International Journal of Bifurcation and Chaos, Vol. 25, No. 10 (2015) 1550136 (13 pages)

DOI: 10.1142/S0218127415501369

Bifurcation and Traveling Wave Solutions for the Fokas Equation*

Jibin Li
Department of Mathematics, Zhejiang Normal University,
Jinhua, Zhejiang 321004, P. R. China
Department of Mathematics,
Kunming University of Science and Technology,
Kunming, Yunnan 650093, P. R. China
lijb@zjnu.cn

Zhijun Qiao
Department of Mathematics, The University of Texas Pan-American,
1201 West University Drive, Edinburg, TX 78541, USA
qiao@utpa.edu

Received February 7, 2015

This paper is devoted to discussing bifurcation and traveling wave solutions for the Fokas equation. By investigating the dynamical behavior with phase space analysis, we may derive all possible exact traveling wave solutions, including compactons, cuspons, periodic cusp wave solutions, and smooth solitary wave solutions.

Keywords: Bifurcation; compacton; cuspon; periodic cusp wave solution; solitary wave solution; traveling wave solution.

1. Introduction

In 1995, Fokas studied the following integrable generalization of the sine-Gordon (sG) equation using the bi-Hamiltonian approach (see [Fokas, 1995]):

$$u_{tx} = (1 + \nu\partial_x^2)\sin(u), \quad x \in \mathbf{R}, \; t > 0, \tag{1}$$

where ν is a real parameter and $u(x,t)$ is a scalar-valued function. For our convenience, we call Eq. (1) the Fokas equation in this paper. Fifteen years later, Lenells and Fokas [2010] showed the integrability of the Fokas equation through a Lax pair and conservation laws, solved its initial-value problem, and analyzed its solitons and traveling wave solutions. Unfortunately, the dynamical behavior of the traveling wave system for the Fokas equation (1) is not studied yet in the literature. Neither explicit traveling wave solutions nor bifurcations are presented when some parameters vary.

To investigate the traveling wave solution of the Fokas equation (1), let $u(x,t) = u(x - ct) = \phi(\xi)$, where $\xi = x - ct$, c is the wave speed. Substituting it into the Fokas equation (1) yields

$$-c\phi_{\xi\xi} = \sin(\phi) + \nu(-\phi_\xi^2\sin(\phi) + \phi_{\xi\xi}\cos(\phi)), \tag{2}$$

which is equivalent to the following planar dynamical system

$$\frac{d\phi}{d\xi} = y, \quad \frac{dy}{d\xi} = \frac{(\nu y^2 - 1)\sin(\phi)}{c + \nu\cos(\phi)}, \tag{3}$$

*This research work was partially supported by the National Natural Science Foundation of China (11471289, 11162020).

with the first integral

$$H(\phi, y) = y^2(c + \nu\cos(\phi))^2 - \left(\nu\cos^2(\phi) + 2c\cos(\phi) - \frac{1}{2}\nu\right) = h. \tag{4}$$

Apparently, the system (3) is 2π-periodic in ϕ. Therefore, the pair (ϕ, y) can be viewed on a phase cylinder $S^1 \times R$, where $S^1 = [-\pi, \pi]$ and $-\pi$ is identified with π (see Fig. 1).

To understand the properties of the traveling wave solutions of the Fokas equation (1), it is necessary to find all possible parametric representations for the system (3). In this paper, we use the method of dynamical systems to investigate the dynamics of solutions to the system (3) and give the parametric representations of all bounded orbits with variance of parameter ratio $\frac{c}{\nu}$, where we assume that the parameter (wave speed) $c > 0$ is fixed.

Apparently, when $\nu > 0$, two straight lines $y = \pm Y_0 = \pm\frac{1}{\sqrt{\nu}}$ are solutions to the system (3).

In addition, when $|\frac{c}{\nu}| \leq 1$, the system (3) is a singular traveling wave system of the first class (see [Li & Chen, 2007; Li, 2013; Li & Qiao, 2013]) with the singular straight line $\phi = \pm\phi_s \equiv \pm\arccos(-\frac{c}{\nu})$. In fact, the existence of a singular straight line leads to a dynamical behavior in two scaling variables. In the above three references, we showed that for a singular nonlinear traveling wave system of the first class, the following two results hold.

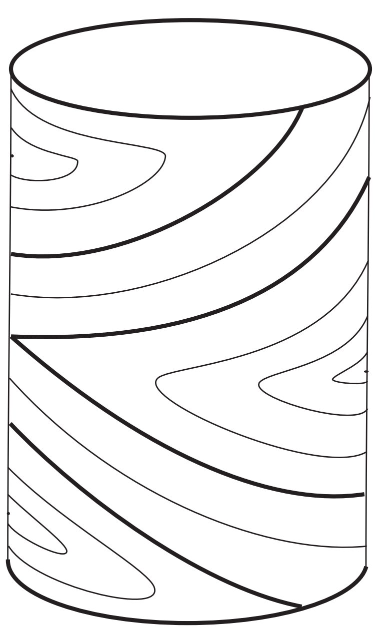


Fig. 1. The phase cylinder $S^1 \times R$ of system (3).

Theorem A (The Rapid-Jump Property of the Derivative near the Singular Straight Line). *Suppose that in a left (or right) neighborhood of a singular straight line, there exists a family of periodic orbits such that along a segment of every orbit near the straight line, the derivative of the wave function jumps down rapidly in a very short time interval.*

Theorem B (Existence of Finite Time Interval(s) of Solutions with Respect to Variables in the Positive or (and) Negative Direction(s)). *For a singular nonlinear traveling wave system of the first class with a possible change of the wave variable, if an orbit transversely intersects to a singular straight line at a point or it approaches the singular straight line but the derivative tends to infinity, then it only takes a finite time interval of the wave variable to make the moving point of the orbit arrive on the singular straight line.*

By Theorem A, all periodic orbits of the system (3), which have segments close to the singular straight line $\phi = \pm\phi_s$ in a period annulus, give rise to periodic cusp wave solutions. These periodic cusp wave solutions have smooth wave profiles, because in a neighborhood of the singular straight line $\phi = \pm\phi_s$, there exist two "time scaling" of wave variables, such that cusp wave profiles appear.

The main result of this paper is as follows.

Theorem 1. *The traveling wave system of the Fokas equation (1) is a dynamical system in a phase cylinder $S^1 \times R$.*

(1) *If $1 < |\frac{c}{\nu}| < \infty$, the Fokas equation (1) has three families of smooth periodic wave solutions given by (8) and (10) as well as two smooth solitary solutions given by (9), which correspond to two heteroclinic orbits of the system (3) in the expanding phase plane given by $H(\phi, y) = h_\pi$.*
(2) *If $\frac{c}{\nu} = -1$, the Fokas equation (1) has two families of compactons given by (11), a cuspon given by (12), and an anti-cuspon given by (13).*
(3) *If $-1 < \frac{c}{\nu} < 0$, the Fokas equation (1) has two or four families of compactons given by (14), (15), (18), (19) and (21), and cuspons given by (16), (17) and (20).*
(4) *If $0 < \frac{c}{\nu} < 1$, the Fokas equation (1) has a family of smooth periodic wave solutions given by (8) and (22), two families of periodic cusp*

wave solutions given by (23), and two or four families of compactons, out of which two parametric representations are given by (25).

(5) *If $\frac{c}{\nu} = 1$, the Fokas equation (1) has a family of periodic wave solutions given by (8) and two families of rotating periodic wave solutions given by (10).*

The proof of the theorem will be seen in the remaining parts of the paper. This paper is organized as follows. In Sec. 2, we consider the bifurcations of phase portraits of system (3). In Secs. 3 and 4, for the two cases of $\nu < 0$ and $\nu > 0$, we give all possible exact solutions of $\phi(\xi)$ under different parameter conditions and different h values defined by $H(\phi, y) = h$ in (4).

