DECOMPOSITION OF THE MODIFIED KADOMTSEV–PETVIASHVILI EQUATION AND ITS FINITE BAND SOLUTION

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The modified Kadomtsev–Petviashvili (mKP) equation is revisited from two $1 + 1$-dimensional integrable equations whose compatible solutions yield a special solution of the mKP equation in view of a transformation. By employing the finite-order expansion of Lax matrix, the mKP equation is reduced to three solvable ordinary differential equations (ODEs). The associated flows induced by the mKP equation are linearized under the Abel–Jacobi coordinates on a Riemann surface. Finally, a finite band solution expressed by Riemann-theta functions for the mKP equation is obtained through the Jacobi inversion.

Keywords: mKP equation; Jacobi inversion; finite band solution.

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1. Introduction

The finite band (algebro-geometric or quasi-periodic) solutions are a remarkable class of exact solutions, which were originally introduced in 1974 by Novikov [21] dedicating to the integration of the Korteweg-de Vries equation with the periodic boundary condition. A feasible theory of finite band solutions was developed with the usage of the spectral technique (see more details in [6, 7, 11, 17, 18]). Later, some well-known soliton equations, such as the Korteweg-de Vries [7, 11], nonlinear Schrödinger [12], sine-Gordon [15], KP [16] equations, were solved with finite band solutions in explicit form. Recently, the nonlinearization of Lax pair [1] has been developed to obtain the algebro-geometric solutions of soliton equations in (1+1)-dimension [22, 23, 29] with the help of algebro-geometric tool. A more extended progress of the nonlinearization method and algebro-geometric scheme is that the finite parametric solutions of two compatible $1 + 1$-dimensional integrable equations generate solutions of a $(2 + 1)$-dimensional integrable equation [2, 4, 8, 13]. Following this idea,

in this paper we find a different decomposition to solve the mKP equation by using the finite-order expansion of Lax matrix [26].

To get finite band solutions of an integrable equation, the crucial point is to choose an appropriate isospectral problem related to the equation. Then, based on the Lax pair of the integrable equation, one may apply the powerful tool of the theory of algebraic curves to derive explicit solutions in terms of Riemann-theta functions. In this paper we decompose the mKP equation into two 1+1-dimensional consistent equations, which are integrable and able to be solved through integrating three solvable ODEs. The Abel–Jacobi coordinates are appropriately chosen to straighten out the phase flows on the complex torus associated with the mKP equation. Furthermore, by employing the Jacobi inversion on the Riemann surface of hyperelliptic curve, the finite band solution of the mKP equation is obtained and expressed in terms of Riemann-theta functions. The whole paper is organized as follows. In Sec. 2, we specify the relation between the mKP equation and two $(1 + 1)$-dimensional integrable equations with the help of a transformation. In Sec. 3, we reduce the mKP equation into three solvable ODEs. In the last section, we present a finite band solution of the mKP equation in explicit form.

2. Decomposition of the mKP Equation

Our starting point is the isospectral problem that was presented in 2001 by Qiao [24],

$$\varphi_x = U\varphi, \quad U = \begin{pmatrix} -\frac{1}{2}\lambda + \frac{1}{2}v & -v \\ \lambda u & \frac{1}{2}\lambda - \frac{1}{2}v \end{pmatrix}, \quad \varphi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}, \tag{2.1}$$

where λ is a spectral parameter; u and v are two spectral potentials. To derive an integrable hierarchy associated with (2.1), let us calculate the stationary zero-curvature equation

$$V_x = [U, V], \quad V = \begin{pmatrix} \lambda a & b \\ \lambda c & -\lambda a \end{pmatrix} = \sum_{j \geq 0} \begin{pmatrix} \lambda a_j & b_j \\ \lambda c_j & -\lambda a_j \end{pmatrix} \lambda^{-j}, \tag{2.2}$$

which is equivalent to

$$\begin{aligned} a_{jx} &= -vc_j - ub_j, \\ b_{jx} &= 2va_{j+1} - b_{j+1} + vb_j, \\ c_{jx} &= 2ua_{j+1} + c_{j+1} - vc_j. \end{aligned} \tag{2.3}$$

Let $S_j = (c_{j+1}, b_{j+1}, a_{j+1})^T$, then (2.3) can be rewritten as

$$KS_{j-1} = JS_j, \quad S_j|_{(u,v)=0} = 0, \quad S_{-1} = \left(-u, v, \frac{1}{2}\right)^T, \quad j \geq 0, \tag{2.4}$$

where

$$J = \begin{pmatrix} 1 & 0 & 2u \\ 0 & 1 & -2v \\ v & u & \partial \end{pmatrix}, \quad K = \begin{pmatrix} \partial + v & 0 & 0 \\ 0 & -\partial + v & 0 \\ v & u & \partial \end{pmatrix}. \tag{2.5}$$

