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ON THE CAUCHY PROBLEM FOR A GENERALIZED CAMASSA-HOLM EQUATION WITH BOTH QUADRATIC AND CUBIC NONLINEARITY

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Abstract. In this paper, we study the Cauchy problem for a generalized integrable Camassa-Holm equation with both quadratic and cubic nonlinearity. By overcoming the difficulties caused by the complicated mixed nonlinear structure, we firstly establish the local well-posedness result in Besov spaces, and then present a precise blow-up scenario for strong solutions. Furthermore, we show the existence of single peakon by the method of analysis.

1. **Introduction.** In this paper, we consider the following partial differential equation with both quadratic and cubic nonlinearity

$$m_t = \frac{1}{2}k_1\big((u^2 - u_x^2)m\big)_x + \frac{1}{2}k_2(um_x + 2mu_x), \quad m = u - u_{xx}, \tag{1}$$

where k_1, k_2 are arbitrary constants. Eq. (1) was first proposed by Fokas in [15, 16]. Very recently, it has been shown that Eq. (1) has a Lax pair, and can be written as bi-Hamiltonian structure [29]

$$m_t = J\frac{\delta H_1}{\delta m} = K\frac{\delta H_2}{\delta m},$$

where

$$J = k_1\partial m\partial^{-1}m\partial + \frac{1}{2}k_2(\partial m + m\partial), \quad K = \partial - \partial^3,$$

and two Hamiltonians are

$$H_1 = \frac{1}{2}\int_{\mathbb{R}}(u^2 + u_x^2)dx,$$

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and

$$H_2 = \frac{1}{8}\int_{\mathbb{R}} (k_1 u^4 + 2k_1 u^2 u_x^2 - \frac{1}{3}k_1 u_x^4 + 2k_2 u^3 + 2k_2 u u_x^2)dx.$$

Thus, Eq. (1) is completely integrable.

Obviously, for $k_1 = 0, k_2 = -2$, Eq. (1) is reduced to the Camassa-Holm (CH) equation [3, 17]

$$m_t + u m_x + 2u_x m = 0, \quad m = u - u_{xx},$$

which describes the unidirectional propagation of waves at the free surface of shallow water under the influence of gravity [3, 4]. $u(t,x)$ stands for the fluid velocity at time t in the spatial x-direction, $x \in \mathbb{R}$, and $m(t,x)$ represents its potential density.

In the past few years, a large amount literature has devoted to the investigation of the CH equation, because it can describe both wave breaking phenomenon [5, 8, 9, 14, 24] (the solution remains bounded while the slope of $u(t,x)$ becomes unbounded in finite time), and solitary waves interacting like solitons [3, 11, 12, 13, 25]. The well-posedness of the CH equation has been shown in [21, 24, 31] with the initial data $u_0 \in H^s(\mathbb{R}), s > \frac{3}{2}$. In particular, Danchin [14] has dealt with the initial-value problem of the CH equation for the initial data in the Besov space $B^s_{p,r}$, with $1 \leq p, r \leq +\infty, s > \max\{1 + \frac{1}{p}, \frac{3}{2}\}$. However, the Cauchy problem of the CH equation is not locally well-posed in $H^s(\mathbb{R}), s < \frac{3}{2}$. Indeed, the solution can not depend uniformly continuously with respect to the initial data [21]. On the other hand, the CH equation has the peaked solitions (peakons) of the form $\varphi_c(t,x) = ce^{-|x-ct|}$ with the traveling speed $c > 0$. For the peakon solution, we know that it replicates a feature that is characteristic for the waves of great height-waves of largest amplitude that are exact solutions of the governing equations for water waves [6, 7, 10, 33]. Constantin and Strauss [11] gave an impressive proof of stability of peakons by using the conservation laws.

For $k_1 = -2, k_2 = 0$, Eq. (1) becomes the following FORQ equation with cubic nonlinearity

$$m_t + \left((u^2 - u_x^2)m\right)_x = 0, \quad m = u - u_{xx}, \tag{2}$$

which was derived independently by Fokas [15], Fuchssteiner [19], Olver and Rosenau [26], and Qiao [27]. Eq. (2) regains attention due to its cuspon and peakon solution property and Lax pair [27], which may allow the initial value problem of (2) to be solved by the inverse scattering transform (IST) method. Unlike the CH equation, Eq. (2) admits not only new cusp solitons (cuspons), but also possesses weak kink solutions (u, u_x, u_t are continuous, but u_{xx} has a jump at its peak point) [28, 29]. It also has significant differences from the CH equation about the dynamics of the two-peakons and peakon-kink solutions [29]. Recently, the so called "white" solitons and "dark" ones of Eq. (2) have been presented in [32] and [22], respectively. In [2], the authors apply the geometric and analytic approaches to give a geometric interpretation to the variable $m(t,x)$ and construct an infinite-dimensional Lie algebra of symmetries to Eq. (2). In [20], the authors consider the formulation of the singularities of solutions and show that some solutions with certain initial date will blow up in finite time, then they discuss the existence of single peakon of the form $\varphi_c(t,x) = \pm\sqrt{\frac{3c}{2}}e^{-|x-ct|}, c > 0$, and multi-peakon solutions for Eq. (2). Very recently, the orbital stability of peakons for Eq. (2) has been proven in [30].

In the present paper, motivated by the study of the CH equation [14], our main work is to prove the local well-posedness to the Cauchy problem (1) in the non-homogeneous Besov spaces. However, one of the differences with [14] is that we are required to deal with cubic nonlinearity in Besov spaces. Moreover, the term "$m_x u_x^2$" makes us have to solve a transport equation satisfied by m, rather than u. In contrast to the case of the CH equation with initial data u_0 in the Sobolev space $H^s(\mathbb{R}), s > \frac{3}{2}$, we can only prove the well-posedness result with the initial profile u_0 in $H^s(\mathbb{R}), s > \frac{5}{2}$. In our procedure, we have overcome the critical index case by the interpolation method when we applied the transport theory to Eq. (1). Another one of the differences with [14, 18] is that Eq. (1) possesses the complicated mixed structure nonlinear structure (with both quadratic and cubic nonlinearity). To get the uniform boundedness of the approximate solutions $\{u^{(n)}\}_{n\in\mathbb{N}}$, we have to handle the quadratic and cubic nonlinearity together in Eq. (1). To overcome these difficulties, we need to consider two cases: the small initial data and the large one. Then with the local well-posedness result, we may naturally present a precise blow-up scenario to Eq. (1) by combining the blow-up criterion of the CH equation and the one of Eq. (2).

The entire paper is organized as follows. In Section 2, we present some facts on Besov spaces, some preliminary properties and the transport equation theory. In Section 3, we establish the local well-posedness result of Eq. (1) in Besov spaces. In Section 4, we derive a blow-up scenario for strong solutions to Eq. (1). In Section 5, we show that the existence of peakons which can be understood as weak solutions for Eq. (1).

Notation. In the following, we denote $C > 0$ a generic constant only depending on p, r, s. Since our discussion about Eq. (1) is mainly on the line $\mathbb{R}$, for simplicity, we omit $\mathbb{R}$ in our notations of function spaces. And we denote the Fourier transform of a function u as $\mathcal{F}u$.

2. Preliminaries.

In this section, we will recall some basic theory of the Littlewood-Paley decomposition and the transport equation theory on Besov spaces, which will play an important role in the sequel. One may get more details from [1, 14].

Proposition 1 (Littlewood-Paley decomposition, [1]). *Let $\mathcal{B} := \{\xi \in \mathbb{R}, |\xi| \le \frac{4}{3}\}$ and $\mathcal{C} := \{\xi \in \mathbb{R}, \frac{3}{4} \le |\xi| \le \frac{8}{3}\}$. Then there exist $\psi(\xi) \in C_c^\infty(\mathcal{B})$ and $\varphi(\xi) \in C_c^\infty(\mathcal{C})$ such that*

$$\psi(\xi) + \sum_{q\in\mathbb{N}} \varphi(2^{-q}\xi) = 1, \quad \forall\, \xi \in \mathbb{R}.$$

and

$$Supp\varphi(2^{-q}\cdot) \cap Supp\varphi(2^{-q'}\cdot) = \emptyset, \quad \text{if } |q - q'| \ge 2,$$
$$Supp\psi(\cdot) \cap Supp\varphi(2^{-q}\cdot) = \emptyset, \quad \text{if } q \ge 1.$$

Then for all $u \in \mathcal{S}'$ ($\mathcal{S}'$ denotes the tempered distribution spaces), we can define the nonhomogeneous Littlewood-Paley decomposition of a distribution u.

$$u = \sum_{q\in\mathbb{Z}} \Delta_q u,$$

where the localization operators are defined as follows:

$$\Delta_q u := 0, \quad \text{for } q \le -2, \quad \Delta_{-1} u := \psi(D)u = \mathcal{F}^{-1}(\psi \mathcal{F}u),$$

and

$$\Delta_q u := \varphi(2^{-q}D)u = \mathcal{F}^{-1}(\varphi(2^{-q}\xi)\mathcal{F}u), \quad \text{for } q \ge 0.$$

Furthermore, we can define the low frequency cut-off operator S_q as follows:

$$S_q u := \sum_{i=-1}^{q-1} \Delta_i u = \psi(2^{-q}D)u = \mathcal{F}^{-1}(\psi(2^{-q}\xi)\mathcal{F}u).$$

Definition 2.1 (Besov spaces, [1])**.** Let $s \in \mathbb{R}$, $1 \leq p, r \leq \infty$. The nonhomogeneous Besov space $B^s_{p,r}(\mathbb{R})$ ($B^s_{p,r}$ for short) is defined by

$$B^s_{p,r} := \{u \in \mathcal{S}'(\mathbb{R});\ \|u\|_{B^s_{p,r}} < \infty\},$$

where

$$\|u\|_{B^s_{p,r}} := \begin{cases} (\sum\limits_{q\in\mathbb{Z}} 2^{qsr}\|\Delta_q u\|^r_{L^p})^{\frac{1}{r}}, & r < \infty, \\ \sup\limits_{q\in\mathbb{Z}} 2^{qs}\|\Delta_q u\|_{L^p}, & r = \infty. \end{cases}$$

If $s = \infty$, $B^\infty_{p,r} := \bigcap\limits_{s\in\mathbb{R}} B^s_{p,r}$.

In order to state the local well-posedness result, we need to define the following spaces.

Definition 2.2. Let $T > 0, s \in \mathbb{R}$ and $1 \leq p \leq \infty$. We define

$$E^s_{p,r}(T) := C([0,T]; B^s_{p,r}) \cap C^1([0,T]; B^{s-1}_{p,r}), \quad \text{for } r < \infty,$$
$$E^s_{p,\infty}(T) := L^\infty([0,T]; B^s_{p,\infty}) \cap Lip([0,T]; B^{s-1}_{p,\infty}),$$

and

$$E^s_{p,r} := \bigcap_{T>0} E^s_{p,r}(T).$$

Next, we list the following useful properties for Besov spaces.