2. Bifurcations of Phase Portraits

Let us consider the following regular system associated with (3)

$$\frac{d\phi}{d\zeta} = y(c + \nu\cos(\phi)), \quad \frac{dy}{d\zeta} = (\nu y^2 - 1)\sin(\phi). \tag{5}$$

It has the same level curves as (4), where $d\xi = (c + \nu\cos(\phi))d\zeta$. The dynamics of the two systems (3) and (5) is different in the neighborhood of the straight line $\phi = \pm\phi_s$. Specially, the variable "ζ" is regarded as a fast variable while the variable "ξ" is slower in the sense of the geometric singular perturbation theory (see [Li, 2013]).

One may easily see that the system (5) always has three equilibrium points $E_1(0,0)$, $E_2(\pm\pi, 0)$, which may be identified in the phase cylinder $S^1 \times R$. When $\nu > 0$, $0 < \frac{c}{\nu} < 1$, the system (5) has four equilibrium points $E_{3\pm}(-\phi_s, \pm Y_0)$ and $E_{4\pm}(\phi_s, \pm Y_0)$.

Let $M(\phi_j, 0)$ be the coefficient matrix of the linearized system of (5) at the equilibrium point E_j. We have

$$J(0,0) = \det M(0,0) = \nu + c,$$

$$J(\pi,0) = \det M(\pi,0) = \nu - c,$$

$$\begin{aligned} J(\pm\phi_s, \pm Y_0) &= \det M(\pm\phi_s, \pm Y_0) \\ &= -2\nu^2 Y_0^2 \sin^2(\pm\phi_s) < 0. \end{aligned}$$

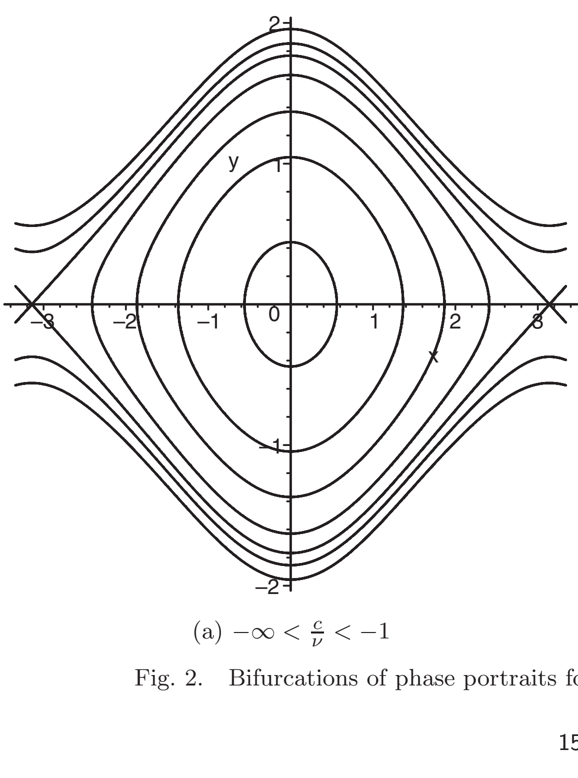
Let

$$h_0 = H(0,0) = -\frac{1}{2}(\nu + 4c),$$

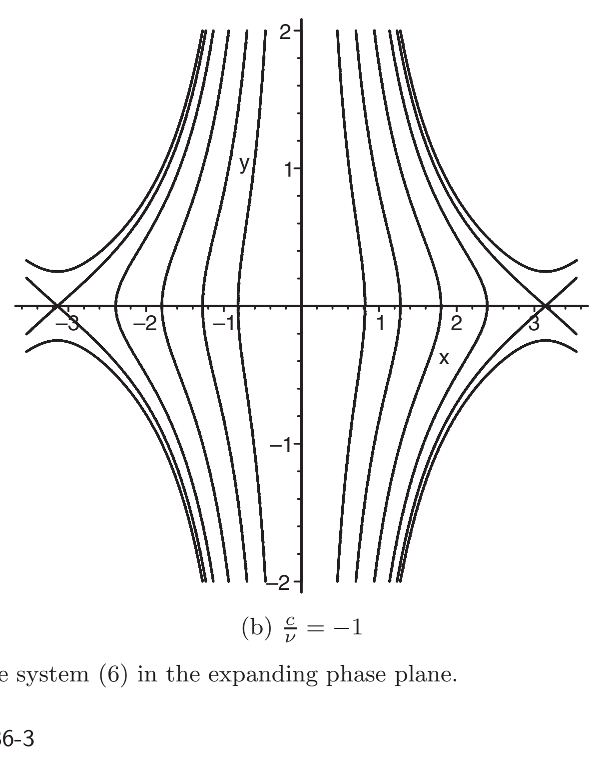
$$h_\pi = H(\pi, 0) = \frac{1}{2}(4c - \nu),$$

$$h_s = H(\pm\phi_s, \pm Y_0) = \frac{c^2}{\nu} + \frac{1}{2}\nu.$$

By the above information, we may do a qualitative analysis and have the following bifurcations of the phase portraits for the system (5), which are shown in Figs. 2(a)–2(f).

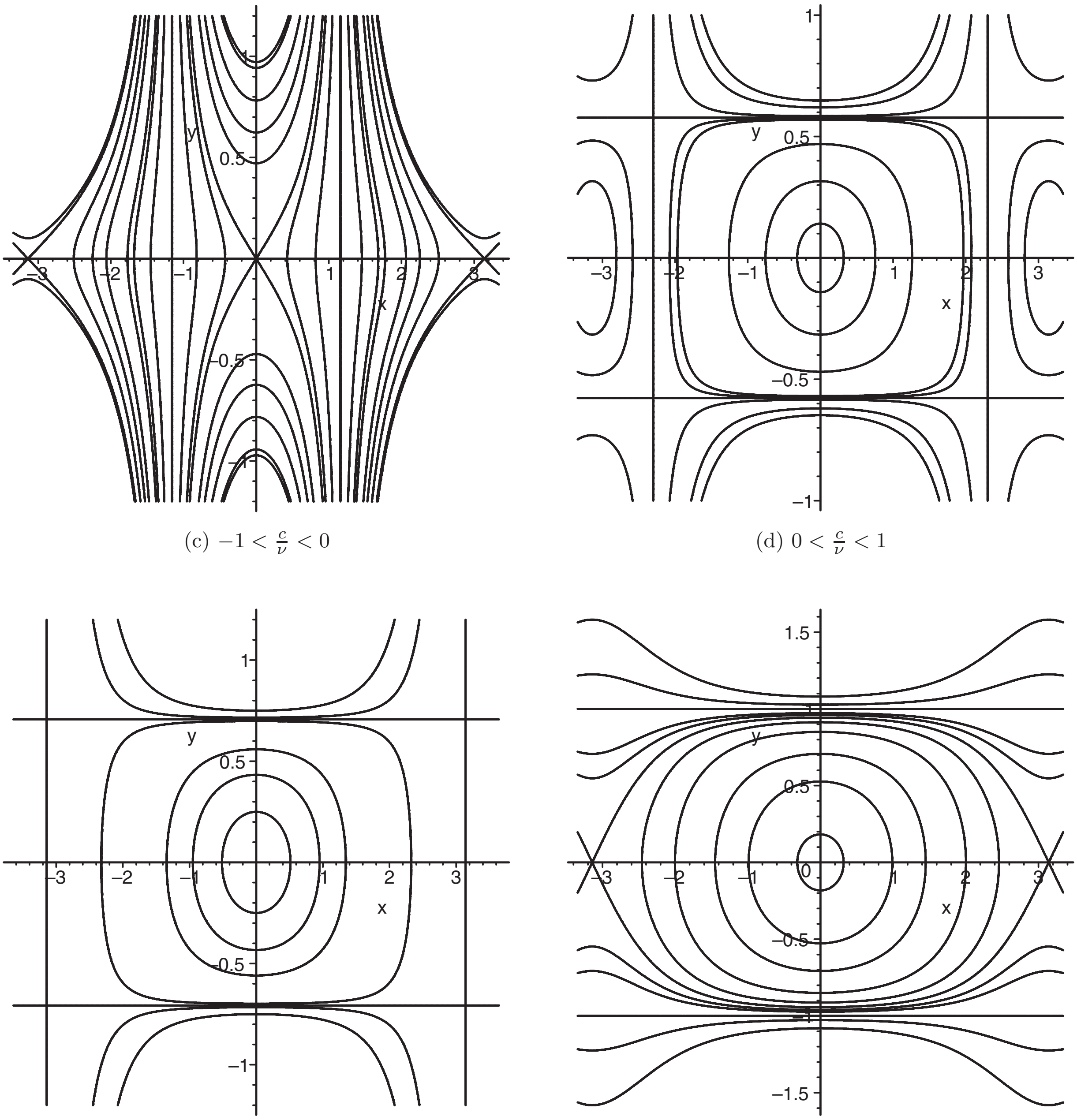


(a) $-\infty < \frac{c}{\nu} < -1$



(b) $\frac{c}{\nu} = -1$

Fig. 2. Bifurcations of phase portraits for the system (6) in the expanding phase plane.



