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Algebro-geometric Solutions for the Derivative Burgers Hierarchy

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Abstract Though completely integrable Camassa–Holm (CH) equation and Degasperis–Procesi (DP) equation are cast in the same peakon family, they possess the second- and third-order Lax operators, respectively. From the viewpoint of algebro-geometrical study, this difference lies in hyper-elliptic and non-hyper-elliptic curves. The non-hyperelliptic curves lead to great difficulty in the construction of algebro-geometric solutions of the DP equation. In this paper, we study algebro-geometric solutions for the derivative Burgers (DB) equation, which is derived by Qiao and Li (2004) as a short wave model of the DP equation with the help of functional gradient and a pair of Lenard operators. Based on the characteristic polynomial of a Lax matrix for the DB equation, we introduce a third order algebraic curve $\mathcal{K}_{r-1}$ with genus $r-1$, from which the associated Baker–Akhiezer functions, meromorphic function, and Dubrovin-type equations are constructed. Furthermore, the theory of algebraic curve is applied to derive explicit representations of the theta function for the Baker–Akhiezer functions and the meromorphic function. In particular, the algebro-geometric solutions are obtained for all equations in the whole DB hierarchy.

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1 Introduction

The derivative Burgers (DB) equation

$$u_{xxt} + 3u_x u_{xx} + uu_{xxx} = 0, \text{ i.e. } (u_t + uu_x)_{xx} = 0, \tag{1.1}$$

was first proposed by Qiao and Li in a search for a short wave model from the Degasperis–Procesi (DP) equation through using asymptotical procedure on integrable PDEs (Qiao and Li 2004). Two years later, Matsuno studied cusp and loop soliton solutions of the short-wave model for the DP equation (Matsuno 2006). Lundmark investigated the formation and dynamics of shock solitary waves for the DP equation (Lundmark 2007) as well as mentioned the dB equation (1.1) belonging to an integrable hierarchy described by Qiao and Li (2004). Thereafter, Kohlenberg et al. (2007) constructed the inverse spectral problem for the discrete cubic string related to a third order spectral problem.

For our convenience of brevity, the dB equation (1.1) may be called the DB equation, and the corresponding system of evolution equations is therefore called the DB hierarchy for short.

Quasi-periodic solutions (also called algebro-geometric solutions or finite gap solutions) of nonlinear equations were originally studied on the KdV equation based on the inverse spectral theory and algebro-geometric method developed by pioneers such as the authors in Ablowitz et al. (1974) ,Belokolos et al. (1994), Krichever (1976), Krichever (1977), Novikov et al. (1984), Dubrovin (1977), Dubrovin (1981), Dubrovin (1983) in the late 1970s. This theory has been extended to the whole hierarchies of nonlinear integrable equations by Gesztesy and Holden using polynomial recursion method (Gesztesy and Ratneseelan 1998; Gesztesy and Holden 2003a, b; Gesztesy et al. 2008; Geronimo et al. 2005; Bulla et al. 1998) and by Zhou and Qiao using the Lax matrix and r-matrix structure (Qiao 2001, 1998; Zhou 1997). As a degenerated case of algebro-geometric solution, the multi-soliton solution and elliptic function solution may be obtained (Belokolos et al. 1994; Novikov et al. 1984; Matveev and Yavor 1979). It is well known that the algebro-geometric solutions of the Camassa–Holm (CH) hierarchy have been obtained with different techniques, see Gesztesy and Holden (2003a), Qiao (2003), and Fan (2009). However, within the authors' knowledge, the algebro-geometric solutions of the DB hierarchy are still not presented yet.

Over the past three decades, integrable equations associated with 2×2 matrix spectral problems have widely been studied. Various methods were developed to construct algebro-geometric solutions for water wave equations, such as KdV, modified KdV, Kadomtsev– Petviashvili, nonlinear Schrödinger, Camassa–Holm, sine-Gordon, Ablowitz–Kaup–Newell–Segur (AKNS), Ablowitz–Ladik lattice, Toda lattice, etc. (Belokolos et al. 1994; Krichever 1976, 1977; Novikov et al. 1984; Dubrovin 1977, 1981, 1983; Gesztesy and Ratneseelan 1998; Gesztesy and Holden 2003a, b; Gesztesy et al. 2008; Geronimo et al. 2005; Bulla et al. 1998; Fan 2009; Hon and Fan 2005; Qiao 2003, 2001, 1998; Ma and Ablowitz 1981; Date and Tanaka 1976). But it is very difficult to extend these methods to soliton equations associated with 3×3 matrix spectral problems. The main reasons for this complexity may get traced back to the associated algebraic curve, which is non-hyperelliptic of the third order typically arising in the 3×3 spectral problem while it is hyper-elliptic of the second order in the 2×2 case.

In 2004, Qiao and Li proposed the DB hierarchy, which may be viewed as a short wave integrable hierarchy of the DP equation through the procedure of functional gradient and recursion operator, connected the DB hierarchy (including the DB equation 1.1) to finite-dimensional integrable systems, and gave the DB equation's peaked traveling wave solutions. Dickson et al. (1999a, b) proposed an unified framework, which yields all algebro-geometric solutions of the entire Boussinesq hierarchy. Geng et al. (2011) further investigated the algebro-geometric solutions of the modified Boussinesq hierarchy in their recent paper.

The purpose of this paper is to construct the algebro-geometric solutions for the DB hierarchy which contains the short wave model of the DP equation—the DB equation (1.1) as a special member. The whole paper is organized as follows. In Sect. 2, based on the Lenard recursion operators and the stationary zero-curvature equation, we derive the DB hierarchy associated with a 3×3 matrix spectral problem. In Sect. 3, we study the meromorphic function ϕ satisfying a second-order nonlinear differential equation. In Sect. 4, we present the explicit theta function representations for the Baker–Akhiezer function and the meromorphic function. In particular, we give the algebro-geometric solutions of the entire stationary DB hierarchy. In Sects. 5 and 6, we extend all the Baker–Akhiezer function, the meromorphic function, the Dubrovin-type equations, and the theta function representations dealt with in Sects. 3 and 4 to the time-dependent cases.

2 The DB Hierarchy

In this section, we derive the DB hierarchy and the corresponding sequence of zero-curvature pairs by using a Lenard recursion formalism (see Qiao and Li (2004) for more details). Throughout this section, we make the following hypothesis.

Hypothesis 2.1 In the stationary case we assume that $u : \mathbb{C} \to \mathbb{C}$ satisfies

$$u \in C^\infty(\mathbb{C}),\ \partial_x^k u \in L^\infty(\mathbb{C}),\ k \in \mathbb{N}_0. \tag{2.1}$$

In the time-dependent case we suppose $u : \mathbb{C}^2 \to \mathbb{C}$ satisfies

$$\begin{aligned} &u(\cdot, t) \in C^\infty(\mathbb{C}),\ \partial_x^k u(\cdot, t) \in L^\infty(\mathbb{C}),\ k \in \mathbb{N}_0, t \in \mathbb{C}, \\ &u(x, \cdot), u_{xx}(x, \cdot) \in C^1(\mathbb{C}),\ x \in \mathbb{C}. \end{aligned} \tag{2.2}$$

We start by the following 3×3 matrix spectral problem

$$\psi_x = U\psi, \quad \psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix}, \quad U = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -mz^{-1} & 0 & 0 \end{pmatrix}, \tag{2.3}$$

where $m = u_{xx}$, the function u is a potential, and z is a constant spectral parameter independent of variable x. Next, we introduce two Lenard operators

$$K = \partial^5, \tag{2.4}$$
$$J = -3(2m\partial + \partial m)\partial^{-3}(m\partial + 2\partial m). \tag{2.5}$$

Obviously, K and J are two skew-symmetric operators. A direct calculation shows that

$$K^{-1} = \partial^{-5},$$
$$J^{-1} = -\frac{1}{27}m^{-2/3}\partial^{-1}m^{-1/3}\partial^3 m^{-1/3}\partial^{-1}m^{-2/3},$$

and we further define an operator

$$\mathscr{L} = K^{-1}J = -3\partial^{-5}(2m\partial + \partial m)\partial^{-3}(m\partial + 2\partial m).$$

Choose $G_0 = \frac{1}{6} \in \ker K$; the Lenard's recursive sequence is defined as follows:

$$G_{j-1} = \mathscr{L}^{-1}G_j, \quad j = 1, 2, \ldots \tag{2.6}$$

Hence G_j are uniquely determined, for example, the first two elements read as

$$G_0 = \frac{1}{6}, \quad G_1 = -\partial^{-4}(uu_{xx} + u_x^2).$$

In order to obtain the DB hierarchy associated with the spectral problem (2.3), we first solve the stationary zero-curvature equation

$$V_x - [U, V] = 0, \quad V = (V_{ij})_{3\times 3} \tag{2.7}$$

with

$$V = \begin{pmatrix} V_{11} & V_{12} & V_{13} \\ V_{21} & V_{22} & V_{23} \\ V_{31} & V_{32} & V_{33} \end{pmatrix}, \tag{2.8}$$

where each entry V_{ij} is a Laurent expansion in z,

$$V_{ij} = \sum_{\ell=0}^{n} V_{ij}^{(\ell)}(G_\ell) z^{2(n-\ell+1)} \quad i, j = 1, \ldots, 3, \quad \ell = 0, \ldots, n. \tag{2.9}$$

Equation (2.7) can be rewritten as

$$\begin{aligned}
V_{11,x} &= V_{21} + z^{-1} m V_{13}, \\
V_{12,x} &= V_{22} - V_{11}, \\
V_{13,x} &= V_{23} - V_{12}, \\
V_{21,x} &= V_{31} + z^{-1} m V_{23}, \\
V_{22,x} &= V_{32} - V_{21}, \\
V_{23,x} &= V_{33} - V_{22}, \\
V_{31,x} &= z^{-1} m (V_{33} - V_{11}), \\
V_{32,x} &= -z^{-1} m V_{12} - V_{31}, \\
V_{33,x} &= -z^{-1} m V_{13} - V_{32}.
\end{aligned} \tag{2.10}$$

Inserting (2.9) into (2.10) yields

$$\begin{aligned}
V_{11}^{(\ell)} &= -z^{-1} G_{\ell,xx} - 3z^{-2} \partial^{-2} (m\partial + 2\partial m) G_\ell, \\
V_{12}^{(\ell)} &= 3z^{-1} G_{\ell,x} + 3z^{-2} \partial^{-3} (m\partial + 2\partial m) G_\ell, \\
V_{13}^{(\ell)} &= -6z^{-1} G_\ell, \\
V_{21}^{(\ell)} &= -z^{-1} G_{\ell,xxx} - 3z^{-2} \partial^{-1} m G_{\ell,x}, \\
V_{22}^{(\ell)} &= 2z^{-1} G_{\ell,xx}, \\
V_{23}^{(\ell)} &= -3z^{-1} G_{\ell,x} + 3z^{-2} \partial^{-3} (m\partial + 2\partial m) G_\ell, \\
V_{31}^{(\ell)} &= -z^{-1} G_{\ell,xxxx} - 3z^{-3} m \partial^{-3} (m\partial + 2\partial m) G_\ell, \\
V_{32}^{(\ell)} &= z^{-1} G_{\ell,xxx} - 3z^{-2} \partial^{-1} m G_{\ell,x}, \\
V_{33}^{(\ell)} &= -z^{-1} G_{\ell,xx} + 3z^{-2} \partial^{-2} (m\partial + 2\partial m) G_\ell.
\end{aligned} \tag{2.11}$$

Substituting (2.10) and (2.11) into (2.7), we can show that Lenard sequence G_ℓ satisfies the Lenard equation

$$K G_\ell = z^{-2} J G_\ell, \quad \ell = 0, 1, \ldots. \tag{2.12}$$

For our use in Theorem 6.2, we introduce the following notation:

$$\begin{aligned}
V_{21}^{(\ell,0)} &= -G_{\ell,xxx}, \qquad & V_{21}^{(\ell,1)} &= -3\partial^{-1} m G_{\ell,x}, \\
V_{22}^{(\ell,0)} &= 2G_{\ell,xx}, \qquad & V_{22}^{(\ell,1)} &= 0, \\
V_{23}^{(\ell,0)} &= -3G_{\ell,x}, \qquad & V_{23}^{(\ell,1)} &= 3\partial^{-3} (m\partial + 2\partial m) G_\ell.
\end{aligned}$$

Let ψ satisfy the spectral problem (2.3) and an auxiliary problem

$$\psi_{t_n} = V\psi, \tag{2.13}$$

where V is defined by (2.8) and (2.9). The compatibility condition between (2.3) and (2.13) yields the zero-curvature equation

$$U_{t_n} - V_x + [U, V] = 0,$$

which is equivalent to the DB hierarchy

$$\mathrm{DB}_n(u) = m_{t_n} - X_n = 0, \qquad n \geq 0, \tag{2.14}$$

where the vector fields are given by

$$X_n = JG_n = J\mathscr{L}^n G_0, \quad n \geq 0.$$

By definition, the set of solutions of (2.14), with n ranging in $\mathbb{N}_0$, represents the class of algebro-geometric DB solutions. At times it is convenient to abbreviate algebro-geometric stationary DB solutions u simply as DB potentials.

The system of equations $\mathrm{DB}_0(u) = 0$ represents the DB equation.

In order to derive the corresponding plane algebraic curve, we consider the stationary zero-curvature equation

$$z^{1/2}V_x = [U, z^{1/2}V], \tag{2.15}$$

which is equivalent to (2.7), but the term $z^{1/2}V$ can ensure that the following algebraic curve is in positive powers of z.