Proposition 2 ([1, 14])**.** *Let $s \in \mathbb{R}$, $1 \leq p, r, p_i, r_i \leq \infty, i = 1, 2$. Then*
(i) Density: if $1 \leq p, r < \infty$, then $\mathcal{C}^\infty_c$ is dense in $B^s_{p,r}$.
(ii) Embedding: $B^s_{p_1,r_1} \hookrightarrow B^{s-(\frac{1}{p_1}-\frac{1}{p_2})}_{p_2,r_2}$, *for $p_1 \leq p_2$ and $r_1 \leq r_2$,*

$$B^{s_2}_{p,r_2} \hookrightarrow B^{s_1}_{p,r_1} \text{ locally compact}, \quad \text{for } s_1 < s_2.$$

(iii) Algebraic properties: if $s > 0$, $B^s_{p,r} \cap L^\infty$ is an algebra. Furthermore, $B^s_{p,r}$ is an algebra, provided that $s > \frac{1}{p}$ or $s \geq \frac{1}{p}$ and $r = 1$.
(iv) Fatou lemma: if $\{u^{(n)}\}_{n\in\mathbb{N}}$ is bounded in $B^s_{p,r}$ and tends to u in $\mathcal{S}'$, then $u \in B^s_{p,r}$. Moreover,

$$\|u\|_{B^s_{p,r}} \leq \lim\inf_{n\to\infty} \|u^{(n)}\|_{B^s_{p,r}}.$$

(v) Complex interpolation: if $u \in B^{s_1}_{p,r} \cap B^{s_2}_{p,r}$, then for all $\theta \in [0,1]$, we have $u \in B^{\theta s_1+(1-\theta)s_2}_{p,r}$. Moreover,

$$\|u\|_{B^{\theta s_1+(1-\theta)s_2}_{p,r}} \leq \|u\|^\theta_{B^{s_1}_{p,r}} \|u\|^{(1-\theta)}_{B^{s_2}_{p,r}}.$$

(vi) One-dimensional Morse-type estimate:
1) If $s > 0$,

$$\|uv\|_{B^s_{p,r}} \leq C(\|u\|_{B^s_{p,r}}\|v\|_{L^\infty} + \|u\|_{L^\infty}\|v\|_{B^s_{p,r}}).$$

2) If $s_1 \leq \frac{1}{p}$, $s_2 > \frac{1}{p}$ ($s_2 \geq \frac{1}{p}$ if $r = 1$) and $s_1 + s_2 > 0$,

$$\|uv\|_{B^{s_1}_{p,r}} \leq C\|u\|_{B^{s_1}_{p,r}}\|v\|_{B^{s_2}_{p,r}},$$

where C is a constant independent of u and v.
(vii) Action of Fourier multipliers on Besov spaces: let $m \in \mathbb{R}$ and f be a S^m-multiplier (i.e., $f : \mathbb{R} \to \mathbb{R}$ is a smooth function and satisfies that for each multi-index α, there exists a constant C_α such that $|\partial^\alpha f(\xi)| \leq C_\alpha(1+|\xi|)^{m-|\alpha|}$, for $\forall \xi \in \mathbb{R}$.) Then the operator $f(D)$ is continuous from $B^s_{p,r}$ to $B^{s-m}_{p,r}$.

Now we state the following transport equation theory that is crucial to prove local well-posedness for Eq. (1).

Lemma 2.3 (A priori estimate, [1, 14])**.** *Let $1 \leq p, r \leq +\infty$ and $s > -\min\{\frac{1}{p}, 1-\frac{1}{p}\}$. Assume that v be a function such that $\partial_x v$ belongs to $L^1([0,T]; B^{s-1}_{p,r})$ if $s > 1+\frac{1}{p}$ or to $L^1([0,T]; B^{\frac{1}{p}}_{p,r} \cap L^\infty)$ otherwise. Suppose also that $f_0 \in B^s_{p,r}$, $F \in L^1([0,T]; B^s_{p,r})$, and that $f \in L^\infty([0,T]; B^s_{p,r}) \cap C([0,T]; \mathcal{S}')$ be the solution of the one-dimensional transport equation*

$$\begin{cases} \partial_t f + v \cdot \partial_x f = F, \\ f\,|_{t=0} = f_0. \end{cases} \tag{3}$$

Then there exists a constant C depending only on s, p, r such that the following statements hold for $t \in [0,T]$
(i) If $r = 1$ or $s \neq 1 + \frac{1}{p}$,

$$\|f(t)\|_{B^s_{p,r}} \leq \|f_0\|_{B^s_{p,r}} + \int_0^t \|F(\tau)\|_{B^s_{p,r}} d\tau + C\int_0^t V'(\tau)\|f(\tau)\|_{B^s_{p,r}} d\tau,$$

or

$$\|f(t)\|_{B^s_{p,r}} \leq e^{CV(t)}(\|f_0\|_{B^s_{p,r}} + \int_0^t e^{-CV(\tau)}\|F(\tau)\|_{B^s_{p,r}} d\tau),$$

where

$$V(t) = \begin{cases} \int_0^t \|\partial_x v(\tau,\cdot)\|_{(B^{\frac{1}{p}}_{p,r}\cap L^\infty)} d\tau, & s < 1+\frac{1}{p}, \\ \int_0^t \|\partial_x v(\tau,\cdot)\|_{B^{s-1}_{p,r}} d\tau, & s > 1+\frac{1}{p}. \end{cases}$$

(ii) If $s \leq 1+\frac{1}{p}$, and $\partial_x f_0, \partial_x f \in L^\infty([0,T]\times\mathbb{R})$ and $\partial_x F \in L^1([0,T]; L^\infty)$, then

$$\begin{aligned} &\|f(t)\|_{B^s_{p,r}} + \|\partial_x f(t)\|_{L^\infty} \\ \leq& e^{CV(t)}(\|f_0\|_{B^s_{p,r}} + \|\partial_x f_0\|_{L^\infty} + \int_0^t e^{-CV(\tau)}(\|F(\tau)\|_{B^s_{p,r}} + \|\partial_x F(\tau)\|_{L^\infty})d\tau), \end{aligned}$$

where $V(t) = \int_0^t \|\partial_x v(\tau,\cdot)\|_{(B^{\frac{1}{p}}_{p,r}\cap L^\infty)} d\tau$.
(iii) If $f = v$, then for all $s > 0$, the estimate in (i) holds with $V(t) = \int_0^t \|\partial_x v(\tau,\cdot)\|_{L^\infty} d\tau$.
(iv) If $r < \infty$, then $f \in C([0,T]; B^s_{p,r})$. If $r = \infty$, then $f \in C([0,T]; B^{s'}_{p,1})$ for all $s' < s$.

Lemma 2.4 (Existence and uniqueness, [14])**.** *Let p, r, s, f_0 and F be as in the statement of Lemma 2.3. Suppose that $v \in L^\rho([0,T]; B^{-M}_{\infty,\infty})$ for some $\rho > 1, M > 0$ and $\partial_x v \in L^1([0,T]; B^{\frac{1}{p}}_{p,\infty} \cap L^\infty)$ if $s < 1+\frac{1}{p}$, and $\partial_x v \in L^1([0,T]; B^{s-1}_{p,r})$ if $s > 1+\frac{1}{p}$ or $s = 1+\frac{1}{p}$ and $r = 1$. Then the transport equation (3) has a unique solution $f \in L^\infty([0,T]; B^s_{p,r}) \bigcap (\bigcap_{s'<s} C([0,T]; B^{s'}_{p,1}))$ and the corresponding inequalities in Lemma 2.3 hold true. Moreover, if $r < \infty$, then $f \in C([0,T]; B^s_{p,r})$.*

3. **Local well-posedness.** In this section, we shall study the local well-posedness of Eq. (1) in the nonhomogeneous Besov spaces. At first, we present a priori estimates about the solutions of Eq. (1), which can be applied to prove the uniqueness and continuity with the initial data in some sense.

Lemma 3.1. *Suppose that $1 \leq p, r \leq \infty$ and $s > \max\{2+\frac{1}{p}, \frac{5}{2}, 3-\frac{1}{p}\}$. Let $u^{(1)}, u^{(2)} \in L^\infty([0,T]; B^s_{p,r}) \cap C([0,T]; \mathcal{S}')$ be two given solutions to Eq. (1) with initial data $u_0^{(1)}, u_0^{(2)} \in B^s_{p,r}$, and let $u^{(12)} := u^{(2)} - u^{(1)}$ and $m^{(12)} := m^{(2)} - m^{(1)}$. Then for all $t \in [0,T]$, we have*
1) If $s > \max\{2+\frac{1}{p}, \frac{5}{2}, 3-\frac{1}{p}\}$ and $s \neq 4+\frac{1}{p}$, then

$$\begin{aligned}\|u^{(12)}\|_{B^{s-1}_{p,r}} \leq &\|u_0^{(12)}\|_{B^{s-1}_{p,r}} \exp\{C\int_0^t (\|u^{(1)}(\tau)\|^2_{B^s_{p,r}} + \|u^{(2)}(\tau)\|^2_{B^s_{p,r}} \\ &+ \|u^{(1)}(\tau)\|_{B^s_{p,r}} + \|u^{(2)}(\tau)\|_{B^s_{p,r}}) d\tau\}.\end{aligned}$$

2) If $s = 4+\frac{1}{p}$, then

$$\begin{aligned}\|u^{(12)}\|_{B^{s-1}_{p,r}} \leq &C\|u_0^{(12)}\|^\theta_{B^{s-1}_{p,r}} (\|u^{(1)}\|_{B^s_{p,r}} + \|u^{(2)}\|_{B^s_{p,r}})^{1-\theta} \exp\{\theta C\int_0^t (\|u^{(1)}(\tau)\|^2_{B^s_{p,r}} \\ &+ \|u^{(2)}(\tau)\|^2_{B^s_{p,r}} + \|u^{(1)}(\tau)\|_{B^s_{p,r}} + \|u^{(2)}(\tau)\|_{B^s_{p,r}}) d\tau\},\end{aligned}$$

with $\theta = \frac{1}{2}(1-\frac{1}{2p}) \in (0,1)$.

Proof. It is obvious that $u^{(12)} \in L^\infty([0,T]; B^s_{p,r}) \cap C([0,T]; \mathcal{S}')$ and $u^{(12)}, m^{(12)}$ solves the following transport equation

$$\begin{cases} m_t^{(12)} - \{\dfrac{k_1}{2}[(u^{(1)})^2 - (u_x^{(1)})^2] + \dfrac{k_2}{2} u^{(1)}\} m_x^{(12)} = F(t,x), & t > 0, x \in \mathbb{R}, \\ m^{(12)}|_{t=0} = m_0^{(12)} := m_0^{(2)} - m_0^{(1)}, \quad m = u - u_{xx}, & x \in \mathbb{R}, \end{cases} \tag{4}$$

where $F(t,x) := \frac{k_1}{2}[u^{(12)}(u^{(1)}+u^{(2)}) - u_x^{(12)}(u_x^{(1)}+u_x^{(2)})]\partial_x m^{(2)} + k_1[u_x^{(12)}(m^{(2)})^2 + u_x^{(1)} m^{(12)}(m^{(1)}+m^{(2)})] + \frac{k_2}{2} u^{(12)} m_x^{(2)} + k_2 u_x^{(12)} m^{(2)} + k_2 u_x^{(1)} m^{(12)}$.