(c) $-1 < \frac{c}{\nu} < 0$ (d) $0 < \frac{c}{\nu} < 1$

(e) $\frac{c}{\nu} = 1$ (f) $1 < \frac{c}{\nu} < \infty$

Fig. 2. (*Continued*)

3. Traveling Wave Solutions of the Fokas Equation in the Case of $\frac{c}{\nu} < 0$

Without loss of generality, we assume that c is fixed and $c > 0$. We know from (4) that

$$y^2 = \frac{h + \nu\cos^2(\phi) + 2\cos(\phi) - \frac{1}{2}\nu}{(c + \nu\cos(\phi))^2}.$$

Thus, by using the first equation of (10), we have

$$\xi = \int_{\phi_0}^{\phi} \frac{(c + \nu\cos(\phi))d\phi}{\sqrt{h - \frac{1}{2}\nu + 2\cos(\phi) + \nu\cos^2(\phi)}}. \tag{6}$$

Letting $\psi = \tan(\frac{\phi}{2})$, (6) becomes

$$\xi = 2(c-\nu)\int_{\psi_0}^{\psi} \frac{d\psi}{\sqrt{A + C\psi^2 - E\psi^4}} + 4\nu\int_{\psi_0}^{\psi} \frac{d\psi}{(1+\psi^2)\sqrt{A + C\psi^2 - E\psi^4}}, \tag{7}$$

where

$$A = h + \frac{1}{2}\nu + 2c = h - h_0,$$

$$C = 2\left(h - \frac{3}{2}\nu\right),$$

$$E = h + \frac{1}{2}\nu - 2c = h_\pi - h.$$

Some technical calculations of (7) will yield exact traveling wave solutions of the Fokas equation (1).

3.1. *Case* $-\infty < \frac{c}{\nu} < 1$ [*see Fig. 2(a)*]

In this case, the system (3) has no singular straight line.

(i) Corresponding to the family of oscillating orbits enclosing the origin $E_1(0,0)$ defined by $H(\phi, y) = h$, $h \in (h_0, h_\pi)$, there exists a family of periodic wave solutions to the Fokas equation (1). Let us now write the function

$$A + C\psi^2 - E\psi^4 \quad \text{as } E(a^2 + \psi^2)(b^2 - \psi^2),$$

where

$$a^2 = \frac{1}{2E}(-C + \sqrt{C^2 + 4AE}),$$

$$b^2 = \frac{1}{2E}(C + \sqrt{C^2 + 4AE}).$$

Then, (7) implies that the Fokas equation (1) has the following parametric representations of the family of smooth periodic wave solutions:

$$\begin{aligned}\phi(\chi) &= 2\arctan(\psi(\chi)) \\ &= 2\arctan(b\mathrm{cn}(\chi, k)), \\ \xi(\chi) &= \frac{2}{(C^2 + 4AE)^{\frac{1}{4}}}\left((c - \nu)\chi \right. \\ &\quad \left. + \frac{2\nu}{1 + b^2}\Pi(\arccos(\mathrm{cn}(\chi, k)), \alpha^2, k)\right),\end{aligned} \tag{8}$$

where $k^2 = \frac{b^2}{a^2+b^2}$, $\alpha^2 = \frac{b^2}{1+b^2}$, $\Pi(\cdot, \alpha^2, k)$ is the elliptic integral of the third kind, and $\mathrm{sn}(u, k), \mathrm{cn}(u, k)$ are the Jacobian elliptic functions (see [Byrd & Fridman, 1971]).

(ii) When $h = h_\pi$, the level curves defined by $H(\phi, y) = h_\pi$ are two homoclinic orbits of the system (3) in the phase cylinder $S^1 \times R$. Usually, in the expanding phase plane, the two curves are heteroclinic orbits connecting to two equilibrium points $E_2(-\pi, 0)$ and $E_2(\pi, 0)$. Now, we have

$$A + C\psi^2 - E\psi^4 = 4c + 4(c - \nu)\psi^2.$$

By (7), we know that the two homoclinic orbits have the following parametric representations:

$$\begin{aligned}\phi(\chi) &= \pm 2\arctan(\psi(\chi)) \\ &= 2\arctan\left(\sqrt{\frac{c}{c - \nu}}\sinh(\chi)\right), \\ \xi(\chi) &= \sqrt{c - \nu}\chi \\ &\quad - 2\sqrt{|\nu|}\,\mathrm{arctanh}\left(\sqrt{\frac{|\nu|}{c - \nu}}\tanh(\chi)\right),\end{aligned} \tag{9}$$

which yield two solitary wave solutions of the Fokas equation (1). Without identification of $E_2(-\pi, 0)$ and $E_2(\pi, 0)$, they are kink and anti-kink wave solutions.

(iii) If $h \in (h_\pi, \infty)$, the level curves defined by $H(\phi, y) = h$ are two families of rotating periodic orbits of the system (3) in the phase cylinder $S^1 \times R$. Since $E < 0$ now, the function

$$A + C\psi^2 - E\psi^4$$

can be rewritten as

$$|E|(a^2 + \psi^2)(b^2 + \psi^2),$$

where

$$a^2 = \frac{1}{2|E|}(C + \sqrt{C^2 - 4A|E|}) \quad \text{and}$$

$$b^2 = \frac{1}{2|E|}(C - \sqrt{C^2 - 4A|E|}).$$

Thus, (7) reads as the following parametric representation:

$$\begin{aligned}\phi(\chi) &= \pm 2\arctan(\psi(\chi)) = \pm 2\arctan(b\mathrm{tn}(\chi, k)), \\ &\qquad \chi \in (-2K(k), 2K(k)), \\ \xi(\chi) &= \frac{2(c - \nu)}{a\sqrt{|E|}}\chi \\ &\quad - \frac{a}{b^4\sqrt{|E|}}\mathrm{dn}(F(\arctan(\mathrm{tn}(\chi, k)), k), k) \\ &\quad \times \mathrm{tn}(F(\arctan(\mathrm{tn}(\chi, k)), k)) \\ &\quad + \frac{1}{ab^2\sqrt{|E|}}E(\arctan(\mathrm{tn}(\chi, k)), k),\end{aligned} \tag{10}$$

where $k^2 = \frac{a^2-b^2}{a^2}$, $F(\cdot,k)$, $E(\cdot,k)$ are the elliptic integral of the first and second kinds, and $\mathrm{tn}(u,k), \mathrm{dn}(u,k)$ are the Jacobian elliptic functions (see [Byrd & Fridman, 1971]). Equation (10) gives rise to two families of periodic wave solutions of the Fokas equation (1).

3.2. *Case* $\frac{c}{\nu} = -1$ [*see Fig. 2(b)*]

In this case, we have $\nu = -c$, $h_0 = h_s = -\frac{3}{2}c$, $h_\pi = \frac{5}{2}c$. The singular straight line $\phi = \phi_s = 0$ passes though the origin $E_1(0,0)$.