It is easy to see that the first equation in (2.3) leads to the identity $vS_j^{(1)} + uS_j^{(2)} + \partial_x S_j^{(3)} = 0$, and each S_j could be determined uniquely by the recursive relation (2.4). For instance, the first two members are

$$S_0 = \begin{pmatrix} -u_x - 2u^2 v - uv \\ -v_x + v^2 + 2uv^2 \\ uv \end{pmatrix},$$

and

$$S_1 = \begin{pmatrix} -u_{xx} - 6u^2 v^2 - 6u^3 v^2 - 6uu_x v - 2u_x v - uv_x - uv^2 \\ v_{xx} - 3vv_x - 6uvv_x + 6u^2 v^3 + 6uv^3 + v^3 \\ u_x v - uv_x + 3u^2 v^2 + 2uv^2 \end{pmatrix}.$$

For any positive integer n, let us choose an auxiliary isospectral problem of (2.1) as follows,

$$\varphi_{t_n} = V^{(n)}\varphi, \quad V^{(n)} = \begin{pmatrix} V_{11}^{(n)} & V_{12}^{(n)} \\ V_{21}^{(n)} & -V_{11}^{(n)} \end{pmatrix}, \quad n \geq 1, \tag{2.6}$$

where

$$V_{11}^{(n)} = -\frac{1}{2}b_n + \sum_{j=0}^{n} a_j \lambda^{n+1-j}, \quad V_{12}^{(n)} = \sum_{j=0}^{n} b_j \lambda^{n-j}, \quad V_{21}^{(n)} = \sum_{j=0}^{n} c_j \lambda^{n+1-j}.$$

Thus the compatibility condition of (2.1) and (2.6), under the isospectral assumption $\lambda_{t_n} = 0$, leads to the zero-curvature equation

$$U_{t_n} - V_x^{(n)} + [U, V^{(n)}] = 0,$$

which generates the desired $1 + 1$-dimensional integrable hierarchy

$$(u, v)_{t_n}^T = (-a_n + c_n, -b_n)_x^T, \quad n \geq 0, \tag{2.7}$$

in the sense of Lax compatibility. Clearly, the first two nontrivial integrable equations are

$$\begin{cases} u_y = -u_{xx} - 4uu_x v - 2u^2 v_x - 2u_x v - 2uv_x, \\ v_y = v_{xx} - 2vv_x - 2u_x v^2 - 4uvv_x, \end{cases} \tag{2.8}$$

and

$$\begin{cases} u_t = -u_{xxx} - 3(u_x v)_x - 3(uv^2)_x - 9(u^2 v^2)_x - 6(u^3 v^2)_x - 6(uu_x v)_x, \\ v_t = -v_{xxx} + 3(vv_x)_x + 6(uvv_x)_x - 6(u^2 v^3)_x - 6(uv^3)_x - 3v^2 v_x, \end{cases} \tag{2.9}$$

where we set $t_1 = y$ and $t_2 = t$ in time variables. And, it is worthwhile to point out that the system (2.8) belongs to the integrable equations of nonlinear Schrödinger type [19].

Let (u, v) be the common solution of (2.8) and (2.9), and introduce a transformation

$$q(x, y, t) = u(x, y, t)v(x, y, t). \tag{2.10}$$

Through some lengthy computations, we have

$$q_y = uv_{xx} - vu_{xx} - 6uu_xv^2 - 6u^2vv_x - 2u_xv^2 - 4uvv_x,$$

$$\partial_x^{-1}q_y = uv_x - vu_x - 3u^2v^2 - 2uv^2,$$

$$\partial_x^{-1}q_{yy} = uv_{xxx} + vu_{xxx} - u_{xx}v_x - v_{xx}u_x - 6u^2v_x^2 - 4uv_x^2 + 6u_x^2v^2$$
$$+ 8uv^2u_{xx} + 36u^2v^3u_x + 36u^3v^2v_x + 28uu_xv^3 + 44u^2v^2v_x$$
$$- 8u^2vv_{xx} + 4v^2u_{xx} + 4u_xv^3 + 12uv^2v_x - 4uvv_{xx} + 4vu_xv_x,$$

$$q_t = -uv_{xxx} - vu_{xxx} - 3u_{xx}v^2 - 3u_xvv_x - 3u_xv^3 - 9uv^2v_x - 24uu_xv^3$$
$$- 36u^2v^2v_x - 30u^2u_xv^3 - 30u^3v^2v_x - 6u_x^2v^2 + 6u^2v_x^2$$
$$- 6uv^2u_{xx} + 6u^2vv_{xx} + 3uv_x^2 + 3uvv_{xx},$$

which retrieve the mKP equation [14]

$$q_t = -\frac{1}{4}(q_{xx} - 2q^3)_x + \frac{3}{4}(2q_x\partial_x^{-1}q_y - \partial_x^{-1}q_{yy}). \tag{2.11}$$

So, the mKP equation (2.11) is revisited through two $(1 + 1)$-dimensional integrable Eqs. (2.8) and (2.9), which are in the same integrable hierarchy (2.7). This implies that compatible solutions of two $(1 + 1)$-dimensional integrable equations can produce a special solution of the mKP equation (2.11) through the transformation (2.10).