Applying Lemma 2.3 to the transport equation (4), we have

$$\begin{aligned}\|m^{(12)}\|_{B^{s-3}_{p,r}} \leq &\|m_0^{(12)}\|_{B^{s-3}_{p,r}} + \int_0^t \|F(\tau)\|_{B^{s-3}_{p,r}} d\tau \\ &+ C\int_0^t \|(u^{(1)})^2 - (u_x^{(1)})^2 + u^{(1)}\|_{B^{s-3}_{p,r}} \|m^{(12)}\|_{B^{s-3}_{p,r}} d\tau.\end{aligned} \tag{5}$$

Indeed, if $\max\{2+\frac{1}{p}, \frac{5}{2}\} < s \leq 3+\frac{1}{p}$, by Proposition 2 (vi), we get

$$\begin{aligned}\|F(\tau)\|_{B^{s-3}_{p,r}} \leq &C\{\|[u^{(12)}(u^{(1)}+u^{(2)}) - u_x^{(12)}(u_x^{(1)}+u_x^{(2)})]\partial_x m^{(2)}\|_{B^{s-3}_{p,r}} \\ &+ \|u_x^{(12)}(m^{(2)})^2\|_{B^{s-3}_{p,r}} + \|u_x^{(1)} m^{(12)}(m^{(1)}+m^{(2)})\|_{B^{s-3}_{p,r}} \\ &+ \|u^{(12)} m_x^{(2)} + u_x^{(12)} m^{(2)} + u_x^{(1)} m^{(12)}\|_{B^{s-3}_{p,r}}\} \\ \leq &C\{\|m_x^{(2)}\|_{B^{s-3}_{p,r}} \|[u^{(12)}(u^{(1)}+u^{(2)}) - u_x^{(12)}(u_x^{(1)}+u_x^{(2)})]\|_{B^{s-2}_{p,r}} \\ &+ \|u_x^{(12)}\|_{B^{s-3}_{p,r}} \|(m^{(2)})^2\|_{B^{s-2}_{p,r}} + \|m^{(12)}\|_{B^{s-3}_{p,r}} \|u_x^{(1)}(m^{(1)}+m^{(2)})\|_{B^{s-2}_{p,r}} \\ &+ \|m_x^{(2)}\|_{B^{s-3}_{p,r}} \|u^{(12)}\|_{B^{s-2}_{p,r}} + \|u_x^{(12)}\|_{B^{s-3}_{p,r}} \|m^{(2)}\|_{B^{s-2}_{p,r}} \\ &+ \|m^{(12)}\|_{B^{s-3}_{p,r}} \|u_x^{(1)}\|_{B^{s-2}_{p,r}}\}.\end{aligned}$$

Since $s>\max\{2+\frac{1}{p},\frac{5}{2}\}$, we know that $B^{s-2}_{p,r}$ is an algebra. Thus, we deduce

$$\|F(\tau)\|_{B^{s-3}_{p,r}}\leq C[\|u^{(12)}\|_{B^{s-1}_{p,r}}(\|u^{(1)}\|^2_{B^s_{p,r}}+\|u^{(2)}\|^2_{B^s_{p,r}}+\|u^{(1)}\|_{B^s_{p,r}}+\|u^{(2)}\|_{B^s_{p,r}}). \quad (6)$$

For $s>3+\frac{1}{p}$, the inequality (6) also holds true in view of the fact that $B^{s-3}_{p,r}$ is an algebra.
Note that

$$\begin{aligned}&\|(u^{(1)})^2-(u^{(1)}_x)^2+u^{(1)}\|_{B^{s-3}_{p,r}}\|m^{(12)}\|_{B^{s-3}_{p,r}}\\ \leq& C[\|u^{(12)}\|_{B^{s-1}_{p,r}}(\|u^{(1)}\|^2_{B^s_{p,r}}+\|u^{(2)}\|^2_{B^s_{p,r}}+\|u^{(1)}\|_{B^s_{p,r}}+\|u^{(2)}\|_{B^s_{p,r}}).\end{aligned}$$

Therefore, inserting the above inequality and (6) into (5), we obtain

$$\begin{aligned}\|u^{(12)}\|_{B^{s-1}_{p,r}}\leq&\|u^{(12)}_0\|_{B^{s-1}_{p,r}}+C\int_0^t[\|u^{(12)}(\tau)\|_{B^{s-1}_{p,r}}\\ &\times(\|u^{(1)}\|^2_{B^s_{p,r}}+\|u^{(2)}\|^2_{B^s_{p,r}}+\|u^{(1)}\|_{B^s_{p,r}}+\|u^{(2)}\|_{B^s_{p,r}})]d\tau.\end{aligned}$$

Then, by Gronwall's inequality, we prove 1).

Since we can not apply Lemma 2.3 to (4) for the critical case $s=4+\frac{1}{p}$, we here use the interpolation method to deal with it.

In fact, we can choose $\theta=\frac{1}{2}(1-\frac{1}{2p})\in(0,1)$, such that $s-1=3+\frac{1}{p}=(1-\theta)(4+\frac{1}{2p})+\theta(2+\frac{1}{2p})$. Then, by Proposition 2 (v), we have

$$\|u^{(12)}\|_{B^{s-1}_{p,r}}=\|u^{(12)}\|_{B^{3+\frac{1}{p}}_{p,r}}\leq\|u^{(12)}\|^{\theta}_{B^{2+\frac{1}{2p}}_{p,r}}\|u^{(12)}\|^{1-\theta}_{B^{4+\frac{1}{2p}}_{p,r}}.$$

Then, from the obtained result of 1) in this lemma, we get

$$\begin{aligned}\|u^{(12)}\|_{B^{s-1}_{p,r}}\leq&[\|u^{(12)}_0\|_{B^{2+\frac{1}{2p}}_{p,r}}\exp(C\int_0^t(\|u^{(1)}\|^2_{B^{3+\frac{1}{2p}}_{p,r}}+\|u^{(2)}\|^2_{B^{3+\frac{1}{2p}}_{p,r}}+\|u^{(1)}\|_{B^{3+\frac{1}{2p}}_{p,r}}\\ &+\|u^{(2)}\|_{B^{3+\frac{1}{2p}}_{p,r}})d\tau)]^{\theta}\times(\|u^{(1)}\|_{B^{4+\frac{1}{2p}}_{p,r}}+\|u^{(2)}\|_{B^{4+\frac{1}{2p}}_{p,r}})^{1-\theta}\\ \leq&C\|u^{(12)}_0\|^{\theta}_{B^{s-1}_{p,r}}(\|u^{(1)}\|_{B^s_{p,r}}+\|u^{(2)}\|_{B^s_{p,r}})^{1-\theta}\exp\{\theta C\int_0^t(\|u^{(1)}(\tau)\|^2_{B^s_{p,r}}\\ &+\|u^{(2)}(\tau)\|^2_{B^s_{p,r}}+\|u^{(1)}(\tau)\|_{B^s_{p,r}}+\|u^{(2)}(\tau)\|_{B^s_{p,r}})d\tau\}.\end{aligned}$$

This completes the proof of Lemma 3.1. □

Next, we use the classical Friedrichs regularization method to construct the approximation solutions to Eq. (1).

Lemma 3.2. *Suppose that p,r and s be as in the statement of Lemma 3.1, $u_0\in B^s_{p,r}$ and $u^{(0)}:=0$. Then there exists a sequence of smooth functions $\{u^{(n)}\}_{n\in\mathbb{N}}\in C(\mathbb{R}^+;B^{\infty}_{p,r})$ solving the following transport equation by induction*

$$\begin{cases}m^{(n+1)}_t-\{\frac{k_1}{2}[(u^{(n)})^2-(u^{(n)}_x)^2]+\frac{k_2}{2}u^{(n)}\}m^{(n+1)}_x=\\ \qquad\qquad k_1u^{(n)}_x(m^{(n)})^2+k_2u^{(n)}_xm^{(n)}, & t>0,x\in\mathbb{R},\\ u^{(n+1)}\mid_{t=0}=u^{(n+1)}_0(x)=S_{n+1}u_0, & x\in\mathbb{R}.\end{cases} \quad (7)$$

Moreover, there exists a $T>0$ such that the solutions satisfying the following properties.
(1) $\{u^{(n)}\}_{n\in\mathbb{N}}$ is uniformly bounded in $E^s_{p,r}(T)$.
(2) $\{u^{(n)}\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $C([0,T];B^{s-1}_{p,r})$.

Proof. By the definition of S_q, we know that all the data $S_{n+1}u_0 \in B^{\infty}_{p,r}$. Thus, from Lemma 2.4, we deduce by induction that for all $n \in \mathbb{N}$, Eq. (7) has a global solution belonging $C(\mathbb{R}^+; B^{\infty}_{p,r})$.

For $s > \max\{2 + \frac{1}{p}, \frac{5}{2}, 3 - \frac{1}{p}\}$ and $s \neq 4 + \frac{1}{p}$, by Lemma 2.3, we obtain

$$\begin{aligned}\|m^{(n+1)}\|_{B^{s-2}_{p,r}} \leq & e^{C\int_0^t \|[(u^{(n)})^2-(u^{(n)}_x)^2+u^{(n)}](\tau)\|_{B^{s-2}_{p,r}}d\tau}(\|m_0\|_{B^{s-2}_{p,r}} \\ & + C\int_0^t e^{-C\int_0^\tau \|[(u^{(n)})^2-(u^{(n)}_x)^2+u^{(n)}](\tau')\|_{B^{s-2}_{p,r}}d\tau'} \\ & \times \|u^{(n)}_x(m^{(n)})^2 + u^{(n)}_x m^{(n)}\|_{B^{s-2}_{p,r}}d\tau). \end{aligned} \tag{8}$$

Since $s > 2 + \frac{1}{p}$, we know that $B^{s-2}_{p,r}$ is an algebra. Thus, we have

$$\begin{aligned}\|u^{(n)}_x(m^{(n)})^2 + u^{(n)}_x m^{(n)}\|_{B^{s-2}_{p,r}} \leq & C\|u^{(n)}_x\|_{B^{s-2}_{p,r}}(\|(m^{(n)})^2\|_{B^{s-2}_{p,r}} + \|(m^{(n)})\|_{B^{s-2}_{p,r}}) \\ \leq & C(\|u^{(n)}\|^3_{B^s_{p,r}} + \|u^{(n)}\|^2_{B^s_{p,r}}),\end{aligned}$$

and

$$\|(u^{(n)})^2 - (u^{(n)}_x)^2 + u^{(n)}\|_{B^{s-2}_{p,r}} \leq C(\|u^{(n)}\|^2_{B^s_{p,r}} + \|u^{(n)}\|_{B^s_{p,r}}).$$

Inserting the above inequalities into (8), we obtain

$$\begin{aligned}\|u^{(n+1)}\|_{B^s_{p,r}} \leq & e^{C\int_0^t(\|u^{(n)}\|^2_{B^s_{p,r}}+\|u^{(n)}\|_{B^s_{p,r}})(\tau)d\tau}\|u_0\|_{B^s_{p,r}} \\ & + C\int_0^t e^{C\int_\tau^t(\|u^{(n)}\|^2_{B^s_{p,r}}+\|u^{(n)}\|_{B^s_{p,r}})(\tau')d\tau'} \\ & \times (\|u^{(n)}\|^3_{B^s_{p,r}} + \|u^{(n)}\|^2_{B^s_{p,r}})d\tau. \end{aligned} \tag{9}$$