(i) The level curves defined by $H(\phi,y) = h$, $h \in (h_0, h_\pi)$ are two families of open orbits for the system (3), which lie in two areas among the straight line $\phi = 0$ and two manifolds of the saddle points $E_2(\pm\pi,0)$ either stable or unstable. By Theorem B, these orbits generate two families of compactons for the Fokas equation (1) [see Fig. 3(a)].

The parametric representations of the two families of compactons in Fig. 3(a) are provided by

$$\phi(\chi) = \pm 2\arctan(\psi(\chi)) = \pm 2\arctan(b\,\mathrm{cn}(\chi,k)),$$

$$\xi(\chi) = \frac{4c}{(C^2+4AE)^{\frac{1}{4}}} \times \left(\chi - \frac{1}{1+b^2}\Pi(\arccos(\mathrm{cn}(\chi,k)),\alpha^2,k)\right),$$

$$\chi \in (-K(k), K(k)), \tag{11}$$

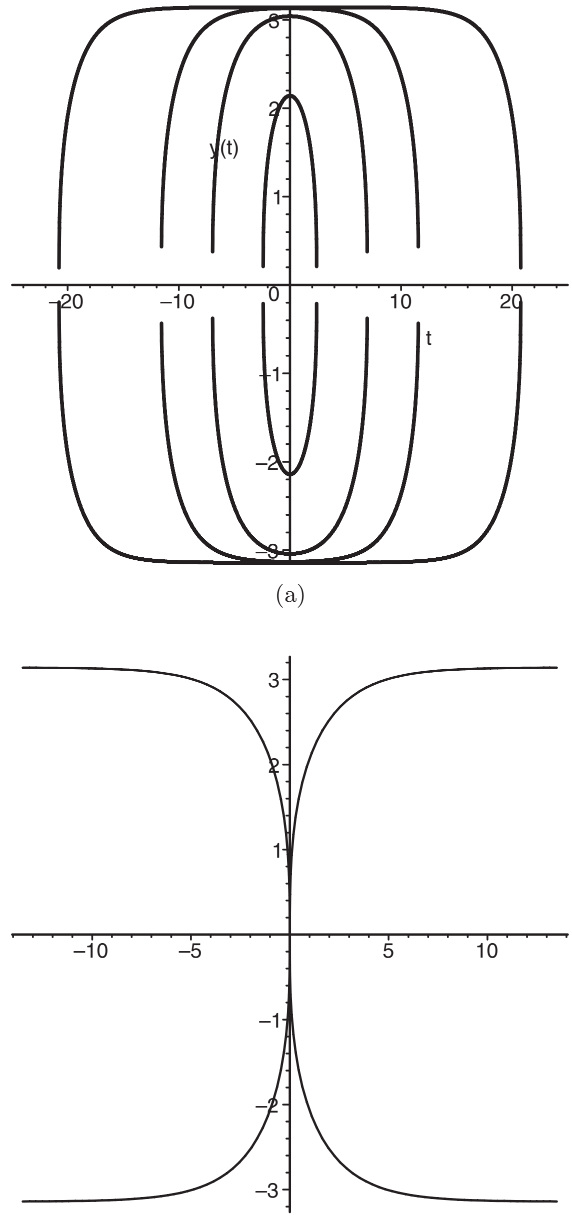
where k and a^2, b^2, α^2 are the same as (8).

(ii) The level curves defined by $H(\phi,y) = h_\pi$ stand for the stable and unstable manifolds of the two saddle points $E_2(\pi,0)$ and $E_2(-\pi,0)$ for the system (3). The unstable manifold of the saddle point $E_2(-\pi,0)$ has the following parametric representation:

$$\phi(\chi) = 2\arctan(\psi(\chi)) = 2\arctan\left(\frac{\sqrt{2}}{2}\sinh(\chi)\right), \quad \chi \in (-\infty,0)$$

$$\xi(\chi) = \sqrt{2c}\,\chi - 2\sqrt{c}\,\mathrm{arctanh}\left(\frac{\sqrt{2}}{2}\tanh(\chi)\right), \tag{12}$$

while the stable manifold of the saddle point $E_2(-\pi,0)$ has parametric representation as follows:



(a)

(b)

Fig. 3. The profiles of two compacton families and cuspons of the Fokas equation (1). (a) Two compacton families and (b) cuspon and anti-cuspon.

$$\phi(\chi) = -2\arctan(\psi(\chi)) = -2\arctan\left(\frac{\sqrt{2}}{2}\sinh(\chi)\right), \quad \chi \in (0,\infty)$$

$$\xi(\chi) = \sqrt{2c}\,\chi - 2\sqrt{c}\,\mathrm{arctanh}\left(\frac{\sqrt{2}}{2}\tanh(\chi)\right). \tag{13}$$

By (12) and (13), we may draw their figures, which are called cuspons of the Fokas equation (1) [see

Fig. 3(b), the lower curves]. Similarly, the stable and unstable manifold of the saddle point $E_2(\pi, 0)$ yield an anti-cuspon of the Fokas equation (1) [see Fig. 3(b), the upper curves].

3.3. *Case* $-1 < \frac{c}{\nu} < 0$ [*see Fig. 2(c)*]

In this case, we have $\nu < -c < 0$, $0 < \phi_s < \frac{\pi}{2}$, $h_s < h_0 < h_\pi$. As h varies, the level curves defined by $H(\phi, y) = h$ are changed as well. Their graphs are shown in Figs. 4(a)–4(e), where we also draw two singular straight lines $\phi = \pm\phi_s$.

(i) If $h \in (h_s, h_0)$, there exist four families of open orbits for the system (3) corresponding to the level curves defined by $H(\phi, y) = h$ [see Fig. 4(a)]. By Theorem B, these orbits give rise to four families of compactons for the Fokas equation (1) [see Fig. 5(a)].

Corresponding to the open orbit on the right-hand side of the singular straight line $\phi = \phi_s$ of Fig. 4(a), we have

$$A + C\psi^2 - E\psi^4 = (a^2 - \psi^2)(\psi^2 - b^2),$$

where

$$a^2 = \frac{1}{2E}(C + \sqrt{C^2 + 4AE}),$$

$$b^2 = \frac{1}{2E}(C + \sqrt{C^2 - 4AE}).$$

Hence, this curve has the following parametric representation:

$$\begin{aligned} \phi(\chi) &= 2\arctan(\psi(\chi)) = 2\arctan(a\,\mathrm{dn}(\chi, k)), \\ \xi(\chi) &= \frac{2}{a}\Bigg((c-\nu)\chi \\ &\quad + \frac{2\nu}{1+a^2}\Pi(\arcsin(\mathrm{sn}(\chi, k)), \alpha_1^2, k)\Bigg), \\ &\qquad \chi \in (-\chi_s, \chi_s), \end{aligned} \tag{14}$$