A direct calculation shows that the matrix $yI - z^{1/2}V$ also satisfies the stationary zero-curvature equation; then we conclude that

$$\frac{\mathrm{d}}{\mathrm{d}x}(\det(yI - z^{1/2}V)) = 0,$$

which implies that the characteristic polynomial $\det(yI - z^{1/2}V)$ of Lax matrix $z^{1/2}V$ is independent of the variable x. Therefore we define the algebraic curve

$$\mathcal{F}_r(z, y) = \det(yI - z^{1/2}V) = y^3 + yS_r(z) - T_r(z), \tag{2.16}$$

where $S_r(z)$ and $T_r(z)$ are polynomials with constant coefficients of z,

$$S_r(z) = z\left(\begin{vmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{vmatrix} + \begin{vmatrix} V_{22} & V_{23} \\ V_{32} & V_{33} \end{vmatrix} + \begin{vmatrix} V_{11} & V_{13} \\ V_{31} & V_{33} \end{vmatrix}\right), \tag{2.17}$$

$$T_r(z) = z^{3/2}\begin{vmatrix} V_{11} & V_{12} & V_{13} \\ V_{21} & V_{22} & V_{23} \\ V_{31} & V_{32} & V_{33} \end{vmatrix}. \tag{2.18}$$

In order to ensure the polynomials with integer powers, we introduce $z = \tilde{z}^2$, and the algebraic curve becomes,

$$\mathcal{F}_r(\tilde{z}, y) = y^3 + yS_r(\tilde{z}) - T_r(\tilde{z}), \tag{2.19}$$

where $S_r(\tilde{z})$ and $T_r(\tilde{z})$ are polynomials with constant coefficients of $\tilde{z}$,

$$S_r(\tilde{z}) = \tilde{z}^2\left(\begin{vmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{vmatrix} + \begin{vmatrix} V_{22} & V_{23} \\ V_{32} & V_{33} \end{vmatrix} + \begin{vmatrix} V_{11} & V_{13} \\ V_{31} & V_{33} \end{vmatrix}\right) = \sum_{j=0}^{4n-1} S_{r,j}\tilde{z}^{8n-2j}, \tag{2.20}$$

$$T_r(\tilde{z}) = \tilde{z}^3 \begin{vmatrix} V_{11} & V_{12} & V_{13} \\ V_{21} & V_{22} & V_{23} \\ V_{31} & V_{32} & V_{33} \end{vmatrix} = \sum_{j=0}^{6n} T_{r,j}\tilde{z}^{12n+1-2j}. \tag{2.21}$$

We note that $T_r(\tilde{z})$ is a polynomial of degree r $(r = 12n + 1)$ with respect to $\tilde{z}$, and then $\mathcal{F}_r(\tilde{z}, y) = 0$ naturally leads to the plane third-order algebraic curve $\mathcal{K}_{r-1}$ of genus $r - 1 \in \mathbb{N}$[1],

$$\mathcal{K}_{r-1} : \mathcal{F}_r(\tilde{z}, y) = y^3 + yS_r(\tilde{z}) - T_r(\tilde{z}) = 0, \quad r = 12n + 1. \tag{2.22}$$

The algebraic curve $\mathcal{K}_{r-1}$ in (2.22) is compactified by joining a point at infinity P_∞, but for notational simplicity the compactification is also denoted by $\mathcal{K}_{r-1}$. Points on $\mathcal{K}_{r-1}\backslash\{P_\infty\}$ are represented as pairs $P = (\tilde{z}, y(P))$, where $y(\cdot)$ is the meromorphic function on $\mathcal{K}_{r-1}$ satisfying $\mathcal{F}_r(\tilde{z}, y(P)) = 0$.

The complex structure on $\mathcal{K}_{r-1}$ is defined in the usual way by introducing local coordinates

$$\zeta_{Q_0} : P \to \zeta = \tilde{z} - \tilde{z}_0$$

near points $Q_0 = (\tilde{z}_0, y(Q_0)) \in \mathcal{K}_{r-1}\backslash\{P_0 = (0, 0)\}$, which are neither branch nor singular points of $\mathcal{K}_{r-1}$; near $P_0 = (0, 0)$, the local coordinate is

$$\zeta_{P_0} : P \to \zeta = \tilde{z}^{\frac{1}{3}}, \tag{2.23}$$

and similarly at branch and singular points of $\mathcal{K}_{r-1}$; near the point $P_\infty \in \mathcal{K}_{r-1}$, the local coordinate is

$$\zeta_{P_\infty} : P \to \zeta = \tilde{z}^{-\frac{1}{3}}. \tag{2.24}$$

The holomorphic map $*$, changing sheets, is defined by

$$* : \begin{cases} \mathcal{K}_{r-1} \to \mathcal{K}_{r-1}, \\ P = (\tilde{z}, y_j(\tilde{z})) \to P^* = (\tilde{z}, y_{j+1(\text{mod } 3)}(\tilde{z})), \quad j = 0, 1, 2, \end{cases}$$
$$P^{**} := (P^*)^*, \quad \text{etc.}, \tag{2.25}$$

[1] The computation of the genus see Dickson et al. (1999a, b).

where $y_j(\tilde{z}),\ j = 0, 1, 2$ denote the three branches of $y(P)$ satisfying $\mathcal{F}_r(\tilde{z}, y) = 0$.

Finally, positive divisors on $\mathcal{K}_{r-1}$ of degree $r-1$ are denoted by

$$\mathcal{D}_{P_1,\ldots,P_{r-1}} : \begin{cases} \mathcal{K}_{r-1} \to \mathbb{N}_0, \\ P \to \mathcal{D}_{P_1,\ldots,P_{r-1}}(P) = \begin{cases} k \text{ if } P \text{ occurs } k \text{ times in } \{P_1, \ldots, P_{r-1}\}, \\ 0 \text{ if } P \notin \{P_1, \ldots, P_{r-1}\}. \end{cases} \end{cases} \tag{2.26}$$

In particular, the divisor $(\phi(\cdot))$ of a meromorphic function $\phi(\cdot)$ on $\mathcal{K}_{r-1}$ is defined by

$$(\phi(\cdot)) : \mathcal{K}_{r-1} \to \mathbb{Z}, \quad P \mapsto \omega_\phi(P), \tag{2.27}$$

where $\omega_\phi(P) = m_0 \in \mathbb{Z}$ if $(\phi \circ \zeta_P^{-1})(\zeta) = \sum_{n=m_0}^{\infty} c_n(P)\zeta^n$ for some $m_0 \in \mathbb{Z}$ by using a chart (U_P, ζ_P) near $P \in \mathcal{K}_{r-1}$.

Remark 2.2 We investigate what happens at the point infinity on our DB-type curve $\mathcal{K}_{r-1}$. Following the treatment in Mumford (1984) we substitute the variable $v = \tilde{z}^{-1}$ into (2.22), which yields

$$\begin{aligned} &v^{12n+1}y^3 + (S_{r,0} + S_{r,1}v^2 + \cdots + S_{r,4n-1}v^{8n-2})v^{4n+1}y \\ &\quad - (T_{r,0} + \cdots + T_{r,6n}v^{12n}) = 0. \end{aligned} \tag{2.28}$$

Let $v_1 = v^{4n+1}y$, and (2.28) becomes

$$\begin{aligned} &v_1^3 + (S_{r,0} + S_{r,1}v^2 + \ldots + S_{r,4n}v^{8n})v^2 v_1 \\ &\quad - (T_{r,0} + \ldots + T_{r,6n}v^{12n})v^2 = 0. \end{aligned} \tag{2.29}$$

Let $v \to 0$ (corresponding to $\tilde{z} \to \infty$) in (2.29) to obtain $v_1^3 = 0$. This corresponds to one point of multiplicity three at infinity, given by $(v, v_1) = (0, 0)$. We therefore use the coordinate (2.24) at the branch point at infinity, denoted by P_∞.

Similarly, near point $P_0 = (0, 0) \in \mathcal{K}_{r-1}$, one finds $y^3 = 0$ by taking $\tilde{z} \to 0$ in (2.22). This corresponds to one point of multiplicity three at $\tilde{z} = 0$. We therefore use the coordinate (2.23) at the branch point P_0.

3 The Stationary DB Formalism

In this section, we are devoted to a detailed study of the stationary DB hierarchy. Our principle tools are derived from a fundamental meromorphic function ϕ on the algebraic curve $\mathcal{K}_{r-1}$. With the help of ϕ we study the Baker–Akhiezer vector ψ and Dubrovin-type equations.

First, we give a brief description about the Baker–Akhiezer functions. The exponential e^z is analytic in $\mathbb{C}$ and has an essential singularity at the point $z = \infty$. If $q(z)$ is a rational function, then $f(z) = e^{q(z)}$ is analytic in $\bar{\mathbb{C}} = \mathbb{CP}^1$ everywhere except at the poles of $q(z)$, where $f(z)$ has essential singular points. In the last century Clebsh and

Gordan considered generalizing functions of exponential type to Riemann surfaces of higher genus. Baker noted that such functions of exponential type can be expressed in terms of theta functions of Riemann surfaces. Akhiezer first directed attention to the fact that under certain conditions functions of exponential type on hyper-elliptic Riemann surfaces are eigenfunctions of second-order linear diferential operators. Following the established tradition, we call functions of exponential type on Riemann surfaces Baker–Akhiezer functions.

Next, we introduce the stationary vector Baker–Akhiezer function $\psi = (\psi_1, \psi_2, \psi_3)^t$

$$\begin{aligned}
&\psi_x(P, x, x_0) = U(u(x), \tilde{z}(P))\psi(P, x, x_0),\\
&\tilde{z}V(u(x), \tilde{z}(P))\psi(P, x, x_0) = y(P)\psi(P, x, x_0),\\
&\psi_2(P, x_0, x_0) = 1, \quad P = (\tilde{z}, y) \in \mathcal{K}_{r-1}\backslash\{P_\infty, P_0\},\ x \in \mathbb{C}.
\end{aligned} \tag{3.1}$$

Closely related to $\psi(P, x, x_0)$ is the meromorphic function $\phi(P, x)$ on $\mathcal{K}_{r-1}$ defined by

$$\phi(P, x) = \tilde{z}\frac{\psi_{2,x}(P, x, x_0)}{\psi_2(P, x, x_0)}, \quad P \in \mathcal{K}_{r-1},\ x \in \mathbb{C} \tag{3.2}$$

such that

$$\psi_2(P, x, x_0) = \exp\left(\tilde{z}^{-1}\int_{x_0}^{x} \phi(P, x')dx'\right), \quad P \in \mathcal{K}_{r-1}\backslash\{P_\infty, P_0\}. \tag{3.3}$$

By using (3.1), a direct calculation gives

$$\phi = \tilde{z}\frac{yV_{31} + C_r}{yV_{21} + A_r} = \frac{\tilde{z}F_r}{y^2V_{31} - yC_r + D_r} = \tilde{z}\frac{y^2V_{21} - yA_r + B_r}{E_r}, \tag{3.4}$$

where

$$\begin{aligned}
A_r &= \tilde{z}(V_{23}V_{31} - V_{33}V_{21}) = \tilde{z}[V_{23}V_{31} + V_{21}(V_{22} + V_{11})],\\
B_r &= \tilde{z}^2[V_{22}(V_{11}V_{21} + V_{23}V_{31}) - V_{21}(V_{12}V_{21} + V_{23}V_{32})],\\
C_r &= \tilde{z}(V_{21}V_{32} - V_{22}V_{31}) = \tilde{z}[V_{21}V_{32} + V_{31}(V_{11} + V_{33})],\\
D_r &= \tilde{z}^2[V_{31}(V_{11}V_{33} - V_{13}V_{31}) + V_{32}(V_{21}V_{33} - V_{23}V_{31})], \qquad (3.5)\\
E_r &= \tilde{z}^2[V_{23}(V_{21}V_{33} - V_{11}V_{21} - V_{23}V_{31}) + V_{13}V_{21}^2],\\
F_r &= \tilde{z}^2[V_{31}(V_{22}V_{32} - V_{11}V_{32} + V_{12}V_{31}) - V_{21}V_{32}^2]. \qquad (3.6)
\end{aligned}$$

The quantities $A_r, \ldots, F_r$ in (3.5) and (3.6) are of course not independent of each other. There exist various interrelationships between them and S_r, T_r, some of which are summarized below.

Lemma 3.1 *Let* $(\tilde{z}, x) \in \mathbb{C}^2$. *Then*

$$
\begin{aligned}
& V_{21} F_r = V_{31} D_r - C_r^2 - V_{31}^2 S_r, \\
& A_r F_r = T_r V_{31}^2 + C_r D_r, \qquad (3.7)\\
& V_{31} E_r = V_{21} B_r - A_r^2 - V_{21}^2 S_r, \\
& E_r C_r = T_r V_{21}^2 + A_r B_r, \qquad (3.8)\\
& V_{21} D_r + V_{31} B_r - V_{21} V_{31} S_r + A_r C_r = 0, \\
& T_r V_{21} V_{31} + S_r C_r V_{21} + S_r A_r V_{31} - A_r D_r - B_r C_r = 0, \\
& E_r F_r = -T_r C_r V_{21} - T_r A_r V_{31} + B_r D_r, \qquad (3.9)\\
& E_{r,x} = -2 S_r V_{21} + 3 B_r, \\
& V_{31} F_{r,x} = 3\tilde{z}^{-2} m V_{33} F_r + \tilde{z}^{-2} m V_{32}(-2 V_{31} S_r + 3 D_r). \qquad (3.10)
\end{aligned}
$$

Proof Relations (3.7)–(3.9) are clear from (2.22) and (3.4). Equation (3.10) is a straightforward consequence of (3.6) and the stationary zero-curvature equation (2.10). □

By inspection of (2.11) and (3.6), one infers that E_r and $\tilde{z}^2 F_r$ are polynomials with respect to $\tilde{z}$ of degree $r-1$. Let $\{\mu_j(x)\}_{j=1,\ldots,r-1}$ and $\{\nu_j(x)\}_{j=1,\ldots,r-1}$ denote the zeros of $E_r(x)$ and $\tilde{z}^2 F_r(x)$, respectively. Hence we may write

$$E_r = E_{r,0} \prod_{j=1}^{r-1} (\tilde{z} - \mu_j(x)), \tag{3.11}$$

$$F_r = \tilde{z}^{-2} F_{r,0} \prod_{j=1}^{r-1} (\tilde{z} - \nu_j(x)). \tag{3.12}$$