In order to prove the uniform boundedness of $\{u^{(n)}\}_{n\in\mathbb{N}}$, we shall divide our discussion into two parts. When $2\|u_0\|_{B^s_{p,r}} < 1$, we choose a $T_1 > 0$ such that

$$T_1 < \min\{\frac{1 - 2\|u_0\|_{B^s_{p,r}}}{8C\|u_0\|_{B^s_{p,r}}}, \frac{1}{4C}\},$$

and suppose by induction that for all $t \in [0, T_1]$

$$\|u^{(n)}\|_{B^s_{p,r}} \leq \frac{2\|u_0\|_{B^s_{p,r}}}{1 - 8C\|u_0\|_{B^s_{p,r}}t}. \tag{10}$$

Noting that $t \leq T_1 < \frac{1-2\|u_0\|_{B^s_{p,r}}}{8C\|u_0\|_{B^s_{p,r}}}$, we have $\frac{2\|u_0\|_{B^s_{p,r}}}{1-8C\|u_0\|_{B^s_{p,r}}t} < 1$. By (10), we obtain

$$\|u^{(n)}\|_{B^s_{p,r}} \leq \frac{2\|u_0\|_{B^s_{p,r}}}{1 - 8C\|u_0\|_{B^s_{p,r}}t} \leq (\frac{2\|u_0\|_{B^s_{p,r}}}{1 - 8C\|u_0\|_{B^s_{p,r}}t})^{\frac{2}{3}} \leq (\frac{2\|u_0\|_{B^s_{p,r}}}{1 - 8C\|u_0\|_{B^s_{p,r}}t})^{\frac{1}{2}}. \tag{11}$$

From (11), we can deduce that

$$\begin{aligned}
&C\int_{\tau}^{t}(\|u^{(n)}\|^2_{B^s_{p,r}}+\|u^{(n)}\|_{B^s_{p,r}})(\tau')d\tau'\\
\leq& C\int_{\tau}^{t}\{[(\frac{2\|u_0\|_{B^s_{p,r}}}{1-8C\|u_0\|_{B^s_{p,r}}\tau'})^{\frac{1}{2}}]^2+\frac{2\|u_0\|_{B^s_{p,r}}}{1-8C\|u_0\|_{B^s_{p,r}}\tau'}\}d\tau'\\
\leq& -\frac{1}{2}\int_{\tau}^{t}\frac{-8C\|u_0\|_{B^s_{p,r}}}{1-8C\|u_0\|_{B^s_{p,r}}\tau'}d\tau'\\
=& \ln\sqrt{1-8C\|u_0\|_{B^s_{p,r}}\tau}-\ln\sqrt{1-8C\|u_0\|_{B^s_{p,r}}t}.
\end{aligned}$$

Inserting the above inequality and (11) into (9) yields

$$\begin{aligned}
\|u^{(n+1)}\|_{B^s_{p,r}}\leq&\frac{\|u_0\|_{B^s_{p,r}}}{\sqrt{1-8C\|u_0\|_{B^s_{p,r}}t}}+\frac{C}{\sqrt{1-8C\|u_0\|_{B^s_{p,r}}t}}\{\int_0^t\sqrt{1-8C\|u_0\|_{B^s_{p,r}}\tau}\\
&\times[((\frac{2\|u_0\|_{B^s_{p,r}}}{1-8C\|u_0\|_{B^s_{p,r}}\tau})^{\frac{2}{3}})^3+((\frac{2\|u_0\|_{B^s_{p,r}}}{1-8C\|u_0\|_{B^s_{p,r}}\tau})^{\frac{1}{2}})^2]\}d\tau\\
\leq&\frac{\|u_0\|_{B^s_{p,r}}}{\sqrt{1-8C\|u_0\|_{B^s_{p,r}}t}}+\frac{\|u_0\|_{B^s_{p,r}}}{\sqrt{1-8C\|u_0\|_{B^s_{p,r}}t}}\int_0^t\frac{4C\|u_0\|_{B^s_{p,r}}}{(1-8C\|u_0\|_{B^s_{p,r}}\tau)^{\frac{3}{2}}}d\tau\\
&+\frac{1}{\sqrt{1-8C\|u_0\|_{B^s_{p,r}}t}}\int_0^t\frac{2C\|u_0\|_{B^s_{p,r}}}{(1-8C\|u_0\|_{B^s_{p,r}}\tau)^{\frac{1}{2}}}d\tau\\
\leq&\frac{\|u_0\|_{B^s_{p,r}}}{\sqrt{1-8C\|u_0\|_{B^s_{p,r}}t}}+\frac{\|u_0\|_{B^s_{p,r}}}{\sqrt{1-8C\|u_0\|_{B^s_{p,r}}t}}[(1-8C\|u_0\|_{B^s_{p,r}}\tau)^{-\frac{1}{2}}|_0^t]\\
&+\frac{1}{\sqrt{1-8C\|u_0\|_{B^s_{p,r}}t}}[-\frac{1}{2}(1-8C\|u_0\|_{B^s_{p,r}}\tau)^{\frac{1}{2}}|_0^t]\\
\leq&\frac{\|u_0\|_{B^s_{p,r}}}{\sqrt{1-8C\|u_0\|_{B^s_{p,r}}t}}(1+\frac{1}{\sqrt{1-8C\|u_0\|_{B^s_{p,r}}t}}-1)+\frac{1}{\sqrt{1-8C\|u_0\|_{B^s_{p,r}}t}}\\
&\times\frac{1}{2}(1-\sqrt{1-8C\|u_0\|_{B^s_{p,r}}t})\\
\leq&\frac{2\|u_0\|_{B^s_{p,r}}}{1-8C\|u_0\|_{B^s_{p,r}}t},
\end{aligned}\tag{12}$$

where we used the following fact that

$$T_1<\frac{1}{4C}\Rightarrow\frac{1}{2}(1-\sqrt{1-8C\|u_0\|_{B^s_{p,r}}t})\leq\frac{\|u_0\|_{B^s_{p,r}}}{\sqrt{1-8C\|u_0\|_{B^s_{p,r}}t}},$$

in the last inequality. Thus, we prove (10) for the case $2\|u_0\|_{B^s_{p,r}}<1$.

On the other hand, when $2\|u_0\|_{B^s_{p,r}}\geq 1$, we choose a $T_2>0$ such that $T_2\leq\frac{1-e^{-1}}{16C\|u_0\|^2_{B^s_{p,r}}}<\frac{1}{16C\|u_0\|^2_{B^s_{p,r}}}$, and suppose by induction that for all $t\in[0,T_2]$

$$\|u^{(n)}\|_{B^s_{p,r}}\leq\frac{2\|u_0\|_{B^s_{p,r}}}{\sqrt{1-16C\|u_0\|^2_{B^s_{p,r}}t}}.\tag{13}$$

Noting that $2\|u_0\|_{B^s_{p,r}} \geq 1$, we get $\frac{2\|u_0\|_{B^s_{p,r}}}{(1-16C\|u_0\|^2_{B^s_{p,r}}t)^{\frac{1}{2}}} \geq 1$. From (13), we obtain

$$\begin{aligned}\|u^{(n)}\|_{B^s_{p,r}} \leq \frac{2\|u_0\|_{B^s_{p,r}}}{\sqrt{1-16C\|u_0\|^2_{B^s_{p,r}}t}} &\leq (\frac{2\|u_0\|_{B^s_{p,r}}}{\sqrt{1-16C\|u_0\|^2_{B^s_{p,r}}t}})^{\frac{3}{2}} \\ &\leq (\frac{2\|u_0\|_{B^s_{p,r}}}{\sqrt{1-16C\|u_0\|^2_{B^s_{p,r}}t}})^2. \end{aligned} \tag{14}$$

By (14), we find that

$$\begin{aligned} &C\int_\tau^t (\|u^{(n)}\|^2_{B^s_{p,r}} + \|u^{(n)}\|_{B^s_{p,r}})(\tau')d\tau' \\ \leq &C\int_\tau^t \{(\frac{2\|u_0\|_{B^s_{p,r}}}{\sqrt{1-16C\|u_0\|^2_{B^s_{p,r}}\tau'}})^2 + \frac{4\|u_0\|^2_{B^s_{p,r}}}{1-16C\|u_0\|^2_{B^s_{p,r}}\tau'}\}d\tau' \\ \leq &-\frac{1}{2}\int_\tau^t \frac{-16C\|u_0\|^2_{B^s_{p,r}}}{1-16C\|u_0\|^2_{B^s_{p,r}}\tau'}d\tau' \\ = &\ln\sqrt{1-16C\|u_0\|^2_{B^s_{p,r}}\tau} - \ln\sqrt{1-16C\|u_0\|^2_{B^s_{p,r}}t}. \end{aligned}$$

Inserting the above inequality and (14) into (9), we obtain

$$\begin{aligned} \|u^{(n+1)}\|_{B^s_{p,r}} \leq &\frac{\|u_0\|_{B^s_{p,r}}}{\sqrt{1-16C\|u_0\|^2_{B^s_{p,r}}t}} + \frac{C}{\sqrt{1-16C\|u_0\|^2_{B^s_{p,r}}t}}\{\int_0^t \sqrt{1-16C\|u_0\|^2_{B^s_{p,r}}\tau} \\ &\times [(\frac{2\|u_0\|_{B^s_{p,r}}}{\sqrt{1-16C\|u_0\|^2_{B^s_{p,r}}\tau}})^3 + ((\frac{2\|u_0\|_{B^s_{p,r}}}{\sqrt{1-16C\|u_0\|^2_{B^s_{p,r}}\tau}})^{\frac{3}{2}})^2]\}d\tau \\ \leq &\frac{\|u_0\|_{B^s_{p,r}}}{\sqrt{1-16C\|u_0\|^2_{B^s_{p,r}}t}} + \frac{\|u_0\|_{B^s_{p,r}}}{\sqrt{1-16C\|u_0\|^2_{B^s_{p,r}}t}}\int_0^t \frac{16C\|u_0\|^2_{B^s_{p,r}}}{(1-16C\|u_0\|^2_{B^s_{p,r}}\tau)}d\tau \\ \leq &\frac{\|u_0\|_{B^s_{p,r}}}{\sqrt{1-16C\|u_0\|^2_{B^s_{p,r}}t}}[1-\ln(1-16C\|u_0\|^2_{B^s_{p,r}}\tau)|_0^t] \\ \leq &\frac{2\|u_0\|_{B^s_{p,r}}}{\sqrt{1-16C\|u_0\|^2_{B^s_{p,r}}t}}, \end{aligned} \tag{15}$$

where we used the following fact that

$$T_2 \leq \frac{1-e^{-1}}{16C\|u_0\|^2_{B^s_{p,r}}} \Rightarrow 1-\ln(1-16C\|u_0\|^2_{B^s_{p,r}}t) \leq 2,$$

in the last inequality. Thus, we prove (13) for the case $2\|u_0\|_{B^s_{p,r}} \geq 1$.