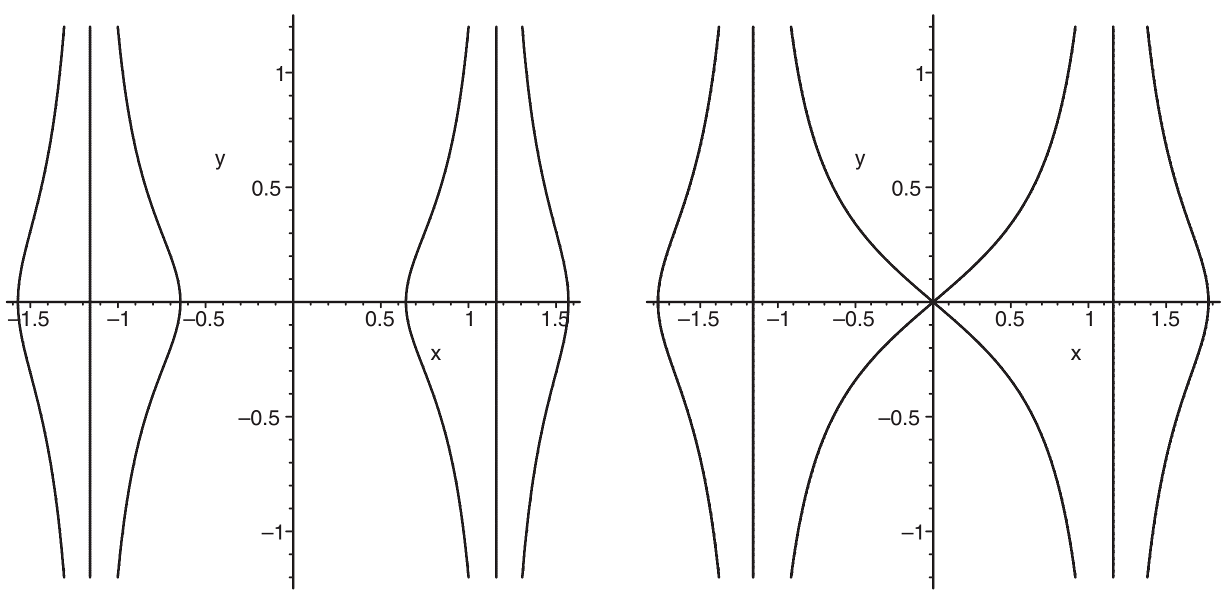
where $k^2 = \frac{a^2-b^2}{a^2}$, $\alpha_1^2 = \frac{a^2-b^2}{1+a^2}$, and χ_s satisfies $2\arctan(a\,\mathrm{dn}(\chi_s, k)) = \phi_s$.

Corresponding to the open orbit on the left-hand side of the singular straight line $\phi = \phi_s$ of Fig. 4(a), we have

$$\begin{aligned} \phi(\chi) &= 2\arctan(\psi(\chi)) \\ &= 2\arctan(b\mathrm{nd}(\chi, k)), \quad \chi \in (-\chi_s, \chi_s) \\ \xi(\chi) &= \frac{2}{a}((c-\nu)\chi + 2\nu(1+b^2) \\ &\quad \times F(\arcsin(\mathrm{sn}(\chi, k)), k) \\ &\quad - b^2\Pi(\arcsin(\mathrm{sn}(\chi, k)), \alpha_2^2, k)), \end{aligned} \tag{15}$$

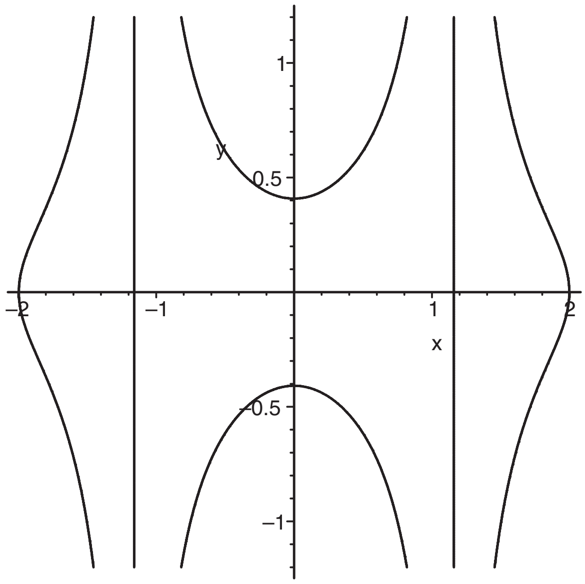
where $k^2 = \frac{a^2-b^2}{a^2}$, $\alpha_2^2 = \frac{k^2}{1+b^2}$, and χ_s satisfies $2\arctan(b\mathrm{nd}(\chi_s, k)) = \phi_s$.

By the symmetric property in Fig. 4(a), we may directly get the parametric representations of

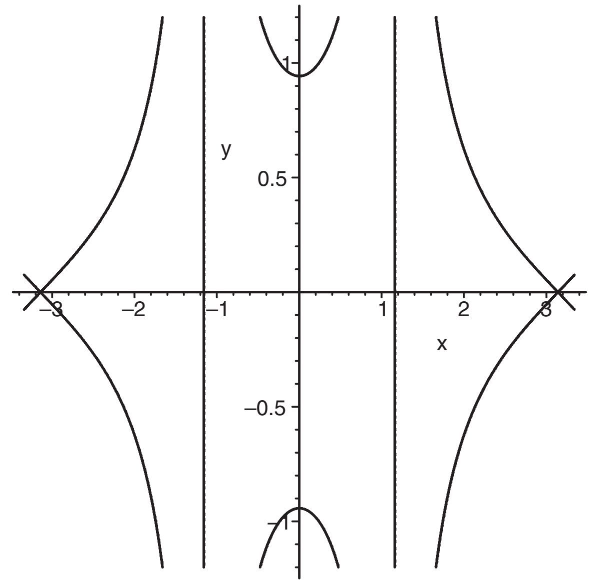


(a) $h \in (h_s, h_0)$ (b) $h = h_0$

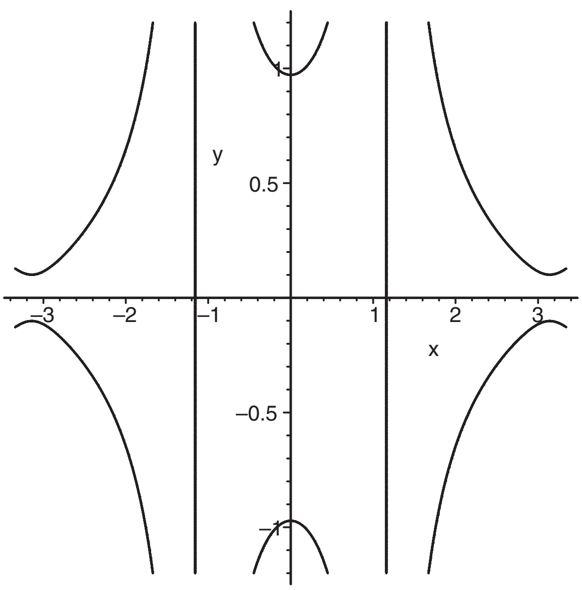
Fig. 4. Graphs of the level curves defined by $H(\phi, y) = h$.



(c) $h \in (h_0, h_\pi)$



(d) $h = h_\pi$



(e) $h \in (h_\pi, \infty)$

Fig. 4. (*Continued*)

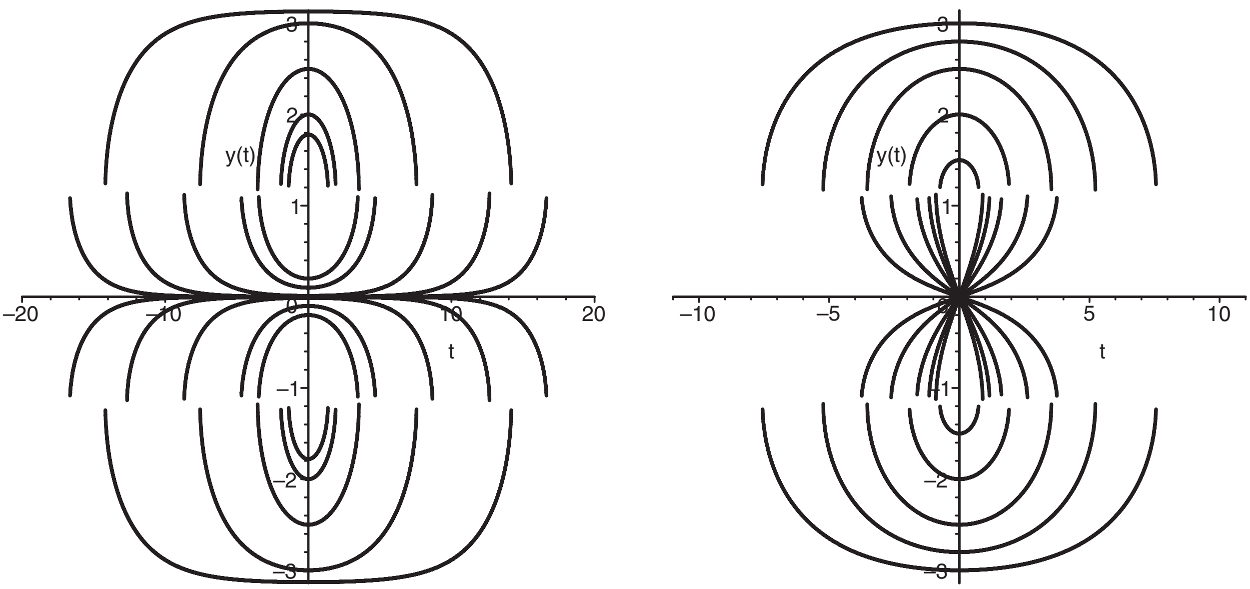
the other two families of compactons for the Fokas equation (1).

