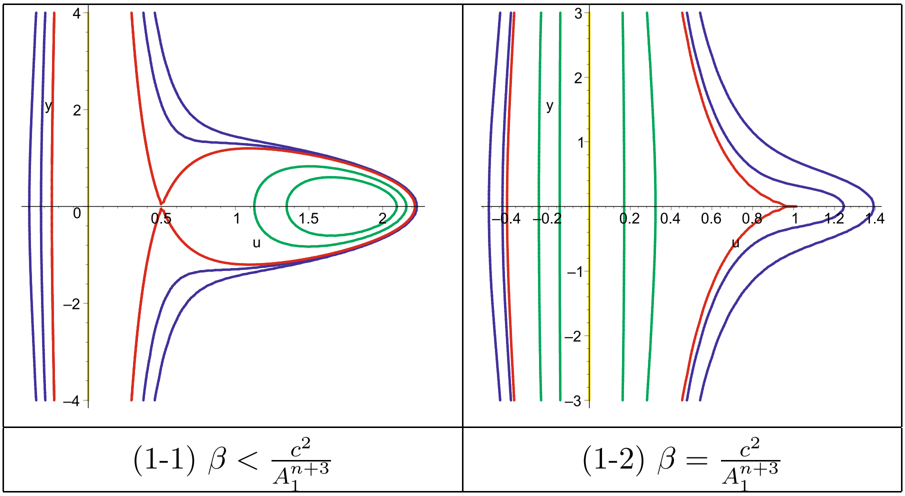


Fig. 1 Phase portraits of system (2.15) for $n = 2m - 1$ and $\sigma > 0$

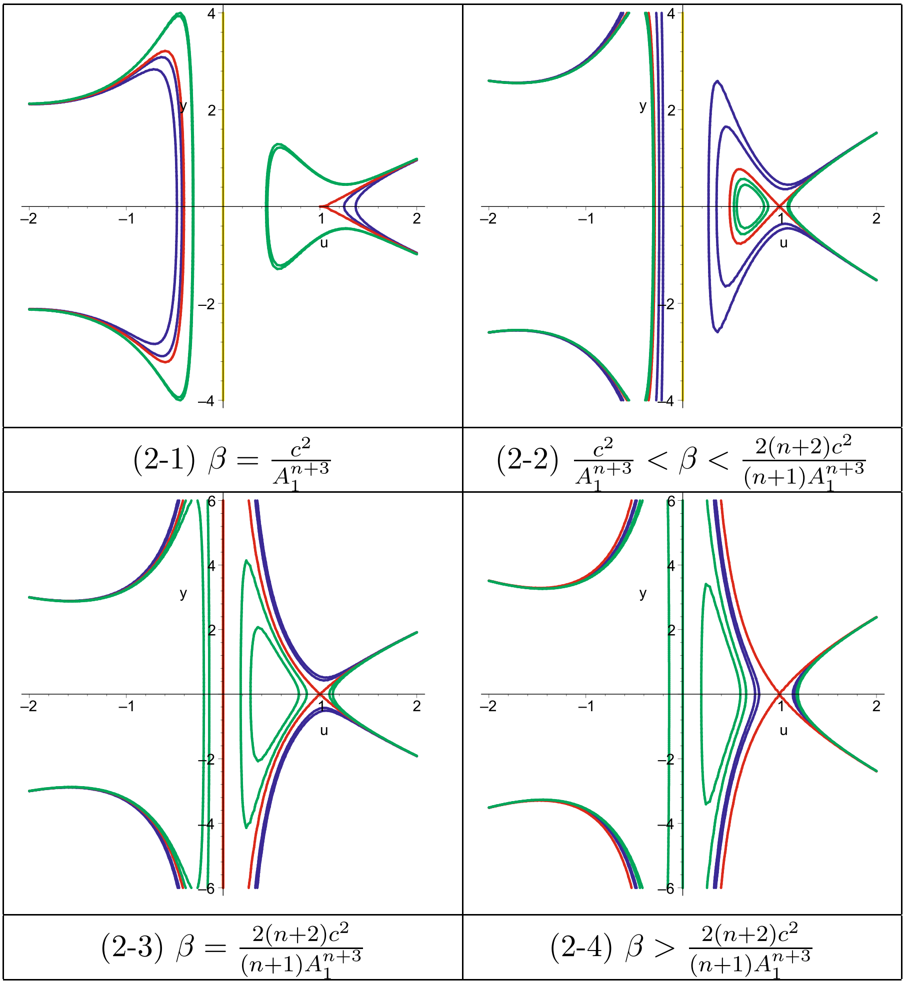


Fig. 2 Phase portraits of system (2.15) for $n = 2m - 1$ and $\sigma < 0$

The invariant algebraic curve $h - H(x, y) = 0$ determines a smooth solitary wave solution satisfying

$$\varphi(0) = B_1, \quad \lim_{\xi \to \pm\infty} \varphi(\xi) = A_1, \quad \varphi'(0) = 0.$$