Therefore, from the above discussion of the two cases, choosing $T = \min\{T_1, T_2\} > 0$, combining (12) and (15), we have proved that $\{u^{(n)}\}_{n\in\mathbb{N}}$ is uniformly bounded in $C([0,T]; B^s_{p,r})$. Using Proposition 2 (vi) and the fact $B^{s-2}_{p,r}$ is an algebra as $s > 2+\frac{1}{p}$, we have

$$\begin{aligned}
&\|[((u^{(n)})^2-(u_x^{(n)})^2)+u^{(n)}]m_x^{(n+1)}+u_x^{(n)}(m^{(n)})^2+u_x^{(n)}m^{(n)}\|_{B^{s-3}_{p,r}}\\
\le& C\{\|m_x^{(n+1)}\|_{B^{s-3}_{p,r}}[(\|(u^{(n)})^2\|_{B^{s-2}_{p,r}}+\|(u_x^{(n)})^2\|_{B^{s-2}_{p,r}})+\|u^{(n)}\|_{B^{s-2}_{p,r}}]\\
&+\|u_x^{(n)}\|_{B^{s-3}_{p,r}}(\|(m^{(n)})^2\|_{B^{s-2}_{p,r}}+\|m^{(n)}\|_{B^{s-2}_{p,r}})\}\\
\le& C[\|u^{(n+1)}\|_{B^s_{p,r}}(\|(u^{(n)})\|^2_{B^s_{p,r}}+\|u^{(n)}\|_{B^s_{p,r}})+(\|u^{(n)}\|^3_{B^s_{p,r}}+\|u^{(n)}\|^2_{B^s_{p,r}})].
\end{aligned}$$

From Eq. (7), we get $\partial_t m^{(n+1)}\in C\in([0,T];B^{s-3}_{p,r})$. Hence, $\partial_t u^{(n+1)}\in C\in([0,T];B^{s-1}_{p,r})$ is uniformly bounded, which yields that the sequence $\{u^{(n)}\}_{n\in\mathbb{N}}$ is uniformly bounded in $E^s_{p,r}(T)$.

Now it suffices to prove that

$$\{u^{(n)}\}_{n\in\mathbb{N}} \text{ is a Cauchy sequence in } C([0,T];B^{s-1}_{p,r}).$$

Indeed, from Eq. (7), for all $n,l\in\mathbb{N}$, we have

$$\{\partial_t-[\frac{k_1}{2}((u^{(n+l)})^2-(u_x^{(n+l)})^2)+\frac{k_2}{2}u^{(n+l)}]\}(m^{(n+l+1)}-m^{(n+1)})=G(t,x),$$

where $G(t,x):=\{\frac{k_1}{2}[(u^{(n+l)}-u^{(n)})(u^{(n+l)}+u^{(n)})-(u_x^{(n+l)}-u_x^{(n)})(u_x^{(n+l)}+u_x^{(n)})]+\frac{k_2}{2}(u^{(n+l)}-u^{(n)})\}\partial_x m^{(n+1)}+k_1u_x^{(n+l)}(m^{(n+l)}-m^{(n)})(m^{(n+l)}+m^{(n)})+k_1(u_x^{(n+l)}-u_x^{(n)})(m^{(n)})^2+k_2u_x^{(n+l)}(m^{(n+l)}-m^{(n)})+k_2(u_x^{(n+l)}-u_x^{(n)})m^{(n)}$.

Applying Lemma 2.3 again, for all $t\in[0,T]$, we have

$$\begin{aligned}
&\|m^{(n+l+1)}-m^{(n+1)}\|_{B^{s-3}_{p,r}}\\
\le& e^{C\int_0^t\|[(u^{(n+l)})^2-(u_x^{(n+l)})^2+u^{(n+l)}](\tau)\|_{B^{s-2}_{p,r}}d\tau}(\|m_0^{(n+l+1)}-m_0^{(n+1)}\|_{B^{s-3}_{p,r}}\\
&+C\int_0^t e^{-C\int_0^\tau\|[(u^{(n+l)})^2-(u_x^{(n+l)})^2+u^{(n+l)}](\tau')\|_{B^{s-2}_{p,r}}d\tau'}\times\|G(\tau)\|_{B^{s-3}_{p,r}}d\tau).
\end{aligned}\tag{16}$$

Similar to the proof of the estimate of $\|F(\tau)\|_{B^{s-3}_{p,r}}$ in Lemma 3.1, for $s>\max\{2+\frac{1}{p},\frac{5}{2},3-\frac{1}{p}\}$ and $s\neq 4+\frac{1}{p}$, we also obtain

$$\begin{aligned}
\|G(\tau)\|_{B^{s-3}_{p,r}}\le& C\|u^{(n+l)}-u^{(n)}\|_{B^{s-1}_{p,r}}(\|u^{(n)}\|^2_{B^s_{p,r}}+\|u^{(n+1)}\|^2_{B^s_{p,r}}\\
&+\|u^{(n+l)}\|^2_{B^s_{p,r}}+\|u^{(n)}\|_{B^s_{p,r}}+\|u^{(n+1)}\|_{B^s_{p,r}}+\|u^{(n+l)}\|_{B^s_{p,r}}).
\end{aligned}$$

Inserting the above inequality into (16), we have

$$\begin{aligned}
&\|u^{(n+l+1)}-u^{(n+1)}\|_{B^{s-1}_{p,r}}\\
\le& e^{C\int_0^t\|[(u^{(n+l)})^2-(u_x^{(n+l)})^2+u^{(n+l)}](\tau)\|_{B^{s-2}_{p,r}}d\tau}(\|u_0^{(n+l+1)}-u_0^{(n+1)}\|_{B^{s-1}_{p,r}}\\
&+C\int_0^t e^{-C\int_0^\tau\|[(u^{(n+l)})^2-(u_x^{(n+l)})^2+u^{(n+l)}](\tau')\|_{B^{s-2}_{p,r}}d\tau'}\|u^{(n+l)}-u^{(n)}\|_{B^{s-1}_{p,r}}\\
&\times(\|u^{(n)}\|^2_{B^s_{p,r}}+\|u^{(n+1)}\|^2_{B^s_{p,r}}+\|u^{(n+l)}\|^2_{B^s_{p,r}}+\|u^{(n)}\|_{B^s_{p,r}}+\|u^{(n+1)}\|_{B^s_{p,r}}\\
&+\|u^{(n+l)}\|_{B^s_{p,r}}).
\end{aligned}$$

Note that $\{u^{(n)}\}_{n\in\mathbb{N}}$ is uniformly bounded in $E^s_{p,r}(T)$ and

$$\begin{aligned}
&\|u_0^{(n+l+1)}-u_0^{(n+1)}\|_{B^{s-1}_{p,r}}\\
=&\|S_{n+l+1}u_0-S_{n+1}u_0\|_{B^{s-1}_{p,r}}=\|\sum_{q=n+1}^{n+l}\Delta_q u_0\|_{B^{s-1}_{p,r}}\\
\le&(\sum_{k\ge -1}2^{k(s-1)r}\|\Delta_k(\sum_{q=n+1}^{n+l}\Delta_q u_0)\|^r_{L^p})^{\frac{1}{r}}\le C(\sum_{k=n}^{n+l+1}2^{-kr}2^{krs}\|\Delta_k u_0\|^r_{L^p})^{\frac{1}{r}}\\
\le&C2^{-n}\|u_0\|_{B^s_{p,r}}.
\end{aligned}$$

Hence, there exists a constant C_T independent of n,l such that for all $t\in[0,T]$

$$\|(u^{(n+l+1)}-u^{(n+1)})(t)\|_{B^{s-1}_{p,r}}\le C_T(2^{-n}+\int_0^t\|(u^{(n+l)}-u^{(n)})(\tau)\|_{B^{s-1}_{p,r}}d\tau).$$

Arguing by induction with respect to the index n, we deduce

$$\begin{aligned}
\|(u^{(n+l+1)}-u^{(n+1)})(t)\|_{B^{s-1}_{p,r}}\le&C_T(2^{-n}\sum_{k=0}^{n}\frac{(2TC_T)^k}{k!}+C_T^{n+1}\int_0^t\frac{(t-\tau)^n}{n!}d\tau)\\
\le&(C_T\sum_{k=0}^{n}\frac{(2TC_T)^k}{k!})2^{-n}+C_T\frac{(TC_T)^{n+1}}{(n+1)!},
\end{aligned}$$

which yields the desired result.

Finally, we can apply the interpolation method, which is similar to the proof in Lemma 3.1, to the critical case $s=4+\frac{1}{p}$. We here omit the details. Therefore, we complete the proof of Lemma 3.2. □

Based on the above preparations, we are in position to state the local existence result of the Cauchy problem (1).

Theorem 3.3. *Suppose that $1\le p,r\le\infty$, $s>\max\{2+\frac{1}{p},\frac{5}{2},3-\frac{1}{p}\}$ and $u_0\in B^s_{p,r}$. Then there exists a time $T>0$ such that the Cauchy problem (1) has a unique solution $u\in E^s_{p,r}(T)$, and the mapping $u_0\to u$ is continuous from $B^s_{p,r}$ into*

$$C([0,T];B^{s'}_{p,r})\cap C^1([0,T];B^{s'-1}_{p,r})$$

for all $s'<s$ if $r=\infty$ and $s'=s$ otherwise.

Proof. According to Lemma 3.2, $\{u^{(n)}\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $C([0,T];B^{s-1}_{p,r})$, so it converges to some function $u\in C([0,T];B^{s-1}_{p,r})$. Thanks to Lemma 3.2 and Proposition 2 (iv) Fatou lemma, we have that $u\in L^\infty([0,T];B^s_{p,r})$. Thus, by the interpolation method, for all $s'<s$, we find that $u\in C([0,T];B^{s'}_{p,r})$.

Taking limit in Eq. (7), we conclude that u solves Eq. (1) in the sense of $u\in C([0,T];B^{s'-1}_{p,r})$, for all $s'<s$. Since $u\in L^\infty([0,T];B^s_{p,r})$ and the fact $B^s_{p,r}$ is an algebra as $s>2+\frac{1}{p}$, the right-hand side of the following equation

$$m_t-[\frac{k_1}{2}(u^2-u_x^2)+\frac{k_2}{2}u]m_x=k_1u_xm^2+k_2u_xm,$$

belongs to $L^\infty([0,T];B^{s-2}_{p,r})$. In particular, for $r<\infty$, Lemma 2.4 enables us to get that $u\in C([0,T];B^{s'}_{p,r})$ for all $s'\le s$. Finally, taking advantage of Eq. (1) again, we obtain that $\partial_t u\in C([0,T];B^{s-1}_{p,r})$ if $r<\infty$, and in $L^\infty([0,T];B^{s-1}_{p,r})$ otherwise.

Moreover, the continuity with respect to the initial data in

$$C([0,T];B^{s'}_{p,r})\cap C^1([0,T];B^{s'-1}_{p,r})\quad(\forall s'<s)$$

can be obtained by Lemma 3.1 and a simple interpolation argument. While the case $s'=s$, a standard of use of a sequence of viscosity approximate solutions $\{u_\varepsilon\}_{\varepsilon>0}$ for Eq. (1) which converges uniformly in $C([0,T];B^s_{p,r})\cap C^1([0,T];B^{s-1}_{p,r})$ gives the proof of the continuity of solutions in $E^s_{p,r}(T)$. This completes the proof of the theorem. □

Remark 1. We know that nonhomogeneous Besov spaces contain Sobolev spaces. In fact, by Fourier-Plancherel formula, we find that the Besov space $B^s_{2,2}(\mathbb{R})$ coincides with the Sobolev space $H^s(\mathbb{R})$. Therefore, Theorem 3.3 implies that under the assumption $u_0\in H^s(\mathbb{R}), s>\frac{5}{2}$, we can obtain the local well-posedness result to Eq. (1).