(ii) When $h = h_0$, the level curves defined by $H(\phi, y) = h_0$ are two stable and two unstable manifolds to the saddle points $E_1(0,0)$ and two open curves [see Fig. 4(b)].

The right unstable manifold of the saddle points $E_1(0,0)$ has the following parametric representation

$$\begin{aligned}
\phi(\chi) &= 2\arctan(\psi(\chi)) \\
&= 2\arctan(M\operatorname{sech}(\chi)), \quad \chi \in (-\infty, -\chi_s)
\end{aligned}$$

$$\begin{aligned}
\xi(\chi) = {} & \frac{c-\nu}{\sqrt{|\nu|-c}}\chi \\
& + \frac{2\nu}{\sqrt{|\nu|-c}}\ln(2(M\cosh(\chi)+\sinh(\chi))) \\
& + 2\sqrt{|\nu|}\operatorname{arctanh}\left(\sqrt{\frac{|\nu|-c}{|\nu|}}\tanh(\chi)\right) \\
& - \Psi_0,
\end{aligned} \tag{16}$$



(a) $h \in (h_s, h_0)$

(b) $h \in (h_0, h_\pi)$

Fig. 5. The wave profiles of four compacton families defined by $H(\phi, y) = h$.

where

$$M = \sqrt{\frac{|\nu| - c}{c}},$$

$$\begin{aligned}\Psi_0 = {} & \frac{c-\nu}{\sqrt{|\nu|-c}}\operatorname{sech}^{-1}\left(\frac{\tan\left(\frac{1}{2}\phi_s\right)}{M}\right) \\ & + \frac{2\nu}{\sqrt{|\nu|-c}} \\ & \times \ln\left(\frac{2M^2 + 2M\sqrt{M^2 - \tan^2\left(\frac{1}{2}\phi_s\right)}}{\tan\left(\frac{1}{2}\phi_s\right)}\right) \\ & + 2\sqrt{|\nu|} \\ & \times \operatorname{arctanh}\left(\sqrt{\frac{c}{|\nu|}\left(M^2 - \tan^2\left(\frac{1}{2}\phi_s\right)\right)}\right)\end{aligned}$$

and χ_s satisfies $\phi(\chi_s) = \phi_s$. The right stable manifold of the saddle points $E_1(0, 0)$ has the following parametric representation

$$\begin{aligned}\phi(\chi) &= 2\arctan(\psi(\chi)) \\ &= 2\arctan(M\operatorname{sech}(\chi)), \quad \chi \in (\chi_s, \infty) \\ \xi(\chi) &= \frac{c-\nu}{\sqrt{|\nu|-c}}\chi \\ &\quad - \frac{2\nu}{\sqrt{|\nu|-c}}\ln(2(M\cosh(\chi) + \sinh(\chi))) \\ &\quad - 2\sqrt{|\nu|}\operatorname{arctanh}\left(\sqrt{\frac{|\nu|-c}{|\nu|}}\tanh(\chi)\right) + \Psi_0.\end{aligned} \tag{17}$$

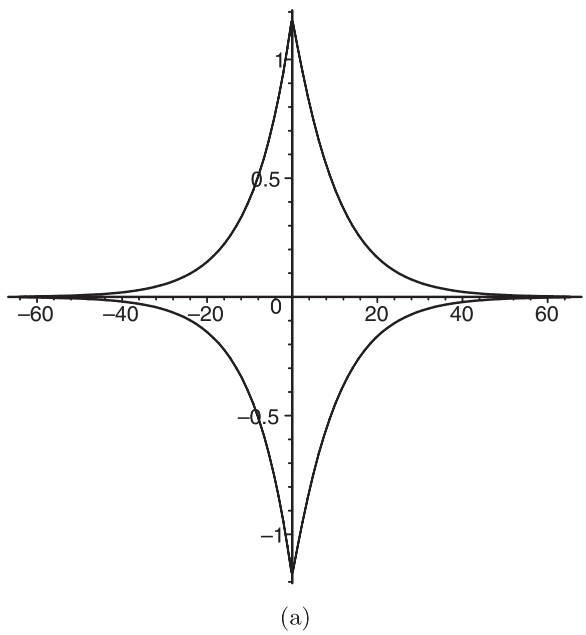
Using (16) and (17) to draw wave profiles leads to a cuspon shown in Fig. 6(a) (upper curve). Making the transformation $\phi \mapsto -\phi$ in (16) and (17), we have the parametric representations of left unstable and stable manifolds of the saddle points $E_1(0, 0)$, which yield an anti-cuspon shown in Fig. 6(a) (lower curve).

In (16) and (17), taking $\chi \in (-\chi_s, 0)$ and $\chi \in (0, \chi_s)$, respectively, we may obtain the parametric representations of two compactons given by two open orbits in Fig. 4(b).

(iii) If $h \in (h_0, h_\pi)$, there exist four families of open orbits of the system (3) corresponding to the level curves defined by $H(\phi, y) = h$ [see Fig. 4(c)]. These orbits give rise to four compacton families of the Fokas equation (1) shown in Fig. 5(b).

The open curves lying on the right-hand side of the singular straight line $\phi = \phi_s$ have the following parametric representations

$$\begin{aligned}\phi(\chi) &= 2\arctan(\psi(\chi)) \\ &= 2\arctan(b\operatorname{cn}(\chi, k)), \quad \chi \in (-\chi_s, \chi_s)\end{aligned}$$



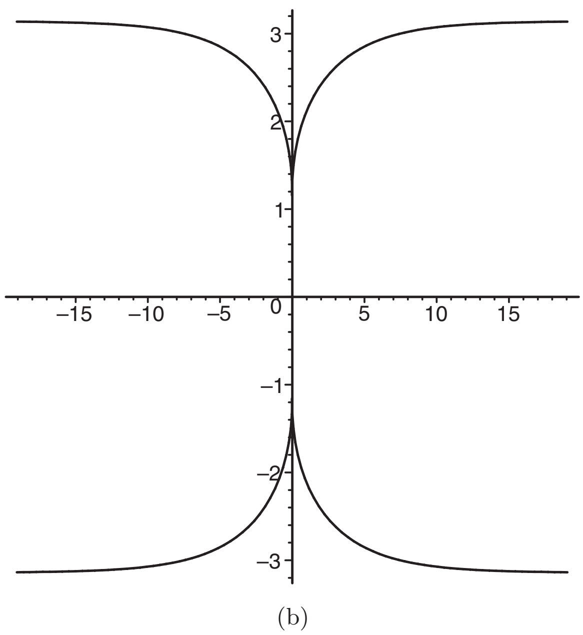


Fig. 6. Four wave profiles of cuspon and anti-cuspon defined by $H(\phi, y) = h_0, h_\pi$. Given by (a) $H(\phi, y) = h_0$ and (b) $H(\phi, y) = h_\pi$.

$$\xi(\chi) = \frac{2}{(C^2+4AE)^{\frac{1}{4}}}\left((c-\nu)\chi + \frac{2\nu}{1+b^2}\Pi(\arccos(\mathrm{cn}(\chi,k)),\alpha^2,k)\right), \tag{18}$$

where k and a^2, b^2, α^2 are the same as (8), and χ_s satisfies $2\arctan(b\mathrm{cn}(\chi_s, k)) = \phi_s$.