Defining

$$\hat{\mu}_j(x) = \left(\mu_j(x), -\frac{A_r(\mu_j(x), x)}{V_{21}(\mu_j(x), x)}\right) \in \mathcal{K}_{r-1}, \ \ j = 1, \ldots, r-1, \ \ x \in \mathbb{C}, \tag{3.13}$$

$$\hat{\nu}_j(x) = \left(\nu_j(x), -\frac{C_r(\nu_j(x), x)}{V_{31}(\nu_j(x), x)}\right) \in \mathcal{K}_{r-1}, \ \ j = 1, \ldots, r-1, \ \ x \in \mathbb{C}. \tag{3.14}$$

One infers from (3.4) that the divisor $(\phi(P, x))$ of $\phi(P, x)$ is given by

$$(\phi(P, x)) = \mathcal{D}_{P_0, \underline{\hat{\nu}}(x)}(P) - \mathcal{D}_{P_\infty, \underline{\hat{\mu}}(x)}(P), \tag{3.15}$$

where

$$\underline{\hat{\nu}}(x) = \{\hat{\nu}_1(x), \ldots, \hat{\nu}_{r-1}(x)\}, \quad \underline{\hat{\mu}}(x) = \{\hat{\mu}_1(x), \ldots, \hat{\mu}_{r-1}(x)\}.$$

Since from (2.25), $y_j(\tilde{z}),\ j=0,1,2$ satisfy $\mathcal{F}_r(\tilde{z},y)=0$, that is,

$$(y-y_0(\tilde{z}))(y-y_1(\tilde{z}))(y-y_2(\tilde{z}))=y^3+yS_r(\tilde{z})-T_r(\tilde{z})=0, \tag{3.16}$$

we can easily get

$$\begin{aligned}&y_0+y_1+y_2=0, \quad y_0y_1+y_0y_2+y_1y_2=S_r(\tilde{z}),\\ &y_0y_1y_2=T_r(\tilde{z}), \quad y_0^2+y_1^2+y_2^2=-2S_r(\tilde{z}),\\ &y_0^3+y_1^3+y_2^3=3T_r(\tilde{z}), \quad y_0^2y_1^2+y_0^2y_2^2+y_1^2y_2^2=S_r^2(\tilde{z}).\end{aligned} \tag{3.17}$$

Further properties of $\phi(P,x)$ and $\psi_2(P,x,x_0)$ are summarized as follows.

Theorem 3.2 *Assume* (3.1), (3.2), $P=(\tilde{z},y)\in\mathcal{K}_{r-1}\backslash\{P_\infty,P_0\}$, *and let* $(\tilde{z},x,x_0)\in\mathbb{C}^3$. *Then*

$$\phi_{xx}(P,x)+3\tilde{z}^{-1}\phi(P,x)\phi_x(P,x)+\tilde{z}^{-2}\phi^3(P,x)-\frac{m_x(x)}{m(x)}\phi_x(P,x)$$

$$-\tilde{z}^{-1}\frac{m_x(x)}{m(x)}\phi^2(P,x)+m(x)\tilde{z}^{-1}=0, \tag{3.18}$$

$$\phi(P,x)\phi(P^*,x)\phi(P^{**},x)=-\tilde{z}^3\frac{F_r(\tilde{z},x)}{E_r(\tilde{z},x)}, \tag{3.19}$$

$$\phi(P,x)+\phi(P^*,x)+\phi(P^{**},x)=\tilde{z}\frac{E_{r,x}(\tilde{z},x)}{E_r(\tilde{z},x)}, \tag{3.20}$$

$$\frac{1}{\phi(P,x)}+\frac{1}{\phi(P^*,x)}+\frac{1}{\phi(P^{**},x)}=\tilde{z}\frac{V_{31}F_{r,x}(\tilde{z},x)}{mV_{32}F_r(\tilde{z},x)}-3\tilde{z}^{-1}\frac{V_{33}}{V_{32}}, \tag{3.21}$$

$$\begin{aligned}&y(P)\phi(P,x)+y(P^*)\phi(P^*,x)+y(P^{**})\phi(P^{**},x)\\ &\quad=\tilde{z}\frac{3T_r(\tilde{z})V_{21}(\tilde{z},x)+2S_r(\tilde{z})A_r(\tilde{z},x)}{E_r(\tilde{z},x)},\end{aligned} \tag{3.22}$$

$$\psi_2(P,x,x_0)\psi_2(P^*,x,x_0)\psi_2(P^{**},x,x_0)=\frac{E_r(\tilde{z},x)}{E_r(\tilde{z},x_0)}, \tag{3.23}$$

$$\psi_{2,x}(P,x,x_0)\psi_{2,x}(P^*,x,x_0)\psi_{2,x}(P^{**},x,x_0)=-\frac{F_r(\tilde{z},x)}{E_r(\tilde{z},x_0)}, \tag{3.24}$$

$$\begin{aligned}\psi_2(P,x,x_0)&=\left[\frac{E_r(\tilde{z},x)}{E_r(\tilde{z},x_0)}\right]^{1/3}\\ &\quad\times\exp\left(\int_{x_0}^{x}\frac{y(P)^2V_{21}(\tilde{z},x')-y(P)A_r(\tilde{z},x')+\frac{2}{3}S_r(\tilde{z})V_{21}(\tilde{z},x')}{E_r(\tilde{z},x')}\mathrm{d}x'\right).\end{aligned} \tag{3.25}$$

Proof Equation (3.18) follows using the definition of (3.2) of ϕ as well as relation (3.1). Equations (3.19)–(3.25) are clear from (3.2), (3.4), (3.7)–(3.10), and (3.17). □

Next, we derive Dubrovin-type equations which are first-order coupled systems of differential equations and govern the dynamics of the zeros $\mu_j(x)$ and $\nu_j(x)$ of $E_r(\tilde{z},x)$ and $F_r(\tilde{z},x)$ with respect to x.

Lemma 3.3 *Assume* (2.14) *to hold in the stationary case.*

(i) *Suppose the zeros* $\{\mu_j(x)\}_{j=1,\dots,r-1}$ *of* $E_r(\tilde{z},x)$ *remain distinct for* $x \in \Omega_\mu$, *where* $\Omega_\mu \subseteq \mathbb{C}$ *is open and connected. Then* $\{\mu_j(x)\}_{j=1,\dots,r-1}$ *satisfy the system of differential equations,*

$$\mu_{j,x}(x) = -\frac{[S_r(\mu_j(x)) + 3y(\hat{\mu}_j(x))^2]V_{21}(\mu_j(x),x)}{E_{r,0}\prod_{\substack{k=1\\k\neq j}}^{r-1}(\mu_j(x)-\mu_k(x))}, \quad j=1,\dots,r-1, \tag{3.26}$$

with initial conditions

$$\{\hat{\mu}_j(x_0)\}_{j=1,\dots,r-1} \in \mathcal{K}_{r-1} \tag{3.27}$$

for some fixed $x_0 \in \Omega_\mu$. *The initial value problem* (3.26), (3.27) *has a unique solution satisfying*

$$\hat{\mu}_j \in C^\infty(\Omega_\mu, \mathcal{K}_{r-1}), \quad j=1,\dots,r-1. \tag{3.28}$$

(ii) *Suppose the zeros* $\{\nu_j(x)\}_{j=1,\dots,r-1}$ *of* $F_r(\tilde{z},x)$ *remain distinct for* $x \in \Omega_\nu$, *where* $\Omega_\nu \subseteq \mathbb{C}$ *is open and connected. Then* $\{\nu_j(x)\}_{j=1,\dots,r-1}$ *satisfy the system of differential equations,*

$$\nu_{j,x}(x) = -\frac{[S_r(\nu_j(x)) + 3y(\hat{\nu}_j(x))^2]m(x)V_{32}(\nu_j(x),x)}{F_{r,0}\prod_{\substack{k=1\\k\neq j}}^{r-1}(\nu_j(x)-\nu_k(x))}, \quad j=1,\dots,r-1 \tag{3.29}$$

with initial conditions

$$\{\hat{\nu}_j(x_0)\}_{j=1,\dots,r-1} \in \mathcal{K}_{r-1} \tag{3.30}$$

for some fixed $x_0 \in \Omega_\nu$. *The initial value problem* (3.29), (3.30) *has a unique solution satisfying*

$$\hat{\nu}_j \in C^\infty(\Omega_\nu, \mathcal{K}_{r-1}), \quad j=1,\dots,r-1. \tag{3.31}$$

Proof By using (3.7), (3.8), and (3.10), one computes

$$E_{r,x}(\mu_j(x),x) = [S_r(\mu_j(x)) + 3y(\hat{\mu}_j(x))^2]V_{21}(\mu_j(x),x), \tag{3.32}$$

$$F_{r,x}(\nu_j(x),x) = [S_r(\nu_j(x)) + 3y(\hat{\nu}_j(x))^2]V_{32}(\nu_j(x),x)m(x)\nu_j(x)^{-2}. \tag{3.33}$$

On the other hand, derivatives of (3.11) and (3.12) with respect to x are

$$E_{r,x}|_{\tilde{z}=\mu_j(x)} = -E_{r,0}\mu_{j,x}(x)\prod_{\substack{k=1\\k\neq j}}^{r-1}(\mu_j(x)-\mu_k(x)), \tag{3.34}$$

$$F_{r,x}|_{\tilde{z}=\nu_j(x)} = -F_{r,0}\nu_j(x)^{-2}\nu_{j,x}(x)\prod_{\substack{k=1\\k\neq j}}^{r-1}(\nu_j(x)-\nu_k(x)). \tag{3.35}$$

Comparing (3.32)–(3.35) leads to (3.26) and (3.29). □

4 Stationary Algebro-geometric Solutions

In this section, we obtain explicit Riemann theta function representations for the meromorphic function ϕ, the Baker–Akhiezer function ψ_2, and especially, for the solutions u of the stationary DB hierarchy.

Lemma 4.1 *Let $x \in \mathbb{C}$.*

(i) *Near $P_\infty \in \mathcal{K}_{r-1}$, in terms of the local coordinate $\zeta = \tilde{z}^{-1/3}$, we have*

$$\phi(P,x) \underset{\zeta\to 0}{=} \frac{1}{\zeta}\sum_{j=0}^{\infty}\kappa_j(x)\zeta^j \quad \text{as } P \to P_\infty, \tag{4.1}$$

where

$$\kappa_0 = u_x(x), \quad \kappa_1 = 0, \tag{4.2}$$

$$\kappa_{2,xx} + 3\kappa_{0,x}\kappa_0 = \frac{m_x}{m}\left(\kappa_{2,x}+\kappa_0^2\right), \quad \kappa_3 = 0, \tag{4.3}$$

$$\ldots\ldots$$

$$\kappa_{2\varsigma,xx} + 3\sum_{i=0}^{\varsigma-1}\kappa_{2i}\kappa_{2\varsigma-2i-2,x} + \sum_{i=0}^{\varsigma-2}\sum_{\ell=0}^{\varsigma-2-i}\kappa_{2i}\kappa_{2\ell}\kappa_{2\varsigma-2i-2\ell-4}$$
$$= \frac{m_x}{m}\left(\kappa_{2\varsigma,x} + \sum_{i=0}^{\varsigma-1}\kappa_{2i}\kappa_{2\varsigma-2i-2}\right) - m\delta_{2\varsigma,4}, \tag{4.4}$$

$$\kappa_{2\varsigma+1} = 0, \quad \varsigma \geq 2, \quad \varsigma \in \mathbb{N}. \tag{4.5}$$

(ii) *Near $P_0 \in \mathcal{K}_{r-1}$, in terms of the local coordinate $\zeta = \tilde{z}^{\frac{1}{3}}$, we have*

$$\phi(P,x) \underset{\zeta\to 0}{=} \sum_{j=0}^{\infty}\iota_j(x)\zeta^{j+1} \quad \text{as } P \to P_0, \tag{4.6}$$

where

$$\iota_0 = -m^{\frac{1}{3}}, \quad \iota_1 = 0, \quad \iota_2 = \frac{(m_x/m)\iota_0^2 - 3\iota_0\iota_{0,x}}{3\iota_0^2} = 0, \quad \iota_3 = 0,$$

$$\iota_4 = \frac{\frac{m_x}{m}\iota_{0,x} - \iota_{0,xx}}{3\iota_0^2}, \quad \iota_5 = 0, \quad \ldots \tag{4.7}$$

$$\iota_{2\varsigma} = \frac{\frac{m_x}{m}\left(\sum_{i=0}^{\varsigma-1}\iota_{2i}\iota_{2\varsigma-2-2i} + \iota_{2\varsigma-4,x}\right) - \iota_{2\varsigma-4,xx} - 3\sum_{i=0}^{\varsigma-1}\iota_{2i}\iota_{2\varsigma-2-2i,x}}{3\iota_0^2},$$

$$\iota_{2\varsigma+1} = 0, \quad \varsigma \geq 3, \quad \varsigma \in \mathbb{N}. \tag{4.8}$$

Proof The existence of these asymptotic expansions (4.1) and (4.6) in terms of local coordinates $\zeta = \tilde{z}^{-1/3}$ near P_∞ and $\zeta = \tilde{z}^{\frac{1}{3}}$ near P_0 is clear from the explicit form of ϕ in (3.4). Insertion of the polynomials V_{ij} $(i, j = 1, 2, 3)$ then, in principle, yields the explicit expansion coefficients in (4.1) and (4.6). For example, $\kappa_0 = u_x(x)$ and $\kappa_1 = 0$ in (4.2). However, this is a cumbersome procedure, especially with regard to the next to leading coefficients in (4.1). Much more efficient is the actual computation of these coefficients utilizing the Riccati-type equation (3.18). Indeed, inserting the ansatz

$$\phi \underset{\zeta\to 0}{=} \sum_{j=0}^{\infty} \iota_j(x)\zeta^{j+1} \quad \text{as } P \to P_0 \tag{4.9}$$

into (3.18) and comparing the same powers of ζ then yields (4.7). Similarly, inserting the ansatz

$$\phi \underset{\zeta\to 0}{=} \frac{1}{\zeta}\sum_{j=0}^{\infty} \kappa_j(x)\zeta^{j} \quad \text{as } P \to P_\infty \tag{4.10}$$

into (3.18) and comparing the same powers of ζ then yields (4.3) and (4.4). Finally, (4.5) and (4.8) arise from the technical treatment in Sect. 2 ($z = \tilde{z}^2$; see (2.19)). □

Remark 4.2 We have derived the explicit expressions for $\kappa_0, \kappa_{2\varsigma+1}, \varsigma \in \mathbb{N}_0$ in Lemma 4.1. However, the coefficients $\kappa_{2\varsigma}, \varsigma \in \mathbb{N}$ in the high-energy expansion of ϕ are still implicit, since (4.3) and (4.4) involve the x-derivatives of $\kappa_{2\varsigma}, \varsigma \in \mathbb{N}$ and hence yield a series of second-order ODEs (or PDEs in the time-dependent case) with variable coefficients. In the process of solving other integrable evolution equations such as classical Thirring system [near the points $P_{0,\pm}$; see Gesztesy and Holden (2003b)], CH hierarchy (near the points $P_{\infty\pm}$; see Gesztesy and Holden (2003a, b)), if we directly insert an ansatz into a Riccati-type equation, an analogous problem will arise.