Remark 2. The existence time for Eq. (1) can be chosen independently of s in the following sense [34]. If

$$u\in C([0,T];H^s)\cap C^1([0,T];H^{s-1})$$

is a solution to Eq. (1) with initial data $H^r, r>\frac{5}{2}, r\neq s$, then

$$u\in C([0,T];H^r)\cap C^1([0,T];H^{r-1})$$

with the same time $T>0$. In particular, if $u_0\in H^\infty$, then $u\in C([0,T];H^\infty)$.

4. **Blow-up scenario.** In this section, by using the local well-posedness result of Theorem 3.3 and energy estimates, we present a precise blow-up scenario for strong solutions to the Cauchy problem (1).

Theorem 4.1. *Let $u_0\in H^s(\mathbb{R}), s>\frac{5}{2}$ be given and assume that T is the maximal existence time of the solution $u(t,x)$ to Eq. (1) with the initial data u_0 guaranteed by Remark 1. When we take k_1,k_2 as non-positive constants, then the corresponding solution $u(t,x)$ blows up in finite time if and only if*

$$\liminf_{t\to T}\inf_{x\in\mathbb{R}}\{mu_x(t,x)\}=-\infty\qquad or\qquad \liminf_{t\to T}\inf_{x\in\mathbb{R}}\{u_x(t,x)\}=-\infty.$$

Proof. From Remark 1-2 and a simple density argument, we only need to prove that Theorem 4.1 holds true for $s=4$. Multiplying Eq. (1) by m, integrating over $\mathbb{R}$ and integration by parts, we get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}}m^2dx=&\frac{k_1}{2}\int_{\mathbb{R}}(u^2-u_x^2)mm_xdx+k_1\int_{\mathbb{R}}u_xm^3dx+k_2\int_{\mathbb{R}}u_xm^2dx+\frac{k_2}{2}umm_xdx\\=&-\frac{k_1}{2}\int_{\mathbb{R}}u_xm^3dx+k_1\int_{\mathbb{R}}u_xm^3dx+k_2\int_{\mathbb{R}}u_xm^2dx-\frac{k_2}{4}\int_{\mathbb{R}}u_xm^2dx\\=&\frac{k_1}{2}\int_{\mathbb{R}}u_xm^3dx+\frac{3k_2}{4}\int_{\mathbb{R}}u_xm^2dx.\end{aligned}\tag{17}$$

Differentiating Eq. (1) with respect to x, we deduce

$$\begin{aligned}m_{tx}=&3k_1u_xmm_x-k_1m^3+k_1um^2+\frac{k_1}{2}(u^2-u_x^2)m_{xx}\\&+\frac{3k_2}{2}m_xu_x+k_2um-k_2m^2+\frac{k_2}{2}um_{xx}.\end{aligned}$$

Multiplying the above equation by m_x, and integrating with respect x over $\mathbb{R}$, we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}} m_x^2 dx =& 3k_1\int_{\mathbb{R}} u_x m m_x^2 dx - k_1\int_{\mathbb{R}} m^3 m_x dx + k_1\int_{\mathbb{R}} u m^2 m_x dx \\ &+ \frac{k_1}{2}\int_{\mathbb{R}} (u^2-u_x^2) m_x m_{xx} dx + \frac{3k_2}{2}\int_{\mathbb{R}} m_x^2 u_x dx \\ &+ k_2\int_{\mathbb{R}} u m m_x dx - k_2\int_{\mathbb{R}} m^2 m_x dx + \frac{k_2}{2}\int_{\mathbb{R}} u m_x m_{xx} dx \\ =& 3k_1\int_{\mathbb{R}} u_x m m_x^2 dx - \frac{k_1}{3}\int_{\mathbb{R}} u_x m^3 dx - \frac{k_1}{4}\int_{\mathbb{R}} m_x^2 (u^2-u_x^2)_x dx \\ &+ \frac{3k_2}{2}\int_{\mathbb{R}} m_x^2 u_x dx - \frac{k_2}{2}\int_{\mathbb{R}} u_x m^2 dx - \frac{k_2}{4}\int_{\mathbb{R}} u - x m_x^2 dx \\ =& \frac{5k_1}{2}\int_{\mathbb{R}} u_x m m_x^2 dx - \frac{k_1}{3}\int_{\mathbb{R}} u_x m^3 dx + \frac{5k_2}{4}\int_{\mathbb{R}} m_x^2 u_x dx - \frac{k_2}{2}\int_{\mathbb{R}} u_x m^2 dx.\end{aligned} \tag{18}$$

From (17)-(18), we get

$$\begin{aligned}&\frac{d}{dt}\int_{\mathbb{R}} (m^2+m_x^2) dx \\ =& 5k_1\int_{\mathbb{R}} u_x m m_x^2 dx + \frac{k_1}{3}\int_{\mathbb{R}} u_x m^3 dx + \frac{5k_2}{2}\int_{\mathbb{R}} m_x^2 u_x dx + \frac{k_2}{2}\int_{\mathbb{R}} u_x m^2 dx.\end{aligned}$$

Assume that $T<\infty$ and there exists $N_1, N_2 > 0$ such that $mu_x \geq -N_1, u_x \geq -N_2$ for all $(t,x) \in [0,T) \times \mathbb{R}$. Let us choose $N, k > 0$ such that $N := \max\{N_1, N_2\}$ and $k := \max\{-k_1, -k_2\}$. It then follows that

$$\frac{d}{dt}\int_{\mathbb{R}} (m^2+m_x^2) dx \leq 10kN \int_{\mathbb{R}} (m^2+m_x^2) dx.$$

Applying Gronwall's lemma to the above inequality implies for $t \in [0,T)$,

$$\|m\|_{H^1}^2 \leq e^{10NkT} \|m_0\|_{H^1}^2. \tag{19}$$

Differentiating Eq. (1) with respect to x twice, we deduce

$$\begin{aligned}m_{txx} =& -6k_1 m^2 m_x + 5k_1 u m m_x + 4k_1 u_x m m_{xx} + 3k_1 u_x m_x^2 \\ &+ k_1 u_x m^2 + \frac{k_1}{2}(u^2-u_x^2) m_{xxx} + 2k_2 u_x m_{xx} - \frac{7k_2}{2} m m_x \\ &+ \frac{5k_2}{2} u m_x + k_2 u_x m + \frac{k_2}{2} u m_{xxx}.\end{aligned}$$

Multiplying the above equation by m_{xx}, integrating with respect to x over $\mathbb{R}$, we have

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}} m_{xx}^2 dx =& -6k_1\int_{\mathbb{R}} m^2 m_x m_{xx}dx+5k_1\int_{\mathbb{R}} umm_x m_{xx}dx+4k_1\int_{\mathbb{R}} u_x m m_{xx}^2 dx\\
&+3k_1\int_{\mathbb{R}} u_x m_x^2 m_{xx}dx + k_1\int_{\mathbb{R}} u_x m^2 m_{xx}dx\\
&+\frac{k_1}{2}\int_{\mathbb{R}}(u^2-u_x^2)m_{xx}m_{xxx}dx+2k_2\int_{\mathbb{R}} u_x m_{xx}^2 dx-\frac{7k_2}{2}\int_{\mathbb{R}} mm_x m_{xx}dx\\
&+\frac{5k_2}{2}\int_{\mathbb{R}} um_x m_{xx}dx + k_2\int_{\mathbb{R}} u_x m m_{xx}dx+\frac{k_2}{2}\int_{\mathbb{R}} um_{xx}m_{xxx}dx\\
=& -6k_1\int_{\mathbb{R}} m^2 m_x m_{xx}dx+5k_1\int_{\mathbb{R}} umm_x m_{xx}dx+4k_1\int_{\mathbb{R}} u_x m m_{xx}^2 dx\\
&+k_1\int_{\mathbb{R}} u_x m m_x^2 dx + 2k_1\int_{\mathbb{R}} umm_x m_{xx}dx - k_1\int_{\mathbb{R}} m^2 m_x m_{xx}dx\\
&-2k_1\int_{\mathbb{R}} u_x m m_x^2 dx + \frac{k_1}{3}\int_{\mathbb{R}} u_x m^3 dx - \frac{k_1}{2}\int_{\mathbb{R}} u_x m m_{xx}^2 dx\\
&+2k_2\int_{\mathbb{R}} u_x m_{xx}^2 dx - \frac{7k_2}{2}\int_{\mathbb{R}} mm_x m_{xx}dx - \frac{5k_2}{4}\int_{\mathbb{R}} u_x m_x^2 dx\\
&-k_2\int_{\mathbb{R}} u_x m_x^2 dx + \frac{k_2}{2}\int_{\mathbb{R}} u_x m^2 dx - \frac{k_2}{4}\int_{\mathbb{R}} u_x m_{xx}^2 dx\\
=& -7k_1\int_{\mathbb{R}} m^2 m_x m_{xx}dx+7k_1\int_{\mathbb{R}} umm_x m_{xx}dx+\frac{7k_1}{2}\int_{\mathbb{R}} u_x m m_{xx}^2 dx\\
&-k_1\int_{\mathbb{R}} u_x m m_x^2 dx + \frac{k_1}{3}\int_{\mathbb{R}} u_x m^3 dx + \frac{7k_2}{4}\int_{\mathbb{R}} u_x m_{xx}^2 dx\\
&-\frac{7k_2}{2}\int_{\mathbb{R}} mm_x m_{xx}dx - \frac{9k_2}{4}\int_{\mathbb{R}} u_x m^2 dx + \frac{k_2}{2}\int_{\mathbb{R}} u_x m^2 dx. \qquad (20)
\end{aligned}$$

Combining (17)-(18) and (20), we obtain

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}}(m^2+m_x^2+m_{xx}^2)dx\\
=& -14k_1\int_{\mathbb{R}} m^2 m_x m_{xx}dx + 14k_1\int_{\mathbb{R}} umm_x m_{xx}dx + 7k_1\int_{\mathbb{R}} u_x m m_{xx}^2 dx\\
&+3k_1\int_{\mathbb{R}} u_x m m_x^2 dx + k_1\int_{\mathbb{R}} u_x m^3 dx + \frac{7k_2}{2}\int_{\mathbb{R}} u_x m_{xx}^2 dx\\
&-7k_2\int_{\mathbb{R}} mm_x m_{xx}dx - 2\int_{\mathbb{R}} u_x m^2 dx + \frac{3k_2}{2}\int_{\mathbb{R}} u_x m^2 dx.
\end{aligned}$$

If mu_x and u_x are bounded from below on $[0,T)\times\mathbb{R}$, *i.e.*, there exists $N_1, N_2 > 0$ such that $mu_x \geq -N_1, u_x \geq -N_2$ for all $(t,x)\in[0,T)\times\mathbb{R}$. Similarly, we can choose $N, k > 0$ such that $N := \max\{N_1, N_2\}$ and $k := \max\{-k_1, -k_2\}$. Then, by (19) and the above equality, we get

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}}(m^2+m_x^2+m_{xx}^2)dx\\
\leq&\frac{21}{2}kN\int_{\mathbb{R}}(m^2+m_x^2+m_{xx}^2)dx+14k(\|m\|_{L^\infty}^2+\|um\|_{L^\infty}\\
&+\|m\|_{L^\infty})\int_{\mathbb{R}}|m_xm_{xx}|dx\\
\leq&\frac{21}{2}kN\int_{\mathbb{R}}(m^2+m_x^2+m_{xx}^2)dx+14k(\|m\|_{H^1}^2+\|m\|_{H^1})\int_{\mathbb{R}}(m_x^2+m_{xx}^2)dx\\
\leq&7k[\frac{3N}{2}+2e^{5NkT}\|m_0\|_{H^1}(e^{5NkT}\|m_0\|_{H^1}+1)]\int_{\mathbb{R}}(m^2+m_x^2+m_{xx}^2)dx.
\end{aligned}$$

Hence, applying Gronwall's inequality implies that for all $t\in[0,T)$

$$\begin{aligned}
\|u\|_{H^4}^2\leq&\|m\|_{H^2}^2=\int_{\mathbb{R}}(m^2+m_x^2+m_{xx}^2)dx\\
\leq&\exp\{7k[\frac{3N}{2}+2e^{5NkT}\|m_0\|_{H^1}(e^{5NkT}\|m_0\|_{H^1}+1)]\}\|m_0\|_{H^2}^2\\
\leq&C\exp\{7k[\frac{3N}{2}+2e^{5NkT}\|m_0\|_{H^1}(e^{5NkT}\|m_0\|_{H^1}+1)]\}\|u_0\|_{H^4}^2.
\end{aligned}$$

The above inequality and Sobolev's embedding theorem ensure that $u(t,x)$ does not blow up in finite time.