The upper open curves lying between the singular straight lines $\phi = -\phi_s$ and $\phi = \phi_s$ have the following parametric representations:

$$\begin{aligned} \phi(\chi) &= 2\arctan(\psi(\chi)) \\ &= 2\arctan(kb\mathrm{sd}(\chi,k)), \quad \chi \in (-\chi_s, \chi_s) \\ \xi(\chi) &= \frac{2}{(C^2+4AE)^{\frac{1}{4}}}\left((c-\nu)\chi + \frac{2\nu}{1-b^2}(F(\arcsin(\mathrm{sn}(\chi,k)),k) - b^2\Pi(\arcsin(\mathrm{sn}(\chi,k)),\alpha^2,k))\right), \end{aligned} \tag{19}$$

where $b^2 = \frac{1}{2E}(-C+\sqrt{C^2+4AE})$, $a^2 = \frac{1}{2E}(C+\sqrt{C^2+4AE})$, $k^2 = \frac{a^2}{a^2+b^2}$, $\alpha^2 = k^2(1-b^2)$, and χ_s satisfies $2\arctan(kb\mathrm{sd}(\chi_s,k)) = \phi_s$.

(iv) If $h = h_\pi$, corresponding to stable and unstable manifolds of the saddle points $E_2(-\pi, 0)$ defined by $H(\phi, y) = h_\pi$, we have the following parametric representations

$$\begin{aligned} \phi(\chi) &= 2\arctan(\psi(\chi)) \\ &= 2\arctan\left(\frac{\sqrt{2}}{2}\sinh(\chi)\right), \quad \chi \in (-\infty, \chi_s) \\ \xi(\chi) &= \pm\frac{1}{\sqrt{2c}}\left((c-\nu)\chi + 2\sqrt{2}\nu\,\mathrm{arctanh}\left(\frac{\sqrt{2}}{2}\sinh(\chi)\right) + \Psi_\pi\right), \end{aligned} \tag{20}$$

where $\Psi_\pi = -(c-\nu)\sinh^{-1}(\sqrt{2}\psi_s) + 2\sqrt{2}\nu \times \mathrm{arctanh}(\frac{\sqrt{2}\psi_s}{\sqrt{2+4\psi_s^2}})$, $\psi_s = \tan(\frac{1}{2}\phi_s)$, and χ_s satisfies $2\arctan(\frac{\sqrt{2}}{2}\sinh(\chi_s)) = \phi_s$.

By the transformation $\phi \mapsto -\phi$ applied to (20), we can have the parametric representations for the stable and unstable manifolds of the saddle points $E_2(\pi, 0)$ defined by $H(\phi, y) = h_\pi$. Using (20) to draw wave profiles, we may obtain the cuspon and anti-cuspon shown in Fig. 6(b).

(v) If $h \in (h_\pi, \infty)$, there exist two families of bounded orbits between two singular straight lines $\phi = \pm\phi_s$ corresponding to the level curves defined by $H(\phi, y) = h$, which lead to two families of compactons for the Fokas equation (1) with the following parametric representations

$$\begin{aligned}\phi(\chi) &= \pm 2\arctan(\psi(\chi)) \\ &= \pm 2\arctan(b\mathrm{tn}(\chi, k)), \quad \chi \in (-\chi_s, \chi_s), \\ \xi(\chi) &= \frac{2(c-\nu)}{a\sqrt{|E|}}\chi - \frac{a}{b^4\sqrt{|E|}} \\ &\quad \times \mathrm{dn}(F(\arctan(\mathrm{tn}(\chi, k)), k), k) \\ &\quad \times \mathrm{tn}(F(\arctan(\mathrm{tn}(\chi, k)), k)) \\ &\quad + \frac{1}{ab^2\sqrt{|E|}}E(\arctan(\mathrm{tn}(\chi, k)), k),\end{aligned} \tag{21}$$

where a^2, b^2, k^2 are the same as (10), and χ_s satisfies $2\arctan(b\mathrm{tn}(\chi_s, k)) = \phi_s$.

4. Exact Traveling Wave Solutions of the Fokas Equation (1) in the Case of $\frac{c}{\nu} > 0$

Let us discuss the following three subcases to give exact traveling wave solutions of the Fokas equation (1) when $\frac{c}{\nu} > 0$.

4.1. *Subcase* $0 < \frac{c}{\nu} < 1$ [*see Fig. 2(d)*]

In this case, the equilibrium points $E_1(0, 0)$ and $E_2(\pm\pi, 0)$ are center points and we have $h_0 < h_\pi < h_s$, and $\frac{\pi}{2} < \phi_s < \pi$.

(i) If $h \in (h_0, h_\pi]$, the system (3) has one family of periodic orbits, enclosing the equilibrium points $E_1(0, 0)$ and the orbit defined by $H(\phi, 0) = h$. It gives rise to a family of periodic wave solutions of the Fokas equation (1), which has the same parametric representation as (8). In particular, when $h = h_\pi$, $A = 4c$, $C = c - \nu < 0$, $E = 0$, the periodic orbit defined by $H(\phi, y) = h_\pi$ has the following parametric representation

$$\begin{aligned}\phi(\chi) &= 2\arctan(\psi(\chi)) \\ &= 2\arctan\left(\sqrt{\frac{c}{\nu - c}}\sin(\chi)\right), \\ \xi(\chi) &= -\sqrt{\nu - c}\,\chi \\ &\quad + 2\sqrt{\nu}\arctan\left(\sqrt{\frac{\nu}{\nu - c}}\tan(\chi)\right).\end{aligned} \tag{22}$$

(ii) If $h \in (h_\pi, h_s)$, the system (3) has two families of periodic orbits, enclosing the equilibrium points $E_1(0, 0)$ and $E_2(\pm\pi, 0)$ and the orbits defined by $H(\phi, y) = h$. Since $E < 0$, the function $A + C\psi^2 - E\psi$ can be written as $|E|(a^2 - \psi^2)(b^2 - \psi^2)$, where $a^2 = a^2 = \frac{1}{2|E|}(-C + \sqrt{C^2 - 4A|E|})$, and $b^2 = \frac{1}{2E}(-C - \sqrt{C^2 - 4A|E|})$.

The family of periodic solutions enclosing the center $E_1(0, 0)$ has the following parametric representation

$$\begin{aligned}\phi(\chi) &= 2\arctan(\psi(\chi)) = 2\arctan(b\mathrm{sn}(\chi, k)), \\ \xi(\chi) &= \frac{2}{a\sqrt{|E|}}(-(\nu - c)\chi \\ &\quad + 2\nu\Pi(\arcsin(\mathrm{sn}(\chi, k)), \alpha^2, k)),\end{aligned} \tag{23}$$

where $k^2 = \frac{b^2}{a^2}$, and $\alpha^2 = -b^2$.

By making the transformation $\psi \mapsto \psi + \pi$, we can obtain the parametric representation of the family of periodic orbits enclosing the centers $E_2(\pi, 0)$, which we omit it here.

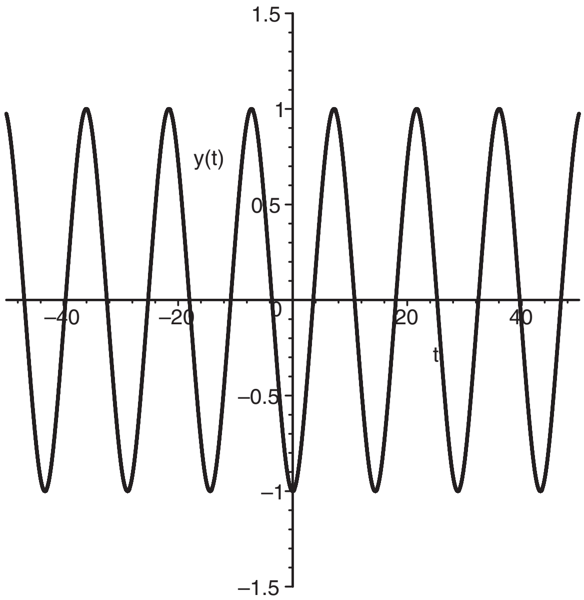
Every periodic orbit of the system (3) defined by $H(\phi, y) = h, h_s - h \ll 1$ has two segments, which are close to the singular straight lines $\phi = \pm\phi_s$. Thus, by Theorem A, they give rise to periodic cusp wave solutions. Corresponding to the two families of periodic solutions enclosing the equilibrium points $E_1(0, 0)$ and $E_2(\pi, 0)$ and the orbits defined by $H(\phi, y) = h$, $h_s - h \ll 1$, we have the wave profiles of periodic cusp wave solutions shown in Figs. 6(b) and 6(c). Corresponding to the family of periodic solutions enclosing the equilibrium points $E_1(0, 0)$ and the orbits defined by $H(\phi, y) = h$, $h \in (h_0, h_\pi)$, we have the wave profile of periodic wave solutions shown in Fig. 6(a).