3. The Solvable Ordinary Differential Equations

In this section, we further decompose the two $(1+1)$-dimensional integrable equations (2.8) and (2.9) into systems of solvable ODEs that are compatible. Let $\psi = (\psi_1, \psi_2)^T$ and $\phi = (\phi_1, \phi_2)^T$ be the basic solutions of linear differential equations (2.1) and (2.6). Let

$$W = \begin{pmatrix} f & g \\ h & -f \end{pmatrix} = \frac{1}{2}(\phi\psi^T + \psi\phi^T)\sigma, \quad \sigma = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{3.1}$$

From (2.1) and (2.6), one can readily verify

$$W_x = [U, W], \quad W_{t_n} = [V^{(n)}, W], \tag{3.2}$$

which imply that the det W of matrix W is a constant of motion along both x- and t_n-flows [27]. Two equations in (3.2) can be rewritten as the component forms:

$$\begin{aligned} f_x &= -vh - \lambda ug, \\ g_x &= 2vf - (\lambda - v)g, \\ h_x &= 2\lambda uf + (\lambda - v)h, \end{aligned} \tag{3.3}$$

and

$$
\begin{aligned}
f_{t_n} &= hV_{12}^{(n)} - gV_{21}^{(n)}, \\
g_{t_n} &= 2gV_{11}^{(n)} - 2fV_{12}^{(n)}, \\
h_{t_n} &= 2fV_{21}^{(n)} - 2hV_{11}^{(n)}.
\end{aligned}
\tag{3.4}
$$

Assume that the functions f, g and h are finite degree polynomials of λ, namely,

$$
f = \sum_{j=0}^{N} f_j \lambda^{n+1-j}, \quad g = \sum_{j=0}^{N} g_j \lambda^{n-j}, \quad h = \sum_{j=0}^{N} h_j \lambda^{n+1-j}.
\tag{3.5}
$$

Substituting the expression (3.5) into (3.3) yields

$$
KG_{j-1} = JG_j, \quad JG_{-1} = 0, \quad KG_{N-1} = 0, \quad G_j = (h_{j+1}, g_{j+1}, f_{j+1})^T.
\tag{3.6}
$$

Apparently, $vh_j + ug_j + \partial_x f_j = 0$; and $JG_{-1} = 0$ admits a general solution

$$
G_{-1} = \alpha_0 S_{-1},
\tag{3.7}
$$

where α_0 is an integral constant. Therefore, G_j can be recursively determined from (3.6). Acting with the operator $J^{-1}K$ on both sides of Eq. (3.7) $k+1$ times results in

$$
G_k = \sum_{j=0}^{k} \alpha_j S_{k-j}, \quad 0 \leq k \leq N,
\tag{3.8}
$$

where α_j are integral constants. Substituting (3.8) into $KG_{N-1} = 0$ yields the following Nth order stationary equation,

$$
\alpha_0 X_N + \alpha_1 X_{N-1} + \cdots + \alpha_N X_0 = 0,
$$

where $X_j = JG_j$ $(j = 1, \ldots, N)$ are vector fields. This means that (u, v) is a finite band solution of the integrable hierarchy (2.7).

Without loss of generality, let us restrict our attention to $\alpha_0 = 1$. Recalling Eq. (3.6), one can easily compute

$$
\begin{aligned}
f_0 &= \frac{1}{2}, \quad g_0 = v, \quad h_0 = -u, \\
f_1 &= uv + \alpha_1, \\
g_1 &= -v_x + v^2 + 2uv^2 + 2\alpha_1 v, \\
h_1 &= -u_x - 2u^2v - uv - 2\alpha_1 u, \\
f_2 &= u_x v - uv_x + 3u^2v^2 + 2uv^2 + 2\alpha_1 uv + \alpha_2, \\
g_2 &= v_{xx} - 3vv_x - 6uvv_x + 6u^2v^3 + 6uv^3 + v^3 \\
&\quad + 4\alpha_1 uv^2 + 2\alpha_2 v + 2\alpha_1 v^2 - 2\alpha_1 v_x, \\
h_2 &= -u_{xx} - 6u^2v^2 - 6u^3v^2 - 6uu_xv - 2u_xv - uv_x \\
&\quad - uv^2 - 2\alpha_1 u_x - 2\alpha_1 uv - 4\alpha_1 u^2v - 2\alpha_2 u.
\end{aligned}
\tag{3.9}
$$