On the other hand, by Sobolev's imbedding theorem, we find that if $\lim\inf_{t\to T}\inf_{x\in\mathbb{R}}\{m\, u_x(t,x)\}=-\infty$ or $\lim\inf_{t\to T}\inf_{x\in\mathbb{R}}\{u_x(t,x)\}=-\infty$, then the solution will blow up in finite time. This completes the proof of the theorem. □

5. **The existence of peaked solutions.** In order to understand the meaning of a peaked solution to Eq. (1), we first rewrite Eq. (1) as

$$\begin{aligned}
&u_t-\frac{k_1}{2}u^2u_x+\frac{k_1}{6}u_x^3-\frac{k_2}{2}uu_x-\frac{k_1}{6}(1-\partial_x^2)^{-1}u_x^3-\frac{1}{2}\partial_x(1-\partial_x^2)^{-1}\big(k_1\\
&\times(uu_x^2+\frac{2}{3}u^3)+k_2(u^2+\frac{1}{2}u_x^2)\big)=0.
\end{aligned}$$

Note that if $p(x):=\frac{1}{2}e^{-|x|}, x\in\mathbb{R}$, then $(1-\partial_x^2)^{-1}f=p*f$ for all $f\in L^2$. From the above two facts, we can then define the notion of weak solutions as follows.

Definition 5.1. Let $u_0\in W^{1,3}$ be given. If $u(t,x)\in L^\infty_{loc}([0,T);W^{1,3}_{loc})$ and satisfies

$$\begin{aligned}
&\int_0^T\int_{\mathbb{R}}\Big(u\phi_t-\frac{1}{6}k_1u^3\phi_x-\frac{1}{6}k_1u_x^3\phi-\frac{1}{4}k_2u^2\phi_x-\frac{1}{2}p*\big(k_1(uu_x^2+\frac{2}{3}u^3)\\
&+k_2(u^2+\frac{1}{2}u_x^2)\big)\phi_x+\frac{1}{6}k_1(p*u_x^3)\phi\Big)dxdt+\int_{\mathbb{R}}u_0\phi(0,x)dx=0,
\end{aligned}$$

for all functions $\phi\in C_c^\infty([0,T)\times\mathbb{R})$, then $u(t,x)$ is called a weak solution to Eq. (1). If u is a weak solution on $[0,T)$ for every $T>0$, then it is called a global weak solution.

Next, we prove the existence of single peakon to Eq. (1).

Theorem 5.2. *The peaked functions of the form*

$$\varphi_c(t,x)=C_1e^{-|x-ct|},$$

where C_1 satisfies $\frac{1}{3}k_1C_1^2+\frac{1}{2}k_2C_1+c=0$, is a global weak solution to Eq. (1) in the sense of Definition 5.1. Moreover, for every time $t\geq 0$, the peaked solutions $\varphi_c(t,x)$ belongs to $H^1\cap W^{1,\infty}$.

Remark 3. (i) For $k_1=0, k_2\neq 0$, we have $C_1=-\frac{2c}{k_2}$. In particular, if $k_1=0, k_2=-2$, then we obtain the single peakon $\varphi_c(t,x)=ce^{-|x-ct|}$ for the CH equation.
(ii) For $k_1\neq 0$, we easily get

$$C_1=\frac{-3(\sqrt{3}k_2\pm\sqrt{3k_2^2-16k_1c})}{4\sqrt{3}k_1}. \tag{21}$$

If $3k_2^2-16k_1c\geq 0$, then the coefficient C_1 of the peakons φ_c is a real number. For example, if we choose $k_1=-2, k_2=0$, and $c>0$, then we obtain the single peakon $\varphi_c(t,x)=\pm\sqrt{\frac{3}{2}c}e^{-|x-ct|}$ of the modified CH equation (2). If $3k_2^2-16k_1c<0$, then the coefficient C_1 of the peakons φ_c is a complex number. In [28], the authors call it as a complex peakon, *i.e.*, the peakon has the complex coefficient. Thus, we can propose here the complex peakon for Eq. (1), which is not presented in both the CH equation and the modified CH equation (2).

Proof. For any test function $\phi(\cdot)\in C_c^\infty(\mathbb{R})$, using integration by parts, we infer

$$\begin{aligned}\int_{\mathbb{R}} e^{-|y|}\phi'(y)dy=&\int_{-\infty}^0 e^y\phi'(y)dy+\int_0^\infty e^{-y}\phi'(y)dy\\ =&e^y\phi(y)\Big|_{-\infty}^0-\int_{-\infty}^0 e^y\phi(y)dy+e^{-y}\phi(y)\Big|_0^\infty+\int_0^\infty e^{-y}\phi(y)dy\\ =&-\int_{-\infty}^0 e^y\phi(y)dy+\int_0^\infty e^{-y}\phi(y)dy=\int_{\mathbb{R}} sign(y)e^{-|y|}\phi(y)dy.\end{aligned}$$

Thus, for all $t\geq 0$, we have

$$\partial_x\varphi_c(t,x)=-\mathrm{sign}(x-ct)\varphi_c(t,x), \tag{22}$$

in the sense of distribution $\mathcal{S}'(\mathbb{R})$. Hence, the peaked solutions $\varphi_c(t,x)$ belongs to $H^1\cap W^{1,\infty}$. The same computation as in (22), for all $t\geq 0$, yields,

$$\partial_t\varphi_c(t,x)=c\ \mathrm{sign}(x-ct)\varphi_c(t,x)\in L^\infty. \tag{23}$$

If denoting $\varphi_{0,c}(x):=\varphi_c(0,x)$, then we get

$$\lim_{t\to 0^+}\|\varphi_c(t,\cdot)-\varphi_{0,c}(x)\|_{W^{1,\infty}}=0. \tag{24}$$

Combining (22)-(24) and integrating by parts, for every test function $\phi(t,x)\in C_c^\infty([0,\infty)\times\mathbb{R})$, we obtain

$$\begin{aligned}&\int_0^\infty\int_{\mathbb{R}}\big(\varphi_c\partial_t\phi-\frac{1}{6}k_1\varphi_c^3\partial_x\phi-\frac{1}{6}k_1(\partial_x\varphi_c)^3\phi-\frac{k_2}{4}\varphi_c^2\partial_x\phi\big)dxdt+\int_{\mathbb{R}}\varphi_{0,c}(x)\phi(0,x)dx\\ =&-\int_0^\infty\int_{\mathbb{R}}\big(\partial_t\varphi_c-\frac{k_1}{2}\varphi_c^2\partial_x\varphi_c+\frac{1}{6}k_1(\partial_x\varphi_c)^3-\frac{k_2}{2}\varphi_c\partial_x\varphi_c\big)\phi dxdt\\ =&-\int_0^\infty\int_{\mathbb{R}}\phi\ \mathrm{sign}(x-ct)\varphi_c(c+\frac{k_1}{3}\varphi_c^2+\frac{k_2}{2}\varphi_c)dxdt.\end{aligned} \tag{25}$$

Form the definition of φ_c and C_1 satisfying $\frac{1}{3}k_1C_1^2+\frac{1}{2}k_2C_1+c=0$, for $x>ct$, we have

$$\begin{aligned}&\mathrm{sign}(x-ct)\varphi_c(c+\frac{k_1}{3}\varphi_c^2+\frac{k_2}{2}\varphi_c)\\=&C_1e^{-(x-ct)}(c+\frac{k_1}{3}C_1^2e^{-2(x-ct)}+\frac{k_2}{2}C_1e^{-(x-ct)})\\=&-\frac{k_1}{3}C_1^3e^{ct-x}-\frac{k_2}{2}C_1^2e^{ct-x}+\frac{k_1}{3}C_1^3e^{3(ct-x)}+\frac{k_2}{2}C_1^2e^{2(ct-x)}.\end{aligned}\tag{26}$$

Similarly, for $x\le ct$, we find

$$\begin{aligned}&\mathrm{sign}(x-ct)\varphi_c(c+\frac{k_1}{3}\varphi_c^2+\frac{k_2}{2}\varphi_c)\\=&-C_1e^{x-ct}(c+\frac{k_1}{3}C_1^2e^{2(x-ct)}+\frac{k_2}{2}C_1e^{x-ct})\\=&\frac{k_1}{3}C_1^3e^{x-ct}+\frac{k_2}{2}C_1^2e^{x-ct}-\frac{k_1}{3}C_1^3e^{3(x-ct)}-\frac{k_2}{2}C_1^2e^{2(x-ct)}.\end{aligned}\tag{27}$$

On the other hand, similar to Definition 5.1, we derive

$$\begin{aligned}&-\int_0^\infty\int_{\mathbb{R}}\frac{1}{2}(1-\partial_x^2)^{-1}\big(k_1\varphi_c(\partial_x\varphi_c)^2+\frac{2}{3}k_1\varphi_c^3+k_2\varphi_c^2+\frac{1}{2}k_2(\partial_x\varphi_c)^2\big)\partial_x\phi\\&+\frac{1}{6}k_1(1-\partial_x^2)^{-1}(\partial_x\varphi_c)^3\phi dxdt=\int_0^\infty\int_{\mathbb{R}}\big[\frac{1}{2}\phi\ \partial_xp*\big(k_1\varphi_c(\partial_x\varphi_c)^2+\frac{k_2}{2}(\partial_x\varphi_c)^2\big)\\&+\phi\ p*\big(k_1\varphi_c^2\partial_x\varphi_c+k_2\varphi_c\partial_x\varphi_c+\frac{k_1}{6}(\partial_x\varphi_c)^3\big)\big]dxdt.\end{aligned}\tag{28}$$

From (22), we have

$$\begin{aligned}&k_1\varphi_c^2\partial_x\varphi_c+k_2\varphi_c\partial_x\varphi_c+\frac{k_1}{6}(\partial_x\varphi_c)^3\\=&-k_1\mathrm{sign}(x-ct)\varphi_c^3-k_2\mathrm{sign}(x-ct)\varphi_c^2-\frac{k_1}{6}\mathrm{sign}^3(x-ct)\varphi_c^3\\=&\frac{k_2}{2}\partial_x(\varphi_c^2)+\frac{7}{18}k_1\partial_x(\varphi_c^3).\end{aligned}\tag{29}$$