By using the first equation of system (2.14) to do the integration, we have

$$\int_{\varphi}^{B_1} \frac{z dz}{(z - A_1)\sqrt{(B_1 - z)(z - B_2)}} = \frac{\sqrt{\beta}}{2\sqrt{\sigma}} \int_{\xi}^{0} d\xi. \tag{4.36}$$

Thus we obtain the following implicit expression of the smooth solitary wave solution.

$$I_1(\varphi) + \frac{A_1}{\sqrt{(A_1 - B_2)(B_1 - A_1)}} I_2(\varphi) = \frac{\sqrt{\beta}}{2\sqrt{\sigma}} |\xi|, \tag{4.37}$$

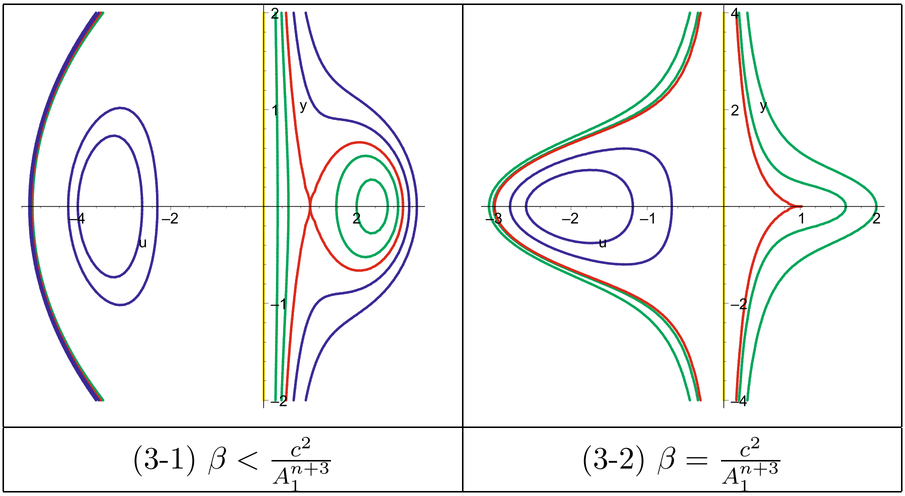


Fig. 3 Phase portraits of system (2.15) for $n = 2m$ and $\sigma > 0$

where

$$I_1(\varphi) = \arctan\left(\frac{B_1+B_2-2\varphi}{2\sqrt{(\varphi-B_2)(B_1-\varphi)}}\right) + \frac{\pi}{2}, \tag{4.38}$$

$$I_2(\varphi) = \ln\left|\frac{A_1B_1 + A_1B_2 - 2B_1B_2 + \varphi B_1 + \varphi B_2 - 2\varphi A_1 + 2\sqrt{(A_1-B_2)(B_1-A_1)(\varphi-B_2)(B_1-\varphi)}}{(B_1-B_2)(\varphi-A_1)}\right|. \tag{4.39}$$

The profile of smooth solitary wave solution $\varphi(\xi)$ and $\phi(\xi)$ is shown in Fig. 6(6-1) and Fig. 7(7-1), respectively.

Case II : $\sigma < 0, \frac{c^2}{A_1^3} < \beta < \frac{4c^2}{A_1^3}$

In this case, we have

$$B_2 < 0 < B_1 < A_1, \quad \varphi(0) = B_1, \quad B_1 \leqslant \varphi < A_1.$$

The invariant algebraic curve $h - H(x, y) = 0$ determines a smooth solitary wave solution satisfying

$$\varphi(0) = B_1, \quad \lim_{\xi\to\pm\infty} \varphi(\xi) = A_1, \quad \varphi'(0) = 0.$$

By using the first equation of system (2.14) to do the integration, we have

$$\int_{B_1}^{\varphi} \frac{zdz}{(A_1-z)\sqrt{(z-B_1)(z-B_2)}} = \frac{\sqrt{\beta}}{2\sqrt{-\sigma}} \int_0^{\xi} d\xi. \tag{4.40}$$

Thus we obtain the following implicit expression of the smooth solitary wave solution.

$$I_3(\varphi) + \frac{A_1}{\sqrt{(A_1-B_2)(A_1-B_1)}} I_4(\varphi) = \frac{\sqrt{\beta}}{2\sqrt{\sigma}}|\xi|, \tag{4.41}$$