Taking into account Eq. (3.5), we define

$$g = v \prod_{j=1}^{N} (\lambda - \mu_j), \quad h = -\lambda u \prod_{j=1}^{N} (\lambda - \nu_j). \tag{3.10}$$

It follows from (3.5) and (3.10) that

$$g_1 = -v \sum_{j=1}^{N} \mu_j \triangleq -v\sigma_1, \quad h_1 = u \sum_{j=1}^{N} \nu_j \triangleq u\sigma_2,$$

$$g_2 = v \sum_{i,j=1,i<j}^{N} \mu_i \mu_j \triangleq v\bar{\sigma}_1, \quad h_2 = -u \sum_{i,j=1,i<j}^{N} \nu_i \nu_j \triangleq -u\bar{\sigma}_2, \tag{3.11}$$

where the symbol $\triangleq$ means that the left-hand side is denoted by the right-hand side, for short. After a direct calculation, the combination of (3.9) and (3.11) gives

$$\partial_x \ln v - v - 2uv - 2\alpha_1 = \sigma_1, \quad -\partial_x \ln u - v - 2uv - 2\alpha_1 = \sigma_2, \tag{3.12}$$

and

$$\begin{cases} \partial_y \ln v + 2uv(3uv - \partial_x \ln v + \partial_x \ln u) + v^2 + 6uv^2 - v_x \\ \quad = \bar{\sigma}_1 + 2\alpha_1\sigma_1 + 4\alpha_1^2 - 2\alpha_2, \\ -\partial_y \ln u + 2uv(3uv - \partial_x \ln v + \partial_x \ln u) + v^2 + 6uv^2 - v_x \\ \quad = \bar{\sigma}_2 + 2\alpha_1\sigma_2 + 4\alpha_1^2 - 2\alpha_2, \end{cases} \tag{3.13}$$

which imply that

$$\partial_x \ln uv = \sigma_1 - \sigma_2, \quad \partial_y \ln uv = \bar{\sigma}_1 - \bar{\sigma}_2 + 2\alpha_1(\sigma_1 - \sigma_2). \tag{3.14}$$

Thus, we have

$$\partial_y \ln uv = \partial_x^2(\ln v - \ln u) + (\partial_x \ln v)^2 - (\partial_x \ln u)^2 - 6uv\partial_x \ln uv - 4v_x - 2vu_x u^{-1}$$

$$= \partial_x(\sigma_1 + \sigma_2) + \sigma_1^2 - \sigma_2^2 + (4\alpha_1 - 2uv)(\sigma_1 - \sigma_2) \tag{3.15}$$

and

$$q = uv = \alpha_1 + \frac{1}{4}(\sigma_1 + \sigma_2) + \frac{1}{4(\sigma_1 - \sigma_2)} \left(2\partial_x(\sigma_1 + \sigma_2) + \sum_{j=1}^{N} (\mu_j^2 - \nu_j^2) \right), \tag{3.16}$$

where the following two identities

$$2\bar{\sigma}_1 = \sigma_1^2 - \sum_{j=1}^{N} \mu_j^2, \quad 2\bar{\sigma}_2 = \sigma_2^2 - \sum_{j=1}^{N} \nu_j^2,$$

are applied in the above calculations.

Since $\det W$ is a $(2N + 2)$-degree polynomial of λ, we define

$$- \det W = f^2 + gh = \frac{1}{4} \prod_{j=1}^{2N+2} (\lambda - \lambda_j) \triangleq \frac{1}{4} R(\lambda), \tag{3.17}$$

where λ_j $(1 \leq j \leq N + 2)$ are roots of the $(2N + 2)$-degree polynomial. Substituting (3.5) into (3.17) and comparing the coefficient of λ^{2N+1} and the one of λ^{2N} in both sides of (3.17), we derive

$$2f_0 f_1 + g_0 h_0 = -\frac{1}{4} \sum_{j=1}^{2N+2} \lambda_j, \quad 2f_0 f_2 + f_1^2 + h_0 g_1 + h_1 g_0 = \frac{1}{4} \sum_{i,j=1,i<j}^{2N+2} \lambda_i \lambda_j, \tag{3.18}$$

which yields

$$\alpha_1 = -\frac{1}{4} \sum_{j=1}^{2N+2} \lambda_j, \quad \alpha_2 = \frac{1}{4} \sum_{i,j=1,i<j}^{2N+2} \lambda_i \lambda_j - \frac{1}{16} \left(\sum_{j=1}^{2N+2} \lambda_j \right)^2. \tag{3.19}$$