The DB hierarchy shares some similarities with the CH hierarchy at this point. Since the concrete expressions $\kappa_j, j \geq 2, j \in \mathbb{N}$ are useless in the process of finding the algebro-geometric solutions of DB hierarchy, we do not intend to write out their explicit forms from (3.4).

We assume $\mathcal{K}_{r-1}$ to be nonsingular for the remainder of this section. We now introduce the holomorphic differentials $\eta_l(P)$ on $\mathcal{K}_{r-1}$ defined by

$$\eta_l(P) = \frac{1}{3y(P)^2 + S_r(\tilde{z})} \begin{cases} \tilde{z}^{l-1} d\tilde{z}, & 1 \le l \le 8n, \\ y(P)\tilde{z}^{l-8n-1} d\tilde{z}, & 8n+1 \le l \le 12n, \end{cases} \tag{4.11}$$

and choose an appropriate fixed homology basis $\{a_j, b_j\}_{j=1}^{r-1}$ on $\mathcal{K}_{r-1}$ in such a way that the intersection matrix of cycles satisfies

$$a_j \circ b_k = \delta_{j,k}, \quad a_j \circ a_k = 0, \quad b_j \circ b_k = 0, \quad j,k = 1, \ldots, r-1.$$

Define an invertible matrix $E \in GL(r-1, \mathbb{C})$ as

$$\begin{aligned} E &= (E_{j,k})_{(r-1)\times(r-1)}, \quad E_{j,k} = \int_{a_k} \eta_j, \\ \underline{e}(k) &= (e_1(k), \ldots, e_{r-1}(k)), \quad e_j(k) = (E^{-1})_{j,k}, \end{aligned} \tag{4.12}$$

and the normalized holomorphic differentials

$$\omega_j = \sum_{l=1}^{r-1} e_j(l)\eta_l, \quad \int_{a_k} \omega_j = \delta_{j,k}, \quad \int_{b_k} \omega_j = \Gamma_{j,k}, \quad j,k = 1, \ldots, r-1. \tag{4.13}$$

Next, choosing a convenient base point $Q_0 \in \mathcal{K}_{r-1} \backslash \{P_\infty, P_0\}$, the vector of Riemann constants $\underline{\Xi}_{Q_0}$ is given by (A.45) (Gesztesy and Holden 2003b, and the Abel maps $\underline{A}_{Q_0}(\cdot)$ and $\underline{\alpha}_{Q_0}(\cdot)$ are defined by

$$\begin{aligned} &\underline{A}_{Q_0} : \mathcal{K}_{r-1} \to J(\mathcal{K}_{r-1}) = \mathbb{C}^{r-1}/L_{r-1}, \\ &P \mapsto \underline{A}_{Q_0}(P) = (A_{Q_0,1}(P), \ldots, A_{Q_0,r-1}(P)) \\ &\quad = \left(\int_{Q_0}^{P} \omega_1, \ldots, \int_{Q_0}^{P} \omega_{r-1} \right) (\text{mod } L_{r-1}), \end{aligned}$$

and

$$\underline{\alpha}_{Q_0} : \text{Div}(\mathcal{K}_{r-1}) \to J(\mathcal{K}_{r-1}), \quad \mathcal{D} \mapsto \underline{\alpha}_{Q_0}(\mathcal{D}) = \sum_{P \in \mathcal{K}_{r-1}} \mathcal{D}(P) \underline{A}_{Q_0}(P),$$

where $L_{r-1} = \{\underline{z} \in \mathbb{C}^{r-1} | \ \underline{z} = \underline{N} + \Gamma \underline{M}, \ \underline{N}, \ \underline{M} \in \mathbb{Z}^{r-1}\}$.

For brevity, define the function $\underline{z} : \mathcal{K}_{r-1} \times \sigma^{r-1}\mathcal{K}_{r-1} \to \mathbb{C}^{r-1}$ by[2]

$$\begin{aligned}&\underline{z}(P, \underline{Q}) = \underline{\Xi}_{Q_0} - \underline{A}_{Q_0}(P) + \underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{Q}}),\\ &\quad P \in \mathcal{K}_{r-1},\ \underline{Q} = (Q_1, \ldots, Q_{r-1}) \in \sigma^{r-1}\mathcal{K}_{r-1};\end{aligned} \tag{4.14}$$

here $\underline{z}(\cdot, \underline{Q})$ is independent of the choice of base point Q_0. The Riemann theta function $\theta(\underline{z})$ associated with $\mathcal{K}_{r-1}$ and the homology is defined by

$$\theta(\underline{z}) = \sum_{\underline{n}\in\mathbb{Z}^{r-1}} \exp\left(2\pi i < \underline{n}, \underline{z} > +\pi i < \underline{n}, \underline{n}\Gamma >\right), \quad \underline{z} \in \mathbb{C}^{r-1},$$

where $< \underline{B}, \underline{C} >= \overline{\underline{B}} \cdot \underline{C}^t = \sum_{j=1}^{r-1} \overline{B}_j C_j$ denotes the scalar product in $\mathbb{C}^{r-1}$.

The normalized differential $\omega^{(3)}_{P_\infty P_0}(P)$ of the third kind is the unique differential holomorphic on $\mathcal{K}_{r-1}\backslash\{P_\infty, P_0\}$ with simple poles at P_∞ and P_0 with residues ± 1, respectively, that is,

$$\begin{aligned}&\omega^{(3)}_{P_\infty P_0}(P) \underset{\zeta\to 0}{=} (\zeta^{-1} + O(1))d\zeta, \quad \text{as } P \to P_\infty,\\ &\omega^{(3)}_{P_\infty P_0}(P) \underset{\zeta\to 0}{=} (-\zeta^{-1} + O(1))d\zeta, \quad \text{as } P \to P_0.\end{aligned} \tag{4.15}$$

In particular,

$$\int_{a_j} \omega^{(3)}_{P_\infty P_0}(P) = 0, \quad j = 1, \ldots, r-1.$$

Then

$$\begin{aligned}&\int_{Q_0}^{P} \omega^{(3)}_{P_\infty P_0}(P) \underset{\zeta\to 0}{=} \ln\zeta + e^{(3)}(Q_0) + O(\zeta), \quad \text{as } P \to P_\infty,\\ &\int_{Q_0}^{P} \omega^{(3)}_{P_\infty P_0}(P) \underset{\zeta\to 0}{=} -\ln\zeta + e^{(3)}(Q_0) + O(\zeta), \quad \text{as } P \to P_0,\end{aligned} \tag{4.16}$$

where $e^{(3)}(Q_0)$ is an integration constant.

The theta function representation of $\phi(P, x)$ then reads as follows.

Theorem 4.3 *Assume that the curve $\mathcal{K}_{r-1}$ is nonsingular. Let $P = (\tilde{z}, y) \in \mathcal{K}_{r-1}\backslash\{P_\infty, P_0\}$ and $x \in \Omega_\mu$, where $\Omega_\mu \subseteq \mathbb{C}$ is open and connected. Suppose that $\mathcal{D}_{\underline{\hat{\mu}}(x)}$, or equivalently $\mathcal{D}_{\underline{\hat{\nu}}(x)}$, is nonspecial[3] for $x \in \Omega_\mu$. Then*

[2] $\sigma^{r-1}\mathcal{K}_{r-1} = \underbrace{\mathcal{K}_{r-1} \times \cdots \times \mathcal{K}_{r-1}}_{r-1}$.

[3] For the definition of a nonspecial divisor, see Farkas and Kra (1992).

$$\phi(P,x) = -m^{\frac{1}{3}}(x)\frac{\theta(\underline{\tilde{z}}(P,\underline{\hat{\nu}}(x)))\theta(\underline{\tilde{z}}(P_0,\underline{\hat{\mu}}(x)))}{\theta(\underline{\tilde{z}}(P_0,\underline{\hat{\nu}}(x)))\theta(\underline{\tilde{z}}(P,\underline{\hat{\mu}}(x)))} exp\left(e^{(3)}(Q_0) - \int_{Q_0}^{P} \omega^{(3)}_{P_\infty P_0}\right). \tag{4.17}$$

Proof Let Φ be defined by the right-hand side of (4.17) with the aim to prove that $\phi = \Phi$. From (4.16) it follows that

$$\begin{aligned} \exp\left(e^{(3)}(Q_0) - \int_{Q_0}^{P} \omega^{(3)}_{P_\infty P_0}\right) &\underset{\zeta\to 0}{=} \zeta^{-1} + O(1), \quad \text{as } P \to P_\infty, \\ \exp\left(e^{(3)}(Q_0) - \int_{Q_0}^{P} \omega^{(3)}_{P_\infty P_0}\right) &\underset{\zeta\to 0}{=} \zeta + O(\zeta^2), \quad \text{as } P \to P_0. \end{aligned} \tag{4.18}$$

Using (3.15) we immediately know that ϕ has simple poles at $\underline{\hat{\mu}}(x)$ and P_∞ and simple zeros at P_0 and $\underline{\hat{\nu}}(x)$. By (4.17) and a special case of Riemann's vanishing theorem (Farkas and Kra 1992; Gesztesy and Holden 2003b; Gesztesy et al. 2008), we see that Φ shares the same properties. Hence, using the Riemann–Roch theorem (Farkas and Kra 1992; Gesztesy and Holden 2003b; Gesztesy et al. 2008) yields the holomorphic function $\Phi/\phi = \gamma$, where γ is a constant with respect to P. Finally, considering the asymptotic expansion of Φ and ϕ near P_0, we obtain

$$\frac{\Phi}{\phi} \underset{\zeta\to 0}{=} \frac{-m^{1/3}(1+O(\zeta))(\zeta+O(\zeta^2))}{-m^{1/3}\zeta + O(\zeta^2)} \underset{\zeta\to 0}{=} 1 + O(\zeta), \quad \text{as } P \to P_0, \tag{4.19}$$

from which we conclude that $\gamma = 1$, where we used (4.18) and (4.6). Hence, we prove (4.17). □

Furthermore, let $\omega^{(2)}_{P_0,3}(P)$ denote the normalized differential of the second kind which is holomorphic on $\mathcal{K}_{r-1}\backslash\{P_0\}$ with a pole of order 3 at P_0,

$$\omega^{(2)}_{P_0,3}(P) = \frac{\tilde{z}^{-1}d\tilde{z}}{3y(P)^2 + S_r(\tilde{z})} + \sum_{j=1}^{r-1}\lambda_j\eta_j(P) \underset{\zeta\to 0}{=} (\zeta^{-3} + O(1))\mathrm{d}\zeta, \quad \text{as } P \to P_0,$$

where the constants $\{\lambda_j\}_{j=1,\dots,r-1} \in \mathbb{C}$ are determined by the normalization condition

$$\int_{a_j} \omega^{(2)}_{P_0,3}(P) = 0, \quad j = 1,\dots,r-1,$$

and the differentials $\{\eta_j(P)\}_{j=1,\dots,r-1}$ (defined in (4.11)) form a basis for the space of holomorphic differentials. Moreover, we define the vector of b-periods of $\omega^{(2)}_{P_0,3}$,

$$\underline{\hat{U}}^{(2)}_3 = (\hat{U}^{(2)}_{3,1},\dots,\hat{U}^{(2)}_{3,r-1}), \quad \hat{U}^{(2)}_{3,j} = \frac{1}{2\pi i}\int_{b_j}\omega^{(2)}_{P_0,3}, \quad j = 1,\dots,r-1. \tag{4.20}$$

Then

$$\int_{Q_0}^{P} \omega_{P_0,3}^{(2)}(P) \underset{\zeta\to 0}{=} -\frac{1}{2}\zeta^{-2} + e_3^{(2)}(Q_0) + O(\zeta), \quad \text{as } P \to P_0,$$
$$\int_{Q_0}^{P} \omega_{P_0,3}^{(2)}(P) \underset{\zeta\to 0}{=} e_3^{(2)}(Q_0) + f_3^{(2)}(Q_0)\zeta^2 + O(\zeta^4), \quad \text{as } P \to P_\infty,$$

where $e_3^{(2)}(Q_0)$, $f_3^{(2)}(Q_0)$ are integration constants.

Similarly, the theta function representation of the Baker–Akhiezer function $\psi_2(P, x, x_0)$ is summarized in the following theorem.