(iii) If $h = h_s$, we have the following explicit solutions of the system (3), $y = \pm 1$, $\phi \in (-\pi, \phi_s)$, $\phi \in (-\phi_s, \phi_s)$ and $\phi \in (\phi_s, \pi)$. These orbits generate six compactons

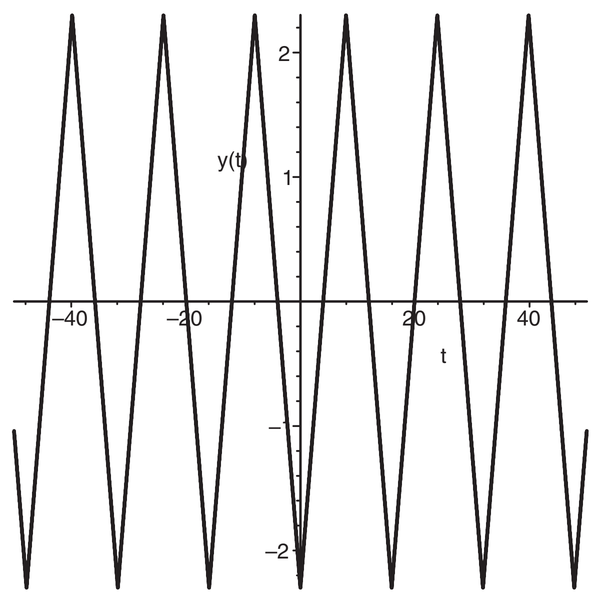
$$\begin{aligned}&\phi = \pm\xi, \\ &\xi \in (-\pi, \phi_s), (-\phi_s, \phi_s) \text{ and } (\phi_s, \pi), \text{ respectively.}\end{aligned} \tag{24}$$

(iv) If $h \in (h_s, \infty)$, the level curves defined by $H(\phi, y) = h$ are four families of rotating periodic orbits of the system (3) in the phase cylinder $S^1 \times R$.

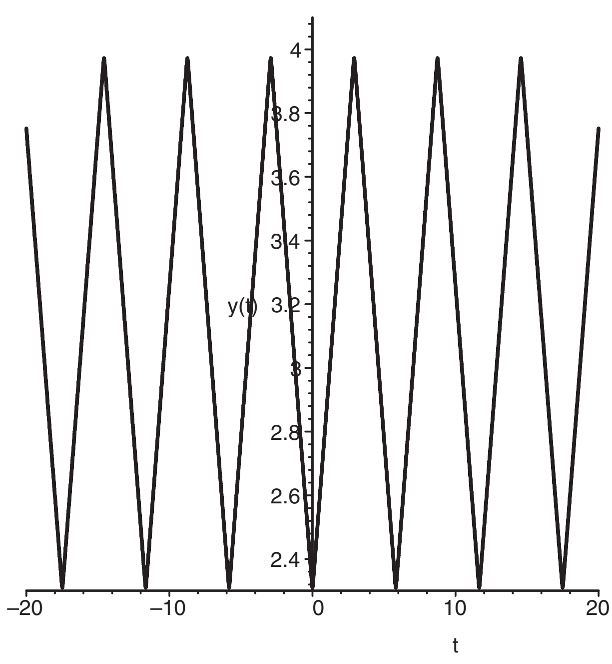
The families of the rotating periodic solutions between two singular straight lines $\phi = \pm\phi_s$ have



(a) $h \in (h_0, h_\pi)$



(b) $h \in (h_\pi, h_S)$, $h_s - h \ll 1$



(c) $h \in (h_\pi, h_s)$, $h_s - h \ll 1$

Fig. 7. The wave profiles of periodic cusp waves defined by $H(\phi, y) = h$.

the following parametric representations

$$\phi(\chi) = \pm 2\arctan(\psi(\chi)) = \pm 2\arctan(b\,\mathrm{sn}(\chi,k)), \quad \chi \in (-\chi_s, \chi_s)$$
$$\xi(\chi) = \frac{2}{a\sqrt{|E|}}(-(\nu - c)\chi + 2\nu\Pi(\arcsin(\mathrm{sn}(\chi,k)), \alpha^2, k)), \tag{25}$$

where k^2, α^2, b^2 are the same as (23), and χ_s satisfies $2\arctan(b\,\mathrm{sn}(\chi_s,k)) = \phi_s$.

4.2. *Subcase* $\frac{c}{\nu} = 1$ [*see Fig. 2(e)*]

In this case, we have $\nu = c$, $h_s = h_\pi$.

(i) If $h \in (h_0, h_\pi)$, the system (3) has one family of periodic orbits, enclosing the equilibrium points $E_1(0,0)$ and the orbits defined by $H(\phi, 0) = h$, which leads to a family of periodic wave solutions of the Fokas equation (1) with the same parametric representation as (8).

(ii) If $h = h_\pi = h_s$, we have the exact solution $y = \pm 1, \phi \in (-\pi, \pi)$ to the system (3), which produces two compactons

$$\phi = \pm\xi, \quad \xi \in (-\pi, \pi). \tag{26}$$

(iii) The level curves defined by $H(\phi, y) = h, h \in (h_s, \infty)$ are two families of rotating orbits for the system (3) in the phase cylinder $S^1 \times R$, which yield the same parametric representation as (10).

4.3. *Subcase* $1 < \frac{c}{\nu} < \infty$ [*see Fig. 2(c)*]

(i) If $h \in (h_0, h_\pi)$, the level curves defined by $H(\phi, y) = h$ are a family of periodic solutions for the system (3), enclosing the centers $E_1(0,0)$. Their parametric representations are the same as (8).

(ii) If $h = h_\pi$, the level curves defined by $H(\phi, y) = h_\pi$ are two homoclinic orbits for the system (3) in the phase cylinder $S^1 \times R$, which lead to the same parametric representation as (9).

(iii) The level curves defined by $H(\phi, y) = h$, $h \in (h_\pi, \infty)$ are two families of rotating orbits for the system (3) in the phase cylinder $S^1 \times R$, which generate the same parametric representation as (10).

Acknowledgments

This work was partially supported by the National Natural Science Foundation of China (Grant Numbers: 11471289, 11162020, 11171295, 61328103 and 61301187), and the Haitian Scholar Plan of Dalian University of Technology.

References

Byrd, P. F. & Fridman, M. D. [1971] *Handbook of Elliptic Integrals for Engineers and Scientists* (Springer, Berlin).

Fokas, A. S. [1995] "On a class of physically important integrable equations," *Physica D* **87**, 145–150.

Lenells, J. & Fokas, A. S. [2010] "On a novel integrable generalization of the sine-Gordon equation," *J. Math. Phys.* **51**, 023519; DOI: 10.1063/1.3272086.

Li, J. & Chen, G. [2007] "On a class of singular nonlinear traveling wave equations," *Int. J. Bifurcation and Chaos* **17**, 4049–4065.

Li, J. [2013] *Singular Nonlinear Traveling Wave Equations: Bifurcations and Exact Solutions* (Science, Beijing).

Li, J. & Qiao, Z. [2013] "Peakon, pseudo-peakon, and cuspon solutions for two generalized Cammasa–Holm equations," *J. Math. Phys.* **54**, 123501-1–13.