Recalling Eqs. (3.17), (3.10) and (3.3), one can check the following formulae

$$f|_{\lambda=\mu_k} = \frac{1}{2} \sqrt{R(\mu_k)}, \quad f|_{\lambda=\nu_k} = \frac{1}{2} \sqrt{R(\nu_k)}, \tag{3.20}$$

$$\begin{cases} g_x|_{\lambda=\mu_k} = -v\mu_{k,x} \displaystyle\prod_{i=1,i\neq k}^{N} (\mu_k - \mu_i), \\ h_x|_{\lambda=\nu_k} = u\nu_k \nu_{k,x} \displaystyle\prod_{i=1,i\neq k}^{N} (\nu_k - \nu_i), \end{cases} \tag{3.21}$$

and

$$g_x|_{\lambda=\mu_k} = v\sqrt{R(\mu_k)}, \quad h_x|_{\lambda=\nu_k} = u\nu_k \sqrt{R(\nu_k)}. \tag{3.22}$$

Therefore, from (3.21) and (3.22) we obtain

$$\mu_{k,x} = -\frac{\sqrt{R(\mu_k)}}{\prod_{i=1,i\neq k}^{N}(\mu_k - \mu_i)}, \quad \nu_{k,x} = \frac{\sqrt{R(\nu_k)}}{\prod_{i=1,i\neq k}^{N}(\nu_k - \nu_i)}, \quad 1 \leq k \leq N. \tag{3.23}$$

Employing a similar procedure for Eqs. (2.6) and (3.10), we have

$$\begin{aligned}
V_{12}^{(1)}|_{\lambda=\mu_k} &= v(\mu_k - \sigma_1 - 2\alpha_1), \\
V_{21}^{(1)}|_{\lambda=\nu_k} &= u\nu_k(-\nu_k + \sigma_2 + 2\alpha_1), \\
V_{12}^{(2)}|_{\lambda=\mu_k} &= v(\mu_k^2 - \mu_k(\sigma_1 + 2\alpha_1) + \bar{\sigma}_1 + 2\alpha_1\sigma_1 + 4\alpha_1^2 - 2\alpha_2), \\
V_{21}^{(2)}|_{\lambda=\nu_k} &= u\nu_k(-\nu_k^2 + \nu_k(\sigma_2 + 2\alpha_1) - \bar{\sigma}_2 - 2\alpha_1\sigma_2 - 4\alpha_1^2 + 2\alpha_2).
\end{aligned} \tag{3.24}$$

A calculation analogous to the result (3.23) leads to

$$\mu_{k,t_n} = \frac{V_{12}^{(n)}\sqrt{R(\mu_k)}}{v \prod_{i=1,i\neq k}^{N}(\mu_k - \mu_i)}, \quad \nu_{k,t_n} = \frac{V_{21}^{(n)}\sqrt{R(\nu_k)}}{u\nu_k \prod_{i=1,i\neq k}^{N}(\nu_k - \nu_i)}, \quad 1 \leq k \leq N. \tag{3.25}$$

We summarize this section with the following conclusion: if constant spectral parameters $\lambda_1, \lambda_2, \ldots, \lambda_{2N+2}$ are given, and let $\mu(x, t_n)$ and $\nu(x, t_n)$ be distinct solutions of two ODEs (3.23) and (3.25); thus u, v determined by Eqs. (3.12)–(3.14) are solutions of integrable equations (2.8) and (2.9). Therefore, $q = uv$ is a special solution of the mKP equation (2.11).

4. The Finite Band Solution

Let us first introduce the Riemann surface of hyperelliptic curve,

$$\Gamma: \; \xi^2 = R(\lambda), \quad R(\lambda) = \prod_{j=1}^{2N+2} (\lambda - \lambda_j),$$

whose genus is N. For the same λ, there are two points $(\lambda, \sqrt{R(\lambda)})$ and $(\lambda, -\sqrt{R(\lambda)})$ on the upper and lower sheets of Γ. Additionally, there exist two points at infinities that are not branch points due to $\deg R(\lambda) = 2N+2$. Under an alternative local coordinate $z = \lambda^{-1}$, the two points are viewed as $\infty_1 = (0, 1)$ and $\infty_2 = (0, -1)$, respectively.