Theorem 4.4 *Assume that the curve $\mathcal{K}_{r-1}$ is nonsingular. Let $P = (\tilde{z}, y) \in \mathcal{K}_{r-1}\backslash\{P_\infty, P_0\}$ and let $x, x_0 \in \Omega_\mu$, where $\Omega_\mu \subseteq \mathbb{C}$ is open and connected. Suppose that $\mathcal{D}_{\underline{\hat{\mu}}(x)}$, or equivalently $\mathcal{D}_{\underline{\hat{\nu}}(x)}$, is nonspecial for $x \in \Omega_\mu$. Then*

$$\psi_2(P, x, x_0) = \frac{\theta(\underline{\tilde{z}}(P, \underline{\hat{\mu}}(x)))\theta(\underline{\tilde{z}}(P_0, \underline{\hat{\mu}}(x_0)))}{\theta(\underline{\tilde{z}}(P_0, \underline{\hat{\mu}}(x)))\theta(\underline{\tilde{z}}(P, \underline{\hat{\mu}}(x_0)))} \times exp\left(\int_{x_0}^{x} 2m^{\frac{1}{3}}(x')dx'\left(\int_{Q_0}^{P} \omega_{P_0,3}^{(2)} - e_3^{(2)}(Q_0)\right)\right). \tag{4.21}$$

Proof Assume temporarily that

$$\mu_j(x) \neq \mu_k(x) \quad \text{for } j \neq k \text{ and } x \in \widetilde{\Omega}_\mu \subseteq \Omega_\mu, \tag{4.22}$$

where $\widetilde{\Omega}_\mu$ is open and connected. For the Baker–Akhiezer function ψ_2 we will use the same strategy as was used in the previous proof. Let Ψ denote the right-hand side of (4.21). We intend to prove $\psi_2 = \Psi$ with ψ_2 given by (3.3). For that purpose we first investigate the local zeros and poles of ψ_2. Since they can come only from simple poles in the integrand in (3.3). By using the definition (3.4) of ϕ, (3.10), and the Dubrovin equations (3.26), one computes

$$\phi(P, x) = -\mu_j \frac{\mu_{j,x}}{\tilde{z} - \mu_j} + O(1), \quad \text{as } \tilde{z} \to \mu_j(x). \tag{4.23}$$

More concisely,

$$\phi(P, x) = \mu_j(x)\frac{\partial}{\partial x}\ln(\tilde{z} - \mu_j(x)) + O(1) \quad \text{for } P \text{ near } \hat{\mu}_j(x), \tag{4.24}$$

which together with (3.3) yields

$$\psi_2(P, x, x_0) = \frac{\tilde{z} - \mu_j(x)}{\tilde{z} - \mu_j(x_0)}O(1) = \begin{cases} (\tilde{z} - \mu_j(x))O(1) & \text{for } P \text{ near } \hat{\mu}_j(x) \neq \hat{\mu}_j(x_0), \\ O(1) & \text{for } P \text{ near } \hat{\mu}_j(x) = \hat{\mu}_j(x_0), \\ (\tilde{z} - \mu_j(x_0))^{-1}O(1) & \text{for } P \text{ near } \hat{\mu}_j(x_0) \neq \hat{\mu}_j(x), \end{cases} \tag{4.25}$$

where $O(1) \neq 0$ in (4.25). Consequently, all zeros and poles of ψ_2 and Ψ on $\mathcal{K}_{r-1}\backslash\{P_\infty, P_0\}$ are simple and coincident. It remains to identify the behavior of ψ_2 and Ψ near P_∞ and P_0. Taking into account (4.1), (4.6), and (4.21), one observes that ψ_2 and Ψ have identical exponential behavior near P_∞ and P_0. The uniqueness result for Baker–Akhiezer functions (Gesztesy and Ratneseelan 1998; Gesztesy and Holden 2003b; Gesztesy et al. 2008; Bulla et al. 1998) then completes the proof $\psi_2 = \Psi$ as both functions share the same singularities and zeros. The extension of this result from $x \in \widetilde{\Omega}_\mu$ to $x \in \Omega_\mu$ then simply follows from the continuity of $\underline{\alpha}_{Q_0}$ and the hypothesis of $\mathcal{D}_{\underline{\hat{\mu}}(x)}$ being nonspecial for $x \in \Omega_\mu$. □

The asymptotic behavior of $y(P)$ and S_r near P_∞ is summarized as follows.

Lemma 4.5

$$y(P) \underset{\zeta\to 0}{=} \varrho\zeta^{-12n-1}(1+\alpha_0\zeta^2+\alpha_1\zeta^4+O(\zeta^6)), \quad \textit{as } P \to P_\infty, \tag{4.26}$$

$$S_r \underset{\zeta\to 0}{=} \zeta^{-24n}(\beta_0+O(\zeta^6)), \quad \textit{as } P \to P_\infty, \tag{4.27}$$

where ϱ is a constant, arising from finding the root of of algebraic equation (2.29) corresponding to the point $P_\infty \in \mathcal{K}_{r-1}$.

Proof From (3.1) and (3.2), we arrive at

$$y(P) = -V_{21}\frac{\tilde{z}^2\phi_x+\tilde{z}\phi^2}{m} + V_{22}\tilde{z} + V_{23}\phi. \tag{4.28}$$

Then, in terms of the local coordinate $\zeta = \tilde{z}^{-1/3}$, insertion of (2.11) and (4.1) into (4.28) yields

$$\begin{aligned} y(P) = \frac{1}{m}\sum_{\ell=0}^{n} V_{21}^{(\ell)}(G_\ell)\zeta^{-12(n+1-\ell)}\Big(-\zeta^{-6}\sum_{j=0}^{\infty}\kappa_{j,x}\zeta^{j-1} - \zeta^{-3}\Big(\sum_{j=0}^{\infty}\kappa_j\zeta^{j-1}\Big)^2\Big) \\ + \sum_{\ell=0}^{n} V_{22}^{(\ell)}(G_\ell)\zeta^{-12(n+1-\ell)-3} + \sum_{\ell=0}^{n} V_{23}^{(\ell)}(G_\ell)\zeta^{-12(n+1-\ell)}\sum_{j=0}^{\infty}\kappa_j\zeta^{j-1} \\ \underset{\zeta\to 0}{=} \varrho\zeta^{-12n-1}(1+\alpha_0\zeta^2+\alpha_1\zeta^4+O(\zeta^6)), \quad \text{as } P \to P_\infty. \end{aligned} \tag{4.29}$$

Similarly, combining (2.20) and (4.1) leads to (4.27). □

A straightforward Laurent expansion of (4.11), (4.12), and (4.13) near P_∞ yields the following results.

Lemma 4.6 *Assume the curve $\mathcal{K}_{r-1}$ to be nonsingular. Then the vector of normalized holomorphic differentials $\underline{\omega}$ have the Laurent series*

$$\underline{\omega} = (\omega_1, \ldots, \omega_{r-1}) \underset{\zeta\to 0}{=} (\underline{\rho}_0 + \underline{\rho}_1\zeta + O(\zeta^2))\mathrm{d}\zeta \tag{4.30}$$

near P_∞ with

$$\underline{\rho}_0 = -\frac{1}{\varrho}\underline{e}(r-1), \quad \underline{\rho}_1 = -\frac{1}{\varrho^2}\underline{e}(8n),$$

where ϱ is given in Lemma 4.5.

Proof In terms of the local coordinate $\zeta = \tilde{z}^{-1/3}$ near P_∞, using (4.11), (4.13), (4.26), and (4.27), we have

$$\begin{aligned}\omega_j &= \sum_{l=1}^{r-1} e_j(l)\eta_l = -3\sum_{l=1}^{8n} e_j(l)\frac{\zeta^{-3l-1}d\zeta}{3y^2+S_r} - 3\sum_{l=8n+1}^{r-1} e_j(l)\frac{y\zeta^{-3l+24n-1}d\zeta}{3y^2+S_r} \\ &\underset{\zeta\to 0}{=} \left(-\frac{1}{\varrho}e_j(r-1) - \frac{1}{\varrho^2}e_j(8n)\zeta + O(\zeta^2)\right)d\zeta, \end{aligned} \tag{4.31}$$

which yields (4.30). □

Theorem 4.7 *Assume that the curve $\mathcal{K}_{r-1}$ is nonsingular and let $x, x_0 \in \mathbb{C}$. Then*

$$\underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\mu}}(x)}) = \underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\mu}}(x_0)}) + \underline{e}(r-1)\int_{x_0}^{x}\frac{A_{r,0}(u(x'))}{E_{r,0}(u(x'))}dx', \tag{4.32}$$

$$\underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\nu}}(x)}) = \underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\nu}}(x_0)}) + \underline{e}(r-1)\int_{x_0}^{x}\frac{A_{r,0}(u(x'))}{E_{r,0}(u(x'))}dx'. \tag{4.33}$$

In particular, the Abel map does not linearize the divisor $\mathcal{D}_{\underline{\hat{\mu}}(\cdot)}$ and $\mathcal{D}_{\underline{\hat{\nu}}(\cdot)}$.

Proof We prove only (4.32) as (4.33) can be obtained from (4.32) and Abel's theorem. Assume temporarily that

$$\mu_j(x) \neq \mu_{j'}(x) \quad \text{for } j \neq j' \text{ and } x \in \widetilde{\Omega}_\mu \subseteq \mathbb{C}, \tag{4.34}$$

where $\widetilde{\Omega}_\mu$ is open and connected. Then using (3.26), (4.11), and (4.13), one computes

$$\begin{aligned}\frac{d}{dx}\alpha_{Q_0,l}(\mathcal{D}_{\underline{\hat{\mu}}(x)}) &= \sum_{k=1}^{8n} e_l(k)\sum_{j=1}^{r-1}\frac{-\left(V_{21}^{(1,0)}\mu_j^{4n-2} + V_{21}^{(1,1)}\mu_j^{4n-4} + \cdots\right)\mu_j^{k-1}}{E_{r,0}\prod_{p=1, p\neq j}^{r-1}(\mu_j - \mu_p)} \\ &\quad + \sum_{k=8n+1}^{r-1} e_l(k)\sum_{j=1}^{r-1}\frac{\left(A_{r,0}\mu_j^{8n-1} + A_{r,1}\mu_j^{8n-3} + \cdots\right)\mu_j^{k-8n-1}}{E_{r,0}\prod_{p=1, p\neq j}^{r-1}(\mu_j - \mu_p)}.\end{aligned}$$

Using the standard Largange interpolation argument then yields

$$\frac{d}{dx}\alpha_{Q_0,l}(\mathcal{D}_{\underline{\hat{\mu}}(x)}) = \frac{A_{r,0}(u(x))}{E_{r,0}(u(x))}e_l(r-1), \tag{4.35}$$

which implies (4.32). The equality (4.33) follows from the linear equivalence $\mathcal{D}_{P_\infty \underline{\hat{\mu}}(x)} \sim \mathcal{D}_{P_0 \underline{\hat{\nu}}(x)}$, and (4.32). The extension of all these results from $x \in \widetilde{\Omega}_\mu$ to $x \in \mathbb{C}$ then simply follows from the continuity of $\underline{\alpha}_{Q_0}$ and the hypothesis of $\mathcal{D}_{\underline{\hat{\mu}}(x)}$ being nonspecial on Ω_μ. □

Next, we provide an explicit representation for the stationary DB solutions u in terms of the Riemann theta function associated with $\mathcal{K}_{r-1}$, assuming the affine part of $\mathcal{K}_{r-1}$ to be nonsingular.

Theorem 4.8 *Assume that u satisfies the n-th stationary DB equation (2.14), that is, $X_n(u) = 0$, and the curve $\mathcal{K}_{r-1}$ is nonsingular. Let $x \in \Omega_\mu$, where $\Omega_\mu \subseteq \mathbb{C}$ is open and connected. Suppose that $\mathcal{D}_{\underline{\hat{\mu}}(x)}$, or equivalently $\mathcal{D}_{\underline{\hat{\nu}}(x)}$, is nonspecial for $x \in \Omega_\mu$. Then*

$$u(x) = u(x_0) \frac{\theta(\underline{\tilde{z}}(P_0, \underline{\hat{\mu}}(x_0)))\theta(\underline{\tilde{z}}(P_\infty, \underline{\hat{\mu}}(x)))}{\theta(\underline{\tilde{z}}(P_\infty, \underline{\hat{\mu}}(x_0)))\theta(\underline{\tilde{z}}(P_0, \underline{\hat{\mu}}(x)))}. \tag{4.36}$$

Proof Using Theorem 4.4, one can write ψ_2 near P_∞ in the coordinate $\zeta = \tilde{z}^{-1/3}$, as

$$\begin{aligned} \psi_2(P, x, x_0) \underset{\zeta \to 0}{=} & \left(\sigma_0(x) + \sigma_1(x)\zeta + \sigma_2(x)\zeta^2 + O(\zeta^3)\right) \\ & \times \exp\left(\left(\int_{x_0}^{x} 2m^{\frac{1}{3}}(x')dx'\right)\left(f_3^{(2)}(Q_0)\zeta^2 + O(\zeta^4)\right)\right), \quad \text{as} P \to P_\infty, \end{aligned} \tag{4.37}$$

where the terms $\sigma_0(x)$, $\sigma_1(x)$ and $\sigma_2(x)$ in (4.37) come from the Taylor expansion about P_∞ of the ratios of the theta functions in (4.21). That is,

$$\frac{\theta\left(\underline{\tilde{z}}(P, \underline{\hat{\mu}}(x))\right)}{\theta\left(\underline{\tilde{z}}(P_0, \underline{\hat{\mu}}(x))\right)} \underset{\zeta \to 0}{=} \frac{\theta_0}{\theta_1} - \frac{\partial_x \theta_0}{\theta_1}\zeta + \frac{\frac{1}{2}\partial_x^2\theta_0 - \partial_{\underline{U}_3^{(2)}}\theta_0}{\theta_1}\zeta^2 + O(\zeta^3), \quad \text{as } P \to P_\infty, \tag{4.38}$$

where

$$\begin{aligned} \theta_0 &= \theta_0(x) = \theta(\underline{\tilde{z}}(P_\infty, \underline{\hat{\mu}}(x))) = \theta\left(\underline{\Xi}_{Q_0} - \underline{A}_{Q_0}(P_\infty) + \underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\mu}}(x)})\right), \\ \theta_1 &= \theta_1(x) = \theta(\underline{\tilde{z}}(P_0, \underline{\hat{\mu}}(x))) = \theta\left(\underline{\Xi}_{Q_0} - \underline{A}_{Q_0}(P_0) + \underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\mu}}(x)})\right), \end{aligned}$$

and $\partial_{\underline{U}_3^{(2)}} = \sum_{j=1}^{r-1} U_{3,j}^{(2)} \frac{\partial}{\partial \tilde{z}_j}$ denotes the directional derivative in the direction of the vector of b-periods $\underline{U}_3^{(2)}$, defined by

$$\underline{U}_3^{(2)} = \left(U_{3,1}^{(2)}, \ldots, U_{3,r-1}^{(2)}\right), \quad U_{3,j}^{(2)} = \frac{1}{2\pi i} \int_{b_j} \omega_{P_\infty,3}^{(2)}, \quad j = 1, \ldots, r-1, \tag{4.39}$$

with $\omega^{(2)}_{P_{\infty,3}}$ holomorphic on $\mathcal{K}_{r-1}\backslash\{P_\infty\}$ with a pole of order 3 at P_∞,

$$\omega^{(2)}_{P_{\infty,3}}(P) \underset{\zeta\to 0}{=} (\zeta^{-3} + O(1))\mathrm{d}\zeta, \quad \text{as } P \to P_\infty. \tag{4.40}$$