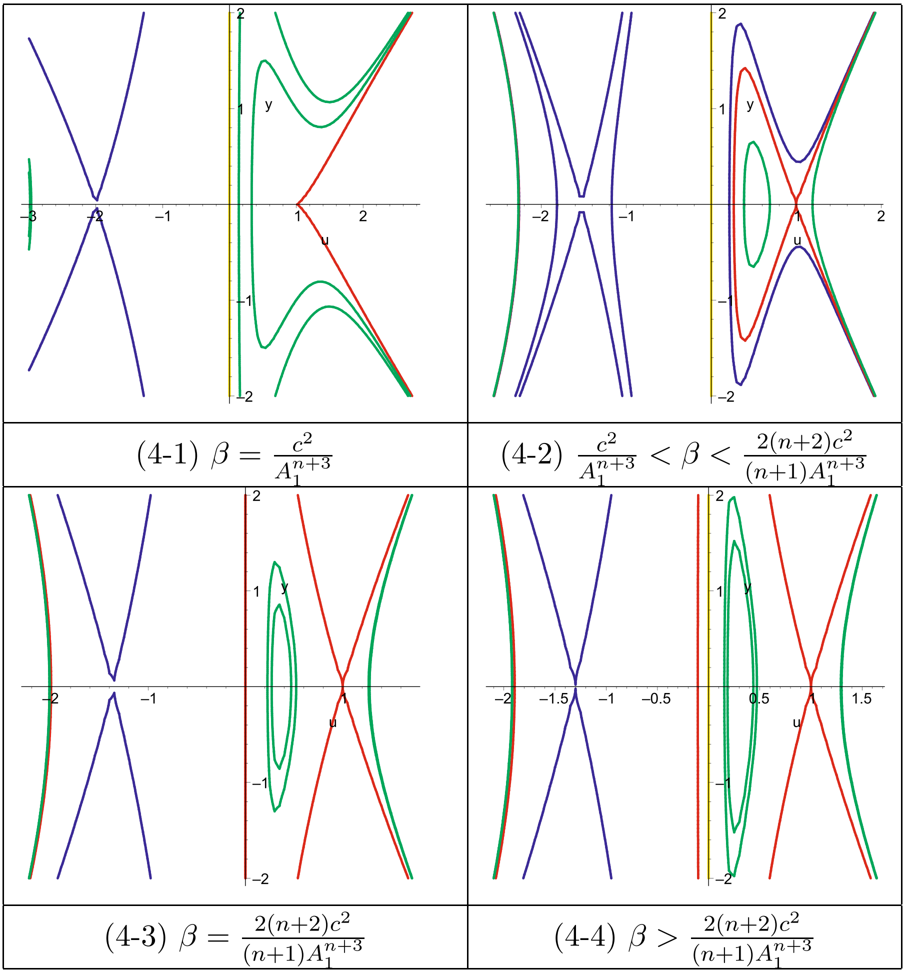


Fig. 4 Phase portraits of system (2.15) for $n = 2m$ and $\sigma < 0$

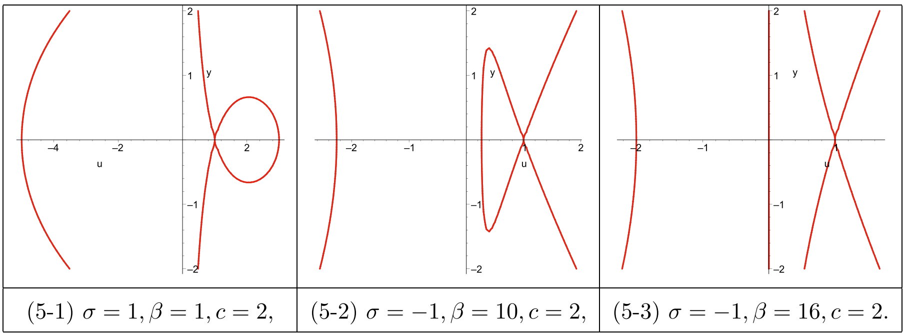
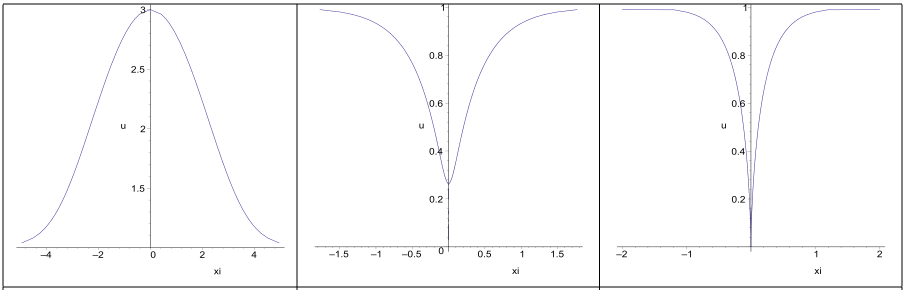