Let us choose a set of canonical basis of cycles: $a_1, a_2, \ldots, a_N; b_1, b_2, \ldots, b_N$ on Γ, which are independent if they have the intersection numbers

$$a_i \circ a_j = b_i \circ b_j = 0, \quad a_i \circ b_j = \delta_{ij}, \quad i, j = 1, 2, \ldots, N.$$

As the following N holomorphic differentials

$$\tilde{w}_l = \frac{\lambda^{l-1} d\lambda}{\sqrt{R(\lambda)}}, \quad 1 \le l \le N,$$

are linearly independent on Γ, we may define

$$A_{ij} = \int_{a_j} \tilde{w}_i, \quad B_{ij} = \int_{b_j} \tilde{w}_i, \quad 1 \le i, j \le N, \tag{4.1}$$

where $A = (A_{ij})_{N \times N}$ is an invertible matrix, and $B = (B_{ij})_{N \times N}$ is a symmetric matrix characterizing Γ. Denote the matrices C and τ by

$$C = (A_{ij})_{N \times N}^{-1}, \quad \tau = A^{-1} B.$$

Then τ is a symmetric matrix ($\tau_{ij} = \tau_{ji}$) with the positive definite imaginary part. We now normalize $\tilde{w}_l$ into a new basis w_j,

$$w_j = \sum_{l=1}^{N} C_{jl} \tilde{w}_l, \quad j = 1, 2, \ldots, N,$$

with properties

$$\int_{a_i} w_j = \sum_{l=1}^{N} C_{jl} \int_{a_i} \tilde{w}_l = \sum_{l=1}^{N} C_{jl} A_{li} = \delta_{ji}, \quad \int_{b_i} w_j = \tau_{ji}.$$

For a fixed point p_0 on Γ, the Abel–Jacobi coordinates are defined by

$$
\begin{aligned}
\rho_j^{(1)}(x,y,t) &= \sum_{k=1}^{N} \int_{p_0}^{\mu_k(x,y,t)} w_j = \sum_{k=1}^{N} \sum_{l=1}^{N} C_{jl} \int_{p_0}^{\mu_k} \frac{\lambda^{l-1} d\lambda}{\sqrt{R(\lambda)}}, \\
\rho_j^{(2)}(x,y,t) &= \sum_{k=1}^{N} \int_{p_0}^{\nu_k(x,y,t)} w_j = \sum_{k=1}^{N} \sum_{l=1}^{N} C_{jl} \int_{p_0}^{\nu_k} \frac{\lambda^{l-1} d\lambda}{\sqrt{R(\lambda)}},
\end{aligned}
\qquad 1 \le j \le N. \qquad (4.2)
$$

Taking derivative on both sides of the first equation in (4.2), we obtain

$$
\partial_x \rho_j^{(1)} = \sum_{l=1}^{N} \sum_{k=1}^{N} C_{jl} \frac{\mu_k^{l-1} \mu_{k,x}}{\sqrt{R(\mu_k)}} = \sum_{l=1}^{N} \sum_{k=1}^{N} C_{jl} \frac{-\mu_k^{l-1}}{\prod_{i=1,i\neq k}^{N}(\mu_k - \mu_i)}. \qquad (4.3)
$$

The following algebraic formulae [20]

$$
\begin{aligned}
I_s &= \sum_{k=1}^{N} \frac{\mu_k^s}{\prod_{i=1,i\neq k}^{N}(\mu_k - \mu_i)} = \delta_{s,N-1}, \quad 1 \le s \le N-1, \\
I_N &= \sigma_1 I_{N-1}, \quad I_{N+1} = \sigma_1 I_N - \bar{\sigma}_1 I_{N-1},
\end{aligned}
\qquad (4.4)
$$

simplifies Eq. (4.3) as

$$
\partial_x \rho_j^{(1)} = -\Omega_j^{(0)}, \quad \Omega_j^{(0)} = C_{jN}, \quad 1 \le j \le N. \qquad (4.5)
$$

A similar procedure can derive

$$
\partial_y \rho_j^{(1)} = \Omega_j^{(1)}, \quad \partial_t \rho_j^{(1)} = \Omega_j^{(2)}, \qquad (4.6)
$$

$$
\partial_x \rho_j^{(2)} = \Omega_j^{(0)}, \quad \partial_y \rho_j^{(2)} = -\Omega_j^{(1)}, \quad \partial_t \rho_j^{(2)} = -\Omega_j^{(2)}, \qquad (4.7)
$$

where

$$
\Omega_j^{(1)} = C_{jN-1} - 2\alpha_1 C_{jN}, \quad \Omega_j^{(2)} = C_{jN-2} - 2\alpha_1 C_{jN-1} - 2\alpha_2 C_{jN} + 4\alpha_1^2 C_{jN}.
$$

Therefore, $\rho_j^{(1)}$ and $\rho_j^{(2)}$ can be directly integrated

$$
\begin{aligned}
\rho_j^{(1)} &= -\Omega_j^{(0)} x + \Omega_j^{(1)} y + \Omega_j^{(2)} t + \gamma_j^{(1)}, \\
\rho_j^{(2)} &= \Omega_j^{(0)} x - \Omega_j^{(1)} y - \Omega_j^{(2)} t + \gamma_j^{(2)},
\end{aligned}
\qquad 1 \le j \le N, \qquad (4.8)
$$

where

$$
\gamma_j^{(1)} = \sum_{k=1}^{N} \int_{p_0}^{\mu_k(0,0,0)} w_j, \quad \gamma_j^{(2)} = \sum_{k=1}^{N} \int_{p_0}^{\nu_k(0,0,0)} w_j,
$$

are two integral constants.