Similarly, we have

$$\frac{\theta\left(\underline{\tilde{z}}(P_0, \underline{\hat{\mu}}(x_0))\right)}{\theta\left(\underline{\tilde{z}}(P, \underline{\hat{\mu}}(x_0))\right)} \underset{\zeta\to 0}{=} \frac{\theta_1(x_0)}{\theta_0(x_0)}\left(1 + \partial_x \ln\theta_0(x)\Big|_{x=x_0}\zeta + O(\zeta^2)\right), \text{as} P \to P_\infty. \tag{4.41}$$

Then, combining (4.21), (4.37), (4.38), and (4.41) leads to

$$\begin{aligned} \sigma_0(x) &= \frac{\theta_1(x_0)}{\theta_0(x_0)}\frac{\theta_0(x)}{\theta_1(x)}, \\ \sigma_1(x) &= \frac{\theta_1(x_0)}{\theta_0(x_0)}\frac{\theta_0(x)}{\theta_1(x)}\left(\partial_x \ln\theta_0(x)\Big|_{x=x_0} - \partial_x \ln\theta_0(x)\right). \end{aligned} \tag{4.42}$$

If we set

$$\psi_2 \underset{\zeta\to 0}{=} (\sigma_0(x) + \sigma_1(x)\zeta + \sigma_2(x)\zeta^2 + O(\zeta^3))\ \exp(\Delta), \quad \text{as } P \to P_\infty \tag{4.43}$$

with

$$\begin{aligned} \exp(\Delta) &= \exp\left(\left(\int_{x_0}^{x} 2m^{\frac{1}{3}}(x')\mathrm{d}x'\right)\left(f_3^{(2)}(Q_0)\zeta^2 + O(\zeta^4)\right)\right) \\ &= \left(1 + \left(f_3^{(2)}(Q_0)\int_{x_0}^{x} 2m^{\frac{1}{3}}(x')\mathrm{d}x'\right)\zeta^2 + O(\zeta^4)\right), \end{aligned}$$

then we compute its x-derivatives as ($P \to P_\infty$)

$$\begin{aligned} \psi_{2,x} &\underset{\zeta\to 0}{=} \left(\sigma_{0,x} + \sigma_{1,x}\zeta + O(\zeta^2)\right)\exp(\Delta) + \left(\left(2f_3^{(2)}(Q_0)m^{\frac{1}{3}}\right)\zeta^2 + O(\zeta^4)\right)\psi_2 \\ &\underset{\zeta\to 0}{=} \sigma_{0,x} + \sigma_{1,x}\zeta + O(\zeta^2), \\ \psi_{2,xx} &\underset{\zeta\to 0}{=} \sigma_{0,xx} + \sigma_{1,xx}\zeta + O(\zeta^2), \\ \psi_{2,xxx} &\underset{\zeta\to 0}{=} \sigma_{0,xxx} + \sigma_{1,xxx}\zeta + O(\zeta^2). \end{aligned} \tag{4.44}$$

By eliminating ψ_1 and ψ_3 in (2.3), we arrive at

$$\psi_{2,xxx} = -m\tilde{z}^{-2}\psi_2 + \frac{m_x}{m}\psi_{2,xx}. \tag{4.45}$$

Substituting (4.44) into (4.45) and comparing the coefficients of ζ^0, we obtain

$$\frac{(\sigma_{0,xx})_x}{\sigma_{0,xx}} = \frac{m_x}{m} = \frac{(u_{xx}(x))_x}{u_{xx}(x)},$$

which together with the first line of (4.42) leads to (4.36). □

Remark 4.9 We note the unusual fact that P_0, as opposed to P_∞, is the essential singularity of ψ_2. What makes matters worse is the intricate x-dependence of the leading-order exponential term in ψ_2, near P_0, as displayed in (4.21). This is in sharp contrast to standard Baker–Akhiezer functions that typically feature a linear behavior with respect to x in connection with their essential singularities. Therefore, in Theorem 4.7, the Abel map does not provide the proper change of variables to linearize the divisor $\mathcal{D}_{\underline{\hat{\mu}}(x)}$ in the DB context, which is in sharp contrast to standard integrable soliton equations such as the Boussinesq hierarchy.

Remark 4.10 The fact that ψ in (3.1) differs from standard Baker–Akhiezer functions. Hence, one can not expect the usual theta function representation of ψ_j, $j = 1, 2, 3$, in terms of ratios of theta functions times an exponential term containing a meromorphic differential with a pole at the essential singularity of ψ_j multiplied by $(x - x_0)$. However, inserting (4.17) and (4.21) into $\psi_3 = \psi_{2,x} = \tilde{z}^{-1}\phi\psi_2$, and $\psi_1 = -m^{-1}\tilde{z}^2\psi_{3,x} = -m^{-1}\tilde{z}(\phi\psi_2)_x$, respectively, we obtain the theta function representations of ψ_1 and ψ_3.

5 The Time-Dependent DB Formalism

In this section, we extend the algebro-geometric analysis of Sect. 3 to the time-dependent DB hierarchy. We employ the notation $\widetilde{G}_j$, $\widetilde{V}$, $\widetilde{V}_{ij}$, etc., in order to distinguish them from G_j, V, V_{ij}, etc. In addition, we indicate that the individual pth DB flow by a separate time variable $t_p \in \mathbb{C}$. In analogy to (3.1), we introduce the time-dependent vector Baker–Akhiezer function $\psi = (\psi_1, \psi_2, \psi_3)^t$ by

$$\begin{aligned}
&\psi_x(P, x, x_0, t_p, t_{0,p}) = U(u(x, t_p), \tilde{z}(P))\psi(P, x, x_0, t_p, t_{0,p}),\\
&\psi_{t_p}(P, x, x_0, t_p, t_{0,p}) = \widetilde{V}(u(x, t_p), \tilde{z}(P))\psi(P, x, x_0, t_p, t_{0,p}),\\
&\tilde{z}V(u(x, t_p), \tilde{z}(P))\psi(P, x, x_0, t_p, t_{0,p}) = y(P)\psi(P, x, x_0, t_p, t_{0,p}),\\
&\psi_2(P, x_0, x_0, t_{0,p}, t_{0,p}) = 1, \quad x, t_p \in \mathbb{C},
\end{aligned} \tag{5.1}$$

where $\widetilde{V} = (\widetilde{V}_{ij})_{3\times 3}$, and

$$\widetilde{V}_{ij} = \sum_{l=0}^{p} \widetilde{V}_{ij}^{(l)}(\widetilde{G}_l)\tilde{z}^{4(p-l+1)} \quad i, j = 1, \ldots, 3, \quad l = 0, \ldots, p \tag{5.2}$$

with $\widetilde{V}_{ij}^{(l)}(\widetilde{G}_l)$ determined by $\widetilde{G}_l$, which is defined in (2.6) by substituting $\widetilde{G}_l$ for G_l.

The compatibility conditions of the first three expressions in (5.1) yield that

$$\begin{aligned} &U_{t_p}(\tilde{z}) - \widetilde{V}_x(\tilde{z}) + [U(\tilde{z}), \widetilde{V}(\tilde{z})] = 0, \\ &-V_x(\tilde{z}) + [U(\tilde{z}), V(\tilde{z})] = 0, \\ &-V_{t_p}(\tilde{z}) + [\widetilde{V}(\tilde{z}), V(\tilde{z})] = 0. \end{aligned} \tag{5.3}$$

A direct calculation shows that $yI - \tilde{z}V(\tilde{z})$ satisfies the last two equations in (5.3). Then the characteristic polynomial of Lax matrix $\tilde{z}V(\tilde{z})$ for the DB hierarchy is an independent constant of variables x and t_p with the expansion

$$\det(yI - \tilde{z}V) = y^3 + yS_r(\tilde{z}) - T_r(\tilde{z}), \tag{5.4}$$

where $S_r(\tilde{z})$ and $T_r(\tilde{z})$ are defined as in (2.20) and (2.18). Then the time-dependent DB curve $\mathcal{K}_{r-1}$ is defined by

$$\mathcal{K}_{r-1} : \mathcal{F}_r(\tilde{z}, y) = y^3 + yS_r(\tilde{z}) - T_r(\tilde{z}) = 0. \tag{5.5}$$

In analogy to (3.2), we can define the following meromorphic function $\phi(P, x, t_p)$ on $\mathcal{K}_{r-1}$, the fundamental ingredient for the construction of algebro-geometric solutions of the time-dependent DB hierarchy,

$$\phi(P, x, t_p) = \tilde{z}\frac{\partial_x \psi_2(P, x, x_0, t_p, t_{0,p})}{\psi_2(P, x, x_0, t_p, t_{0,p})}, \quad P \in \mathcal{K}_{r-1}, \ x \in \mathbb{C}. \tag{5.6}$$

Using (5.1), a direct calculation shows that

$$\begin{aligned} \phi(P, x, t_p) &= \tilde{z}\frac{yV_{31}(\tilde{z}, x, t_p) + C_r(\tilde{z}, x, t_p)}{yV_{21}(\tilde{z}, x, t_p) + A_r(\tilde{z}, x, t_p)} \\ &= \tilde{z}\frac{F_r(\tilde{z}, x, t_p)}{y^2V_{31}(\tilde{z}, x, t_p) - yC_r(\tilde{z}, x, t_p) + D_r(\tilde{z}, x, t_p)} \\ &= \tilde{z}\frac{y^2V_{21}(\tilde{z}, x, t_p) - yA_r(\tilde{z}, x, t_p) + B_r(\tilde{z}, x, t_p)}{E_r(\tilde{z}, x, t_p)}, \end{aligned} \tag{5.7}$$

where $P = (\tilde{z}, y) \in \mathcal{K}_{r-1}$, $(x, t_p) \in \mathbb{C}^2$, and $A_r(\tilde{z}, x, t_p)$, $B_r(\tilde{z}, x, t_p)$, $C_r(\tilde{z}, x, t_p)$, $D_r(\tilde{z}, x, t_p)$, $E_r(\tilde{z}, x, t_p)$, and $F_r(\tilde{z}, x, t_p)$ are defined as in (3.5) and (3.6). Hence the interrelationships among them, (3.7)–(3.10), also hold in the time-dependent case.

Similarly, we denote by $\{\mu_j(x, t_p)\}_{j=1,\ldots,r-1}$ and $\{\nu_j(x, t_p)\}_{j=1,\ldots,r-1}$ the zeros of $E_r(\tilde{z}, x, t_p)$ and $\tilde{z}^2F_r(\tilde{z}, x, t_p)$, respectively. Thus, we may write

$$E_r(\tilde{z}, x, t_p) = E_{r,0}(x, t_p)\prod_{j=1}^{r-1}(\tilde{z} - \mu_j(x, t_p)), \tag{5.8}$$

$$F_r(\tilde{z}, x, t_p) = \tilde{z}^{-2}F_{r,0}(x, t_p)\prod_{j=1}^{r-1}(\tilde{z} - \nu_j(x, t_p)). \tag{5.9}$$

Defining

$$\hat{\mu}_j(x,t_p) = \Big(\mu_j(x,t_p), y(\hat{\mu}_j(x,t_p))\Big) = \left(\mu_j(x,t_p), -\frac{A_r(\mu_j,x,t_p)}{V_{21}(\mu_j,x,t_p)}\right) \in \mathcal{K}_{r-1},$$
$$j = 1,\ldots,r-1,\ (x,t_p) \in \mathbb{C}^2, \tag{5.10}$$
$$\hat{\nu}_j(x,t_p) = \Big(\nu_j(x,t_p), y(\hat{\nu}_j(x,t_p))\Big) = \left(\nu_j(x,t_p), -\frac{C_r(\nu_j,x,t_p)}{V_{31}(\nu_j,x,t_p)}\right) \in \mathcal{K}_{r-1},$$
$$j = 1,\ldots,r-1,\ (x,t_p) \in \mathbb{C}^2. \tag{5.11}$$

One infers from (5.7) that the divisor $(\phi(P,x,t_p))$ of $\phi(P,x,t_p)$ is given by

$$(\phi(P,x,t_p)) = \mathcal{D}_{P_0,\underline{\hat{\nu}}(x,t_p)}(P) - \mathcal{D}_{P_\infty,\underline{\hat{\mu}}(x,t_p)}(P), \tag{5.12}$$

where

$$\underline{\hat{\nu}}(x,t_p) = \{\hat{\nu}_1(x,t_p),\ldots,\hat{\nu}_{r-1}(x,t_p)\}, \quad \underline{\hat{\mu}}(x,t_p) = \{\hat{\mu}_1(x,t_p),\ldots,\hat{\mu}_{r-1}(x,t_p)\}.$$

Further properties of $\phi(P,x,t_p)$ are summarized as follows.