Fig. 5 The invariant algebraic curve $h - H(x, y) = 0$ for $n = 0$ and $A_1 = 1$



(6-1) $\sigma = 1, \beta = 1, c = 2,$ (6-2) $\sigma = -1, \beta = 10, c = 2,$ (6-3) $\sigma = -1, \beta = 16, c = 2.$

Fig. 6 The graphs of functions $\varphi(\xi)$ for $n = 0$ and $A_1 = 1$

where

$$I_3(\varphi) = -\ln|\frac{2\varphi - B_1 - B_2 + 2\sqrt{(\varphi - B_2)(\varphi - B_1)}}{B_1 - B_2}|, \tag{4.42}$$

$$I_4(\varphi) = \ln|\frac{-A_1B_1 - A_1B_2 + 2B_1B_2 - \varphi B_1 - \varphi B_2 + 2\varphi A_1 + 2\sqrt{(A_1 - B_2)(A_1 - B_1)(\varphi - B_2)(\varphi - B_1)}}{(B_2 - B_1)(\varphi - A_1)}|. \tag{4.43}$$

The profile of smooth solitary wave solution $\varphi(\xi)$ and $\phi(\xi)$ is shown in Fig. 6(6-2) and Fig. 7(7-2), respectively.

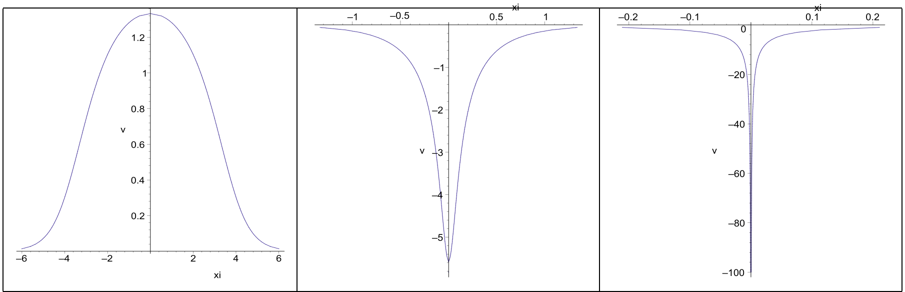
Case III $: \sigma < 0, \beta = \frac{4c^2}{A_1^3}.$

In this case, we have

$$B_2 < 0 = B_1 < A_1, \quad \varphi(0) = 0, \quad 0 \leqslant \varphi < A_1.$$

The invariant algebraic curve $h - H(x, y) = 0$ determines a cuspon solution satisfying

$$\varphi(0) = 0, \quad \lim_{\xi \to \pm\infty} \varphi(\xi) = A_1, \quad \varphi'(-0) = -\infty, \quad \varphi'(+0) = +\infty.$$



(7-1) $\sigma = 1, \beta = 1, c = 2,$ (7-2) $\sigma = -1, \beta = 10, c = 2,$ (7-3) $\sigma = -1, \beta = 16, c = 2.$

Fig. 7 The graphs of functions $\phi(\xi)$ for $n = 0$ and $A_1 = 1$

By using the first equation of system (2.14) to do the integration, we have

$$\int_0^{\varphi} \frac{\sqrt{z}dz}{(A_1 - z)\sqrt{z - B_2}} = \frac{\sqrt{\beta}}{2\sqrt{-\sigma}} \int_0^{\xi} d\xi. \tag{4.44}$$

Thus we obtain the following implicit expression of the cuspon solution.

$$I_5(\varphi) + \frac{\sqrt{A_1(A_1 - B_2)}}{A_1 - B_2} I_6(\varphi) = \frac{\sqrt{\beta}}{2\sqrt{\sigma}}|\xi|, \tag{4.45}$$

where

$$I_5(\varphi) = -\ln|\tfrac{2\varphi - B_2 + 2\sqrt{\varphi(\varphi - B_2)}}{B_2}|, \tag{4.46}$$

$$I_6(\varphi) = \ln|\tfrac{-A_1B_2 - \varphi B_2 + 2\varphi A_1 + 2\sqrt{A_1\varphi(A_1 - B_2)(\varphi - B_2)}}{B_2(\varphi - A_1)}|. \tag{4.47}$$

The profile of cuspon solution $\varphi(\xi)$ and unbounded solution $\phi(\xi)$ is shown in Fig. 6(6-3) and Fig. 7(7-3), respectively.

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