In what follows, we discuss the Jacobi inversion that converts the explicit solution (4.8), i.e. Abel–Jacobi coordinates to the original coordinate q. Let T be the lattice generated

by $2N$ periodic vectors $\{\delta_i, \tau_j\}$, and $J(\Gamma) = \mathbb{C}^N/T$ be the Jacobian of Γ. The Abel map is defined by

$$\mathcal{A} : \mathrm{Div}(\Gamma) \to J(T), \quad \mathcal{A}(p) = \left(\int_{p_0}^{p} w_1, \ldots, \int_{p_0}^{p} w_N \right),$$

where p is an arbitrary point. Moreover, $\mathcal{A}$ can be linearly extended into the factor group

$$\mathrm{Div}(\Gamma) : \mathcal{A}\left(\sum n_k p_k \right) = \sum n_k \mathcal{A}(p_k).$$

Then the Riemann-theta function on Γ is given by [10, 25]

$$\theta(\zeta) = \sum_{z \in Z^N} \exp\left(\pi i \langle Bz, z \rangle + 2\pi i \langle \zeta, z \rangle \right), \quad \zeta \in \mathbb{C}^N,$$

$$\langle Bz, z \rangle = \sum_{i,j=1}^{N} B_{ij} z_i z_j, \quad \langle \zeta, z \rangle = \sum_{i=1}^{N} z_i \zeta_i, \quad i^2 = -1.$$

Consider two special divisors $\sum_{k=1}^{N} p_k^{(m)}$,

$$\mathcal{A}\left(\sum_{k=1}^{N} p_k^{(m)} \right) = \sum_{k=1}^{N} \mathcal{A}(p_k^{(m)}) = \sum_{k=1}^{N} \int_{p_0}^{p_k^{(m)}} w = \rho^{(m)}, \quad m = 1, 2,$$

where $p_k^{(1)} = (\mu_k, \zeta(\mu_k))$ and $p_k^{(2)} = (\nu_k, \zeta(\nu_k))$. The component form is

$$\sum_{k=1}^{N} \int_{p_0}^{p_k^{(m)}} w_j = \rho_j^{(m)}, \quad 1 \leq j \leq N, \quad m = 1, 2.$$

According to the Riemann theorem [10], there exist two Riemann constants $M^{(1)}, M^{(2)} \in \mathbb{C}^N$ determined by Γ such that

- $f^{(1)}(\lambda) = \theta(\mathcal{A}(\zeta(\lambda)) - \rho^{(1)} - M^{(1)})$ has N simple zeros at $\mu_1, \ldots, \mu_N$,
- $f^{(2)}(\lambda) = \theta(\mathcal{A}(\zeta(\lambda)) - \rho^{(2)} - M^{(2)})$ has N simple zeros at $\nu_1, \ldots, \nu_N$.

To make the functions single valued, the Riemann surface Γ is cut along with all paths a_k, b_k to form a simply connected region, whose boundary is denoted by γ. From the residue formulae, we obtain

$$\begin{aligned}
\sum_{j=1}^{N} \mu_j^k &= \frac{1}{2\pi i} \oint_\gamma \lambda^k d\ln f^{(1)}(\lambda) - \sum_{s=1, \lambda=\infty_s}^{2} \mathrm{Res}\, \lambda^k d\ln f^{(1)}(\lambda), \\
\sum_{j=1}^{N} \nu_j^k &= \frac{1}{2\pi i} \oint_\gamma \lambda^k d\ln f^{(2)}(\lambda) - \sum_{s=1, \lambda=\infty_s}^{2} \mathrm{Res}\, \lambda^k d\ln f^{(2)}(\lambda).
\end{aligned} \tag{4.9}$$