Theorem 5.1 *Assume* (5.1), (5.6)*,* $P = (\tilde{z}, y) \in \mathcal{K}_{r-1} \backslash \{P_\infty, P_0\}$*, and let* $(\tilde{z}, x, t_p) \in \mathbb{C}^3$*. Then*

$$\phi_{xx}(P,x,t_p) + 3\tilde{z}^{-1}\phi(P,x,t_p)\phi_x(P,x,t_p) + \tilde{z}^{-2}\phi^3(P,x,t_p) + m(x,t_p)\tilde{z}^{-1}$$
$$-\frac{m_x(x,t_p)}{m(x,t_p)}\phi_x(P,x,t_p) - \tilde{z}^{-1}\frac{m_x(x,t_p)}{m(x,t_p)}\phi^2(P,x,t_p) = 0, \tag{5.13}$$

$$\phi_{t_p}(P,x,t_p) = \tilde{z}\partial_x\bigg(\frac{\widetilde{V}_{21}(\tilde{z},x,t_p)}{m(x,t_p)}(-\tilde{z}\phi_x(P,x,t_p) - \phi^2(P,x,t_p))$$
$$+\widetilde{V}_{22}(\tilde{z},x,t_p) + \widetilde{V}_{23}(\tilde{z},x,t_p)\tilde{z}^{-1}\phi(P,x,t_p)\bigg), \tag{5.14}$$

$$\phi(P,x,t_p)\phi(P^*,x,t_p)\phi(P^{**},x,t_p) = -\tilde{z}^3\frac{F_r(\tilde{z},x,t_p)}{E_r(\tilde{z},x,t_p)}, \tag{5.15}$$

$$\phi(P,x,t_p) + \phi(P^*,x,t_p) + \phi(P^{**},x,t_p) = \tilde{z}\frac{E_{r,x}(\tilde{z},x,t_p)}{E_r(\tilde{z},x,t_p)}, \tag{5.16}$$

$$\frac{1}{\phi(P,x,t_p)} + \frac{1}{\phi(P^*,x,t_p)} + \frac{1}{\phi(P^{**},x,t_p)} = -\frac{3V_{33}(\tilde{z},x,t_p)}{\tilde{z}V_{32}(\tilde{z},x,t_p)}$$
$$+\frac{\tilde{z}V_{31}(\tilde{z},x,t_p)F_{r,x}(\tilde{z},x,t_p)}{m(x,t_p)V_{32}(\tilde{z},x,t_p)F_r(\tilde{z},x,t_p)}, \tag{5.17}$$

$$y(P)\phi(P,x,t_p) + y(P^*)\phi(P^*,x,t_p) + y(P^{**})\phi(P^{**},x,t_p) =$$
$$\tilde{z}\frac{3T_r(\tilde{z})V_{21}(\tilde{z},x,t_p) + 2S_r(\tilde{z})A_r(\tilde{z},x,t_p)}{E_r(\tilde{z},x,t_p)}. \tag{5.18}$$

Proof Equation (5.13) follows from (5.1) and (5.7). Relation (5.14) is clear from (5.1) and (5.6). Moreover, (5.15)–(5.18) can be derived as in Theorem 3.2. □

Next, we consider the t_p-dependence of E_r and F_r.

Lemma 5.2 *Assume* (5.1) and (5.3) *and let* $(\tilde{z}, x, t_p) \in \mathbb{C}^3$. *Then*

$$E_{r,t_p}(\tilde{z}, x, t_p) = E_{r,x}(\tilde{z}, x, t_p)\Big(\widetilde{V}_{23} - \frac{\widetilde{V}_{21}}{V_{21}} V_{23}\Big) + E_r(\tilde{z}, x, t_p) 3\Big(\widetilde{V}_{22} - \frac{\widetilde{V}_{21}}{V_{21}} V_{22}\Big), \tag{5.19}$$

$$\begin{aligned} F_{r,t_p}(\tilde{z}, x, t_p) = {} & F_{r,x}(\tilde{z}, x, t_p)\frac{\tilde{z}^2}{m}\Big(-\widetilde{V}_{31} + \frac{V_{31}}{V_{32}}\widetilde{V}_{32}\Big) + 3F_r(\tilde{z}, x, t_p) \\ & \times \Big(\widetilde{V}_{22} + \widetilde{V}_{23,x} - \frac{V_{33}}{V_{32}}\widetilde{V}_{32}\Big). \end{aligned} \tag{5.20}$$

Proof Differentiating (5.16) with respect to t_p, one infers that

$$\begin{aligned} \partial_{t_p}(\phi(P, x, t_p) + \phi(P^*, x, t_p) + \phi(P^{**}, x, t_p)) &= \partial_{t_p}\Big(\tilde{z}\frac{E_{r,x}(\tilde{z}, x, t_p)}{E_r(\tilde{z}, x, t_p)}\Big) \\ &= \tilde{z}\partial_{t_p}\partial_x(\ln E_r(\tilde{z}, x, t_p)) = \tilde{z}\partial_x\partial_{t_p}(\ln E_r(\tilde{z}, x, t_p)). \end{aligned} \tag{5.21}$$

Without loss of generality, taking the integration constants as zero and using (5.14) and (5.16), one finds

$$\begin{aligned} & \partial_{t_p}(\ln E_r(\tilde{z}, x, t_p)) \\ &= -\frac{\widetilde{V}_{21}}{m}\tilde{z}(\phi_x(P, x, t_p) + \phi_x(P^*, x, t_p) + \phi_x(P^{**}, x, t_p)) \\ &\quad - \frac{\widetilde{V}_{21}}{m}(\phi^2(P, x, t_p) + \phi^2(P^*, x, t_p) + \phi^2(P^{**}, x, t_p)) \\ &\quad + \tilde{z}^{-1}\widetilde{V}_{23}(\phi(P, x, t_p) + \phi(P^*, x, t_p) + \phi(P^{**}, x, t_p)) + 3\widetilde{V}_{22} \\ &= \tilde{z}^{-1}\Big(\widetilde{V}_{23} - \frac{\widetilde{V}_{21}}{V_{21}}V_{23}\Big)(\phi(P) + \phi(P^*) + \phi(P^{**})) + 3\widetilde{V}_{22} - 3\frac{\widetilde{V}_{21}}{V_{21}}V_{22} \\ &\quad \times\Big(\widetilde{V}_{23} - \frac{\widetilde{V}_{21}}{V_{21}}V_{23}\Big)\Big(\frac{E_{r,x}}{E_r}\Big) + 3\widetilde{V}_{22} - 3\frac{\widetilde{V}_{21}}{V_{21}}V_{22}, \end{aligned} \tag{5.22}$$

which implies (5.19). Then taking into account (5.3), (5.13), (5.15), and (5.17), one obtains

$$\begin{aligned} \partial_{t_p}\Big(-\tilde{z}^3\frac{F_r(\tilde{z}, x, t_p)}{E_r(\tilde{z}, x, t_p)}\Big) &= -\tilde{z}^3\frac{F_r}{E_r}\Big[-\frac{\tilde{z}^2}{m}\widetilde{V}_{31}\Big(\frac{F_{r,x}}{F_r} - \frac{E_{r,x}}{E_r}\Big) - \frac{\tilde{z}^2\widetilde{V}_{21,x}}{m}\frac{E_{r,x}}{E_r} \\ &\quad + 3\frac{\widetilde{V}_{21}}{V_{21}}V_{22} + \frac{\widetilde{V}_{21}}{V_{21}}V_{23}\frac{E_{r,x}}{E_r} + \Big(\widetilde{V}_{21} + \widetilde{V}_{22,x}\Big)\Big(-3\frac{V_{33}}{V_{32}} + \frac{\tilde{z}^2 V_{31}F_{r,x}}{mV_{32}F_r}\Big) + 3\widetilde{V}_{23,x}\Big], \end{aligned} \tag{5.23}$$

which implies that

$$\frac{F_{r,t_p}}{E_r} - \frac{F_r E_{r,t_p}}{E_r^2} = \frac{F_{r,x}}{E_r}\Big(-\frac{\tilde{z}^2}{m}\widetilde{V}_{31} + \frac{\tilde{z}^2 V_{31}}{mV_{32}}(\widetilde{V}_{21} + \widetilde{V}_{22,x})\Big) + \frac{E_{r,x}F_r}{E_r^2}\Big(\frac{\tilde{z}^2}{m}\widetilde{V}_{31} - \frac{\tilde{z}^2\widetilde{V}_{21,x}}{m} + \frac{\widetilde{V}_{21}}{V_{21}}V_{23}\Big) + \frac{F_r}{E_r}\Big(3\frac{V_{22}}{V_{21}}\widetilde{V}_{21} + 3\widetilde{V}_{23,x} - \frac{3V_{33}}{V_{32}}(\widetilde{V}_{21} + \widetilde{V}_{22,x})\Big). \quad (5.24)$$

Then combining (5.19) and (5.24) readily leads to (5.20). □

The properties of $\psi_2(P, x, x_0, t_p, t_{0,p})$ are summarized as follows.

Theorem 5.3 *Assume* (5.1), (5.6), *and* $P = (\tilde{z}, y) \in \mathcal{K}_{r-1}\backslash\{P_\infty, P_0\}$, *and let* $(\tilde{z}, x, x_0, t_p, t_{0,p}) \in \mathbb{C}^5$. *Then*

$$\psi_{2,t_p}(P, x, x_0, t_p, t_{0,p}) = \Big(\frac{\widetilde{V}_{21}(\tilde{z}, x, t_p)}{m(x, t_p)}(-\tilde{z}\phi_x(P, x, t_p) - \phi^2(P, x, t_p)) + \widetilde{V}_{22}(\tilde{z}, x, t_p) + \widetilde{V}_{23}(\tilde{z}, x, t_p)\tilde{z}^{-1}\phi(P, x, t_p)\Big)\psi_2(P, x, x_0, t_p, t_{0,p}), \quad (5.25)$$

$$\psi_2(P, x, x_0, t_p, t_{0,p}) = \exp\Bigg(\tilde{z}^{-1}\int_{x_0}^{x}\phi(P, x', t_p)\mathrm{d}x' + \int_{t_{0,p}}^{t_p}\Bigg[\frac{\tilde{z}^{-1}y(P) - V_{22}(\tilde{z}, x_0, t')}{V_{21}(\tilde{z}, x_0, t')} \times \widetilde{V}_{21}(\tilde{z}, x_0, t') + \Big(\widetilde{V}_{23}(\tilde{z}, x_0, t') - \frac{\widetilde{V}_{21}(\tilde{z}, x_0, t')}{V_{21}(\tilde{z}, x_0, t')}V_{23}(\tilde{z}, x_0, t')\Big) \times \tilde{z}^{-1}\phi(P, x_0, t') + \widetilde{V}_{22}(\tilde{z}, x_0, t')\Bigg]\mathrm{d}t'\Bigg), \quad (5.26)$$

Proof Relation (5.25) follows from (5.1) and (5.6). Equation (5.26) is clear from (5.14) and (5.25). □

In analogy to Lemma 3.3, the dynamics of $\mu_j(x, t_p)$ and $\nu_j(x, t_p)$ with respect to variations of x and t_p are described in terms of the following Dubrovin-type equations.

Lemma 5.4 *Assume* (5.1)–(5.7).

(i) *Suppose the zeros* $\{\mu_j(x, t_p)\}_{j=1,\ldots,r-1}$ *of* $E_r(\tilde{z}, x, t_p)$ *remain distinct for* $(x, t_p) \in \Omega_\mu$, *where* $\Omega_\mu \subseteq \mathbb{C}^2$ *is open and connected. Then* $\{\mu_j(x, t_p)\}_{j=1,\ldots,r-1}$ *satisfy the system of differential equations,*

$$\mu_{j,x}(x, t_p) = -\frac{[S_r(\mu_j(x, t_p)) + 3y(\hat{\mu}_j(x, t_p))^2]V_{21}(\mu_j(x, t_p), x, t_p)}{E_{r,0}\prod_{\substack{k=1\\k\neq j}}^{r-1}(\mu_j(x, t_p) - \mu_k(x, t_p))}, \quad j = 1, \ldots, r-1, \quad (5.27)$$

$$\mu_{j,t_p}(x,t_p) = -[V_{21}(\mu_j(x,t_p),x,t_p)\widetilde{V}_{23}(\mu_j(x,t_p),x,t_p) - \widetilde{V}_{21}(\mu_j(x,t_p),x,t_p) \times V_{23}(\mu_j(x,t_p),x,t_p)] \frac{[S_r(\mu_j(x,t_p)) + 3y(\hat{\mu}_j(x,t_p))^2]}{E_{r,0}\prod_{\substack{k=1\\k\neq j}}^{r-1}(\mu_j(x,t_p)-\mu_k(x,t_p))},$$
$$j = 1,\ldots,r-1, \tag{5.28}$$

with initial conditions

$$\{\hat{\mu}_j(x_0,t_{0,p})\}_{j=1,\ldots,r-1} \in \mathcal{K}_{r-1} \tag{5.29}$$

for some fixed $(x_0,t_{0,p}) \in \Omega_\mu$. *The initial value problem* (5.28), (5.29) *has a unique solution satisfying*

$$\hat{\mu}_j \in C^\infty(\Omega_\mu, \mathcal{K}_{r-1}), \quad j = 1,\ldots,r-1. \tag{5.30}$$

(ii) *Suppose the zeros* $\{\nu_j(x,t_p)\}_{j=1,\ldots,r-1}$ *of* $F_r(\tilde{z},x,t_p)$ *remain distinct for* $(x,t_p) \in \Omega_\nu$, *where* $\Omega_\nu \subseteq \mathbb{C}^2$ *is open and connected. Then* $\{\nu_j(x,t_p)\}_{j=1,\ldots,r-1}$ *satisfy the system of differential equations,*

$$\nu_{j,x}(x,t_p) = -\frac{[S_r(\nu_j(x,t_p)) + 3y(\hat{\nu}_j(x,t_p))^2]m(x,t_p)V_{32}(\nu_j(x,t_p),x,t_p)}{F_{r,0}\prod_{\substack{k=1\\k\neq j}}^{r-1}(\nu_j(x,t_p)-\nu_k(x,t_p))},$$
$$j = 1,\ldots,r-1, \tag{5.31}$$

$$\nu_{j,t_p}(x,t_p) = -[V_{31}(\nu_j(x,t_p),x,t_p)\widetilde{V}_{32}(\nu_j(x,t_p),x,t_p) - V_{32}(\nu_j(x,t_p),x,t_p) \times \widetilde{V}_{31}(\nu_j(x,t_p),x,t_p)] \frac{[S_r(\nu_j(x,t_p)) + 3y(\hat{\nu}_j(x,t_p))^2]\nu_j(x,t_p)^2}{F_{r,0}\prod_{\substack{k=1\\k\neq j}}^{r-1}(\nu_j(x,t_p)-\nu_k(x,t_p))},$$
$$j = 1,\ldots,r-1, \tag{5.32}$$

with initial conditions

$$\{\hat{\nu}_j(x_0,t_{0,p})\}_{j=1,\ldots,r-1} \in \mathcal{K}_{r-1} \tag{5.33}$$

for some fixed $(x_0,t_{0,p}) \in \Omega_\nu$. *The initial value problem* (5.32), (5.33) *has a unique solution satisfying*

$$\hat{\nu}_j \in C^\infty(\Omega_\nu, \mathcal{K}_{r-1}), \quad j = 1,\ldots,r-1. \tag{5.34}$$

Proof For obvious reasons it suffices to focus on (5.27) and (5.28). But the proof of (5.27) is identical to that in Lemma 3.3. We now prove (5.28). From (5.8), we have

$$E_{r,t_p}(\tilde{z},x,t_p)|_{\tilde{z}=\mu_j(x,t_p)} = -E_{r,0}\mu_{j,t_p}(x,t_p)\prod_{\substack{k=1\\k\neq j}}^{r-1}(\mu_j(x,t_p)-\mu_k(x,t_p)). \tag{5.35}$$

On the other hand, using (5.19) and (5.27), one computes

$$E_{r,t_p}(\tilde{z},x,t_p)|_{\tilde{z}=\mu_j(x,t_p)} = E_{r,x}(\mu_j(x,t_p),x,t_p)\Big(\widetilde{V}_{23}-\frac{\widetilde{V}_{21}}{V_{21}}V_{23}\Big)$$
$$= [S_r(\mu_j(x,t_p))+3y(\hat{\mu}_j(x,t_p))^2](V_{21}\widetilde{V}_{23}-\widetilde{V}_{21}V_{23}), \tag{5.36}$$

which together with (5.35) yields (5.28). □

6 Time-Dependent Algebro-geometric Solutions

In our final section, we extend the results of Sect. 4 from the stationary DB hierarchy to the time-dependent case. We obtain Riemann theta function representations for the Baker–Akhiezer function ψ_2, the meromorphic function ϕ, and especially, for the algebro-geometric solutions u of the whole DB hierarchy.