Inserting (29) into (28), we obtain

$$\begin{aligned}&-\int_0^\infty\int_{\mathbb{R}}\frac{1}{2}(1-\partial_x^2)^{-1}\big(k_1\varphi_c(\partial_x\varphi_c)^2+\frac{2}{3}k_1\varphi_c^3+k_2\varphi_c^2+\frac{1}{2}k_2(\partial_x\varphi_c)^2\big)\partial_x\phi\\&+\frac{1}{6}k_1(1-\partial_x^2)^{-1}(\partial_x\varphi_c)^3\phi dxdt=\int_0^\infty\int_{\mathbb{R}}\phi\ \partial_xp*(\frac{k_1}{2}\varphi_c(\partial_x\varphi_c)^2+\frac{k_2}{4}(\partial_x\varphi_c)^2\\&+\frac{k_2}{2}\varphi_c^2+\frac{7}{18}k_1\varphi_c^3)dxdt.\end{aligned}\tag{30}$$

Note that $\partial_xp(x)=-\frac{1}{2}\mathrm{sign}(x)e^{-|x|}, x\in\mathbb{R}$, we deduce

$$\begin{aligned}&\partial_xp*(\frac{k_1}{2}\varphi_c(\partial_x\varphi_c)^2+\frac{k_2}{4}(\partial_x\varphi_c)^2+\frac{k_2}{2}\varphi_c^2+\frac{7}{18}k_1\varphi_c^3)(t,x)\\=&-\frac{1}{2}\int_{-\infty}^\infty\mathrm{sign}(x-y)e^{-|x-y|}\big((\frac{k_1}{2}\mathrm{sign}^2(y-ct)+\frac{7}{18}k_1)C_1^3e^{-3|y-ct|}\\&+(\frac{k_2}{4}\mathrm{sign}^2(y-ct)+\frac{k_2}{2})C_1^2e^{-2|y-ct|}\big)dy.\end{aligned}\tag{31}$$

When $x > ct$, we can split the right hand side of (31) into the following three parts

$$\begin{aligned}&\partial_x p * \big(\frac{k_1}{2}\varphi_c(\partial_x\varphi_c)^2 + \frac{k_2}{4}(\partial_x\varphi_c)^2 + \frac{k_2}{2}\varphi_c^2 + \frac{7}{18}k_1\varphi_c^3\big)(t,x)\\
=&-\frac{1}{2}\big(\int_{-\infty}^{ct} + \int_{ct}^{x} + \int_{x}^{\infty}\big)\mathrm{sign}(x-y)e^{-|x-y|}\big(\big(\frac{k_1}{2}\mathrm{sign}^2(y-ct) + \frac{7}{18}k_1\big)\\
&\times C_1^3 e^{-3|y-ct|} + \big(\frac{k_2}{4}\mathrm{sign}^2(y-ct) + \frac{k_2}{2}\big)C_1^2 e^{-2|y-ct|}\big)dy\\
:=&I_1 + I_2 + I_3.\end{aligned}$$

A direct calculation for each one of the terms $I_i, 1 \le i \le 3$, yields

$$\begin{aligned}I_1 =& -\frac{1}{2}\int_{-\infty}^{ct} e^{-(x-y)}\big(\frac{8}{9}k_1C_1^3e^{3(y-ct)} + \frac{3}{4}k_2C_1^2e^{2(y-ct)}\big)dy\\
=& -\frac{4}{9}k_1C_1^3e^{-(x+3ct)}\int_{-\infty}^{ct} e^{4y}dy - \frac{3}{8}k_2C_1^2e^{-(x+2ct)}\int_{-\infty}^{ct} e^{3y}dy\\
=& -\frac{k_1}{9}C_1^3e^{ct-x} - \frac{k_2}{8}C_1^2e^{ct-x},\end{aligned}$$

$$\begin{aligned}I_2 =& -\frac{1}{2}\int_{ct}^{x} e^{-(x-y)}\big(\frac{8}{9}k_1C_1^3e^{-3(y-ct)} + \frac{3}{4}k_2C_1^2e^{-2(y-ct)}\big)dy\\
=& -\frac{4}{9}k_1C_1^3e^{-(x-3ct)}\int_{ct}^{x} e^{-2y}dy - \frac{3}{8}k_2C_1^2e^{-(x-2ct)}\int_{ct}^{x} e^{-y}dy\\
=&\frac{2}{9}k_1C_1^3(e^{3(ct-x)} - e^{ct-x}) + \frac{3}{8}k_2C_1^2(e^{2(ct-x)} - e^{ct-x}),\end{aligned}$$

and

$$\begin{aligned}I_3 =&\frac{1}{2}\int_{x}^{\infty} e^{x-y}\big(\frac{8}{9}k_1C_1^3e^{-3(y-ct)} + \frac{3}{4}k_2C_1^2e^{-2(y-ct)}\big)dy\\
=&\frac{4}{9}k_1C_1^3e^{x+3ct}\int_{x}^{\infty} e^{-4y}dy + \frac{3}{8}k_2C_1^2e^{x+2ct}\int_{x}^{\infty} e^{-3y}dy\\
=&\frac{k_1}{9}C_1^3e^{3(ct-x)} + \frac{k_2}{8}C_1^2e^{2(ct-x)}.\end{aligned}$$

By the above equalities I_1-I_3, for $x > ct$, we have

$$\begin{aligned}&\partial_x p * \big(\frac{k_1}{2}\varphi_c(\partial_x\varphi_c)^2 + \frac{k_2}{4}(\partial_x\varphi_c)^2 + \frac{k_2}{2}\varphi_c^2 + \frac{7}{18}k_1\varphi_c^3\big)(t,x)\\
=&-\frac{k_1}{3}C_1^3e^{ct-x} + \frac{k_1}{3}C_1^3e^{3(ct-x)} - \frac{k_2}{2}C_1^2e^{ct-x} + \frac{k_2}{2}C_1^2e^{2(ct-x)} \qquad (32)\end{aligned}$$

While for the case $x \le ct$, we can also split the right hand side of (31) into the following three parts

$$\begin{aligned}&\partial_x p * \big(\frac{k_1}{2}\varphi_c(\partial_x\varphi_c)^2 + \frac{k_2}{4}(\partial_x\varphi_c)^2 + \frac{k_2}{2}\varphi_c^2 + \frac{7}{18}k_1\varphi_c^3\big)(t,x)\\
=&-\frac{1}{2}\big(\int_{-\infty}^{x} + \int_{x}^{ct} + \int_{ct}^{\infty}\big)\mathrm{sign}(x-y)e^{-|x-y|}\big(\big(\frac{k_1}{2}\mathrm{sign}^2(y-ct) + \frac{7}{18}k_1\big)\\
&\times C_1^3 e^{-3|y-ct|} + \big(\frac{k_2}{4}\mathrm{sign}^2(y-ct) + \frac{k_2}{2}\big)C_1^2 e^{-2|y-ct|}\big)dy\\
:=&II_1 + II_2 + II_3.\end{aligned}$$

We now directly compute each one of the terms $II_i, 1 \leq i \leq 3$, as follows

$$
\begin{aligned}
II_1 =& -\frac{1}{2}\int_{-\infty}^{x} e^{-(x-y)}\big(\frac{8}{9}k_1C_1^3e^{3(y-ct)} + \frac{3}{4}k_2C_1^2e^{2(y-ct)}\big)dy\\
=& -\frac{4}{9}k_1C_1^3e^{-(x+3ct)}\int_{-\infty}^{x} e^{4y}dy - \frac{3}{8}k_2C_1^2e^{-(x+2ct)}\int_{-\infty}^{x} e^{3y}dy\\
=& -\frac{k_1}{9}C_1^3e^{3(x-ct)} - \frac{k_2}{8}C_1^2e^{2(x-ct)},
\end{aligned}
$$

$$
\begin{aligned}
II_2 =& \frac{1}{2}\int_{x}^{ct} e^{x-y}\big(\frac{8}{9}k_1C_1^3e^{3(y-ct)} + \frac{3}{4}k_2C_1^2e^{2(y-ct)}\big)dy\\
=& \frac{4}{9}k_1C_1^3e^{x-3ct}\int_{x}^{ct} e^{2y}dy + \frac{3}{8}k_2C_1^2e^{x-2ct}\int_{x}^{ct} e^{y}dy\\
=& \frac{2}{9}k_1C_1^3(e^{x-ct} - e^{3(x-ct)}) + \frac{3}{8}k_2C_1^2(e^{x-ct} - e^{2(x-ct)}),
\end{aligned}
$$

and

$$
\begin{aligned}
II_3 =& \frac{1}{2}\int_{ct}^{\infty} e^{x-y}\big(\frac{8}{9}k_1C_1^3e^{-3(y-ct)} + \frac{3}{4}k_2C_1^2e^{-2(y-ct)}\big)dy\\
=& \frac{4}{9}k_1C_1^3e^{x+3ct}\int_{ct}^{\infty} e^{-4y}dy + \frac{3}{8}k_2C_1^2e^{x+2ct}\int_{ct}^{\infty} e^{-3y}dy\\
=& \frac{k_1}{9}C_1^3e^{x-ct} + \frac{k_2}{8}C_1^2e^{x-ct}.
\end{aligned}
$$

By the above equalities II_1-II_3, for $x \leq ct$, we obtain

$$
\begin{aligned}
&\partial_x p * \big(\frac{k_1}{2}\varphi_c(\partial_x\varphi_c)^2 + \frac{k_2}{4}(\partial_x\varphi_c)^2 + \frac{k_2}{2}\varphi_c^2 + \frac{7}{18}k_1\varphi_c^3\big)(t,x)\\
=& \frac{k_1}{3}C_1^3e^{x-ct} - \frac{k_1}{3}C_1^3e^{3(x-ct)} + \frac{k_2}{2}C_1^2e^{x-ct} - \frac{k_2}{2}C_1^2e^{2(x-ct)}. \qquad (33)
\end{aligned}
$$

Combining (25)-(27) with (30)-(33), we infer that

$$
\begin{aligned}
&\int_0^\infty\int_{\mathbb{R}} \big[\varphi_c\partial_t\phi - \frac{1}{6}k_1\varphi_c^3\partial_x\phi - \frac{1}{6}k_1(\partial_x\varphi_c)^3\phi - \frac{k_2}{4}\varphi_c^2\partial_x\phi - \frac{1}{2}(1-\partial_x^2)^{-1}\big(k_1\\
&\times \varphi_c(\partial_x\varphi_c)^2 + \frac{2}{3}k_1\varphi_c^3 + k_2\varphi_c^2 + \frac{1}{2}k_2(\partial_x\varphi_c)^2\big)\partial_x\phi + \frac{1}{6}k_1(1-\partial_x^2)^{-1}(\partial_x\varphi_c)^3\\
&\times \phi\big]dxdt + \int_{\mathbb{R}} \varphi_{0,c}(x)\phi(0,x)dx = 0
\end{aligned}
$$

for every test function $\phi(t,x) \in C_c^\infty([0,\infty)\times\mathbb{R})$. This completes the proof of Theorem 5.2. □

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