In light of [5], we know that integrals

$$\frac{1}{2\pi i} \oint_\gamma \lambda^k d\ln f^{(m)}(\lambda) = \sum_{j=1}^{N} \int_{a_j} \lambda^k w_j \triangleq I_k(\Gamma), \quad m = 1, 2,$$

are constants that are independent of $\rho^{(m)}$. The only remaining step is to figure out the residues

$$f^{(m)}(\lambda)|_{\lambda=\infty_s} = \theta\left(\int_{p_0}^p w - \rho^{(m)} - M^{(m)}\right)$$

$$= \theta\left(\int_{\infty_s}^p w - \pi_s - \rho^{(m)} - M^{(m)}\right)$$

$$= \theta\left(\ldots, \int_{\infty_s}^p w_j - \pi_{sj} - \rho_j^{(m)} - M_j^{(m)}, \ldots\right)$$

$$= \theta\left(\ldots, \rho_j^{(m)} + M_j^{(m)} + \pi_{sj} + (-1)^s\left(C_{jN}z + \frac{1}{2}(C_{jN-1} + \frac{1}{2}R_1 C_{jN})z^2 + \cdots\right), \ldots\right)$$

$$= \theta_s^{(m)}(\rho^{(m)} + M^{(m)} + \pi_s) + (-1)^{s+m}\theta_{s,x}^{(m)}z + \frac{1}{2}((-1)^{s+m-1}\theta_{s,y}^{(m)} + (\theta_{s,x}^{(m)})^2)z^2 + \ldots,$$

where

$$\pi_{sj} = \int_{\infty_s}^{p_0} w_j, \quad R_1 = \sum_{i=1}^{2N+2} \lambda_i, \quad s, m = 1, 2.$$

And then, we arrive at

$$\mathrm{Res}_{\lambda=\infty_s}\lambda d\ln f^{(m)}(\lambda) = (-1)^{s+m}\partial_x \ln\theta_s^{(m)},$$
$$\mathrm{Res}_{\lambda=\infty_s}\lambda^2 d\ln f^{(m)}(\lambda) = (-1)^{s+m-1}\partial_y \ln\theta_s^{(m)} + \partial_x^2 \ln\theta_s^{(m)},$$
(4.10)

where

$$\theta_s^{(1)} = \theta(-\Omega^{(0)}x + \Omega^{(1)}y + \Omega^{(2)}t + \Upsilon_s),$$
$$\theta_s^{(2)} = \theta(\Omega^{(0)}x - \Omega^{(1)}y - \Omega^{(2)}t + \Lambda_s),$$

with

$$\Upsilon_{sj} = \gamma_j^{(1)} + M_j^{(1)} + \pi_{sj}, \quad \Lambda_{sj} = \gamma_j^{(2)} + M_j^{(2)} + \pi_{sj}, \quad 1 \le j \le N.$$

Therefore, from (4.9) and (4.10) we get

$$\sum_{l=1}^N \mu_l = I_1(\Gamma) + \partial_x \ln\frac{\theta_2^{(1)}}{\theta_1^{(1)}}, \quad \sum_{l=1}^N \nu_l = I_1(\Gamma) + \partial_x \ln\frac{\theta_1^{(2)}}{\theta_2^{(2)}}, \quad (4.11)$$

and

$$\sum_{l=1}^N \mu_l^2 = I_2(\Gamma) + \partial_y \ln\frac{\theta_1^{(1)}}{\theta_2^{(1)}} - \partial_x^2 \ln\theta_1^{(1)}\theta_2^{(1)},$$

$$\sum_{l=1}^N \nu_l^2 = I_2(\Gamma) + \partial_y \ln\frac{\theta_2^{(2)}}{\theta_1^{(2)}} - \partial_x^2 \ln\theta_1^{(2)}\theta_2^{(2)}.$$
(4.12)

Substituting (4.11) and (4.12) back into (3.16), we finally obtain a finite band solution of the mKP equation (2.11)

$$q = \alpha_1 + \frac{1}{2}I_1(\Gamma) + \frac{1}{4}\partial_x \ln \frac{\theta_2^{(1)}\theta_1^{(2)}}{\theta_1^{(1)}\theta_2^{(2)}} + \frac{1}{4\partial_x \ln \frac{\theta_2^{(1)}\theta_2^{(2)}}{\theta_1^{(1)}\theta_1^{(2)}}} \left\{ \partial_y \ln \frac{\theta_1^{(1)}\theta_1^{(2)}}{\theta_2^{(1)}\theta_2^{(2)}} + \partial_x^2 \ln \frac{\theta_2^{(1)}(\theta_1^{(2)})^3}{(\theta_1^{(1)})^3\theta_2^{(2)}} \right\}.$$

$$(4.13)$$

From the procedure discussed in the above three sections, we provide an effective way to construct finite band solutions of $(2 + 1)$-dimensional integrable equations, which are involved in the finite-order expansion of Lax matrix [3, 9, 26, 28]. In particular, our paper provides a distinct decomposition for the mKP equation that consisted of two consistent $1+1$-dimensional integrable systems in view of a transformation. As a result, a special finite band solution expressed by Riemann-theta functions to the mKP equation is presented through this decomposition.

Acknowledgments

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