We start with the theta function representation of the meromorphic function $\phi(P,x,t_p)$.

Theorem 6.1 *Assume that the curve $\mathcal{K}_{r-1}$ is nonsingular. Let $P=(\tilde{z},y)\in\mathcal{K}_{r-1}\backslash\{P_\infty,P_0\}$ and let $(x,t_p),(x_0,t_{0,p})\in\Omega_\mu$, where $\Omega_\mu\subseteq\mathbb{C}^2$ is open and connected. Suppose that $\mathcal{D}_{\underline{\hat{\mu}}(x,t_p)}$, or equivalently $\mathcal{D}_{\underline{\hat{\nu}}(x,t_p)}$, is nonspecial for $(x,t_p)\in\Omega_\mu$. Then*

$$\phi(P,x,t_p) = -m^{\frac{1}{3}}(x,t_p)\frac{\theta(\underline{\tilde{z}}(P,\underline{\hat{\nu}}(x,t_p)))\theta(\underline{\tilde{z}}(P_0,\underline{\hat{\mu}}(x,t_p)))}{\theta(\underline{\tilde{z}}(P_0,\underline{\hat{\nu}}(x,t_p)))\theta(\underline{\tilde{z}}(P,\underline{\hat{\mu}}(x,t_p)))}$$
$$\times\exp\bigg(e^{(3)}(Q_0)-\int_{Q_0}^{P}\omega^{(3)}_{P_\infty P_0}\bigg). \tag{6.1}$$

Since the proof of Theorem 6.1 is identical to the corresponding stationary results in Theorem 4.3, we omit the corresponding details.

Motivated by (5.25), we define the meromorphic function $I_s(P,x,t_p)$ on $\mathcal{K}_{r-1}\times\mathbb{C}^2$ by

$$I_s(P,x,t_p)=\frac{\widetilde{V}_{21}}{m}(-\tilde{z}\phi_x-\phi^2)+\widetilde{V}_{22}+\widetilde{V}_{23}\tilde{z}^{-1}\phi. \tag{6.2}$$

The asymptotic properties of $I_s(P,s,t_p)$ are summarized as follows.

Theorem 6.2 *Let $s=12p-2$, $p\in\mathbb{N}_0$, $(x,t_p)\in\mathbb{C}^2$. Then*[4]

$$I_s(P,x,t_p)\underset{\zeta\to 0}{=}\sum_{j=1}^{\frac{s}{2}}\tilde{\alpha}_j\zeta^{-s+2j-2}+O(1),\quad \zeta=\tilde{z}^{-1/3},\quad \text{as } P\to P_\infty, \tag{6.3}$$

[4] Here sums with upper limits strictly less than their lower limits are interpreted as zero.

$$I_{-2}(P,x,t_0) \underset{\zeta\to 0}{=} -u(x,t_0)u_x(x,t_0)\zeta^2 \underset{\zeta\to 0}{=} O(\zeta^2), \quad \zeta=\tilde{z}^{-1/3}, \quad as\ P\to P_\infty,$$
$$I_{-2}(P,x,t_0) \underset{\zeta\to 0}{=} u(x,t_0)m^{1/3}(x,t_0)\zeta^{-2}+O(\zeta^2), \quad \zeta=\tilde{z}^{1/3}, \quad as\ P\to P_0, \tag{6.4}$$

where $\{\tilde{\alpha}_j\}_{j=1,\dots,\frac{s}{2}}\in\mathbb{C}$.

Proof Treating t_p as a parameter, we note that the asymptotic expansions of $\phi(P)$ near P_∞ and near P_0 in (4.1) and (4.6) still apply in the present time-dependent context. In terms of local coordinate $\zeta=\tilde{z}^{-1/3}$ near P_∞, recall the definitions of $\widetilde{V}_{21}$, $\widetilde{V}_{22}$, $\widetilde{V}_{23}$ in (5.2), we may write

$$\widetilde{V}_{21}=\sum_{j=1}^{\infty}V_{21}^{\left([\frac{j+1}{2}],j+1-2[\frac{j+1}{2}]\right)}\zeta^{-(12p-6j)},$$
$$\widetilde{V}_{22}=\sum_{j=1}^{\infty}V_{22}^{\left([\frac{j+1}{2}],j+1-2[\frac{j+1}{2}]\right)}\zeta^{-(12p-6j)},$$
$$\widetilde{V}_{23}=\sum_{j=0}^{\infty}V_{23}^{\left([\frac{j+1}{2}],j+1-2[\frac{j+1}{2}]\right)}\zeta^{-(12p-6j)},$$

where

$$V_{21}^{(\beta_1,\beta_2)}=V_{23}^{(\beta_1,\beta_2)}=0 \quad \text{for} \quad \beta_1\geq p+1, \quad \beta_1,\beta_2\in\mathbb{N},$$
$$V_{22}^{(\beta_1,\beta_2)}=0 \quad \text{for} \quad \beta_1\geq p+1 \quad \text{or} \quad \beta_2=1, \quad \beta_1,\beta_2\in\mathbb{N}$$

and the function $[\,\cdot\,]$ returns the value of a number rounded downward to the nearest integer. Moreover, from (4.1), we find

$$-\tilde{z}\phi_x(P,x,t_p)-\phi^2(P,x,t_p)$$
$$\underset{\zeta\to 0}{=}\sum_{j=-2}^{\infty}\vartheta_{2j}\zeta^{2j} \underset{\zeta\to 0}{=}\sum_{j=-2}^{\infty}\left(\vartheta_{2j}\zeta^{2j}+\vartheta_{2j+1}\zeta^{2j+1}\right), \quad \text{as } P\to P_\infty,$$

with

$$\vartheta_{-4}=-\kappa_{0,x}=-u_{xx}, \quad \vartheta_{-3}=0,$$
$$\vartheta_{2j}=-\kappa_{2j+4,x}-\sum_{i=0}^{2j+2}\kappa_i\kappa_{2j+2-i}, \quad \vartheta_{2j+1}=0, \quad j\geq -1.$$

Therefore, in terms of local coordinate $\zeta=\tilde{z}^{-1/3}$ near P_∞, we obtain

$$I_s(P,x,t_p)=\sum_{j=1}^{6p-1}\chi_j\zeta^{-12p+2j}+\chi_{6p}+\sum_{j=6p+1}^{\infty}\chi_j\zeta^{-12p+2j}, \tag{6.5}$$

where

$$\chi_j = m^{-1} \sum_{k=-2}^{j-3} \vartheta_{2k} V_{21}^{\left(\left[\frac{j-k+3}{6}\right], \frac{j-k+3}{3}-2\left[\frac{j-k+3}{6}\right]\right)} + V_{22}^{\left(\left[\frac{j+3}{6}\right], \frac{j+3}{3}-2\left[\frac{j+3}{6}\right]\right)}$$
$$+ \sum_{\ell=0}^{2j-2} \kappa_\ell V_{23}^{\left(\left[\frac{2j-\ell+4}{12}\right], \frac{2j-\ell+4}{6}-2\left[\frac{2j-\ell+4}{12}\right]\right)} \quad \text{for} \quad 3 \leq j \leq 6p, \quad j \in \mathbb{N}_0,$$
$$\chi_j = m^{-1} \sum_{k=-2}^{j-3} \vartheta_{2k} V_{21}^{\left(\left[\frac{j-k+3}{6}\right], \frac{j-k+3}{3}-2\left[\frac{j-k+3}{6}\right]\right)} + \sum_{\ell=0}^{2j-2} \kappa_\ell V_{23}^{\left(\left[\frac{2j-\ell+4}{12}\right], \frac{2j-\ell+4}{6}-2\left[\frac{2j-\ell+4}{12}\right]\right)}$$
$$\text{for} \quad j = 1, 2, \quad \text{and} \quad j \geq 6p+1, \quad j \in \mathbb{N}_0.$$

Then inserting (6.5) into (5.14) and comparing the coefficients of the same powers of ζ^ℓ $(\ell < 0)$ yields

$$\chi_{j,x} = 0 \quad \text{for } 1 \leq j \leq 6p, \quad j \in \mathbb{N}_0.$$

Hence, we conclude that

$$\chi_1 = \gamma_1(t_p), \quad \chi_2 = \gamma_2(t_p), \quad \dots \quad \chi_{6p} = \gamma_{6p}(t_p),$$

where $\gamma_j(t_p)$ $(j = 1, 2, \dots)$ are integration constants. Next we note that the coefficients κ_j $(j = 0, 1, \dots)$ of the power series for $\phi(P, x, t_p)$ in the coordinate ζ near P_∞ are the ratios of two functions closely related to u. Meanwhile, the coefficients of the homogeneous polynomials $\widetilde{V}_{ij}$ $(i, j = 1, 2, 3)$ are differential polynomials in u. From these considerations it follows that $\gamma_j = \tilde{\alpha}_j \in \mathbb{C}$. Hence, we obtain (6.3). Finally, (6.4) follows from (4.1), (4.6), and (6.2). □

Let $\omega^{(2)}_{P_\infty,j}$, $j \in \mathbb{N}_0$, be the Abel differentials of the second kind normalized by the vanishing of all their a-periods,

$$\int_{a_k} \omega^{(2)}_{P_\infty,j} = 0, \quad k = 1, \dots, r-1$$

and holomorphic on $\mathcal{K}_{r-1} \backslash \{P_\infty\}$, with a pole of order j at P_∞,

$$\omega^{(2)}_{P_\infty,j}(P) \underset{\zeta \to 0}{=} (\zeta^{-j} + O(1)) d\zeta, \quad \text{as } P \to P_\infty. \tag{6.6}$$

Furthermore, define the normalized differential of the second kind by

$$\widetilde{\Omega}^{(2)}_{P_\infty,s+1} = -\sum_{j=1}^{\frac{s}{2}} (s - 2j + 2) \tilde{\alpha}_j \omega^{(2)}_{P_\infty,s-2j+3} \tag{6.7}$$

and

$$\widetilde{\Omega}^{(2)}_{P_0,3} = 2\omega^{(2)}_{P_0,3}, \tag{6.8}$$

where $s = 12p - 2$, $p \in \mathbb{N}_0$. Thus, one infers

$$\int_{a_k} \widetilde{\Omega}^{(2)}_{P_\infty,s+1} = 0, \qquad \int_{a_k} \widetilde{\Omega}^{(2)}_{P_0,3} = 0, \qquad k = 1, \ldots, r-1.$$

In addition, we define the vector of b-periods of the differential of the second kind $\widetilde{\Omega}^{(2)}_{P_\infty,s+1}$,

$$\underline{\widetilde{U}}^{(2)}_{s+1} = (\widetilde{U}^{(2)}_{s+1,1}, \ldots, \widetilde{U}^{(2)}_{s+1,r-1}), \quad \widetilde{U}^{(2)}_{s+1,j} = \frac{1}{2\pi i}\int_{b_j} \widetilde{\Omega}^{(2)}_{P_\infty,s+1}, \quad j = 1, \ldots, r-1 \tag{6.9}$$

with $s = 12p - 2$, $p \in \mathbb{N}_0$. Integrating (6.7) and (6.8) yields

$$\begin{aligned}
\int_{Q_0}^{P} \widetilde{\Omega}^{(2)}_{P_\infty,s+1} &\underset{\zeta\to 0}{=} -\sum_{j=1}^{\frac{s}{2}} (s-2j+2)\tilde{\alpha}_j \int_{\zeta_0}^{\zeta} \omega^{(2)}_{P_\infty,s-2j+3} \\
&\underset{\zeta\to 0}{=} -\sum_{j=1}^{\frac{s}{2}} (s-2j+2)\tilde{\alpha}_j \int_{\zeta_0}^{\zeta} \frac{1}{\zeta^{s-2j+3}} d\zeta + O(1) \\
&\underset{\zeta\to 0}{=} \sum_{j=1}^{\frac{s}{2}} \tilde{\alpha}_j \frac{1}{\zeta^{s-2j+2}} + O(1), \qquad \text{as } P \to P_\infty,
\end{aligned} \tag{6.10}$$

and

$$\int_{Q_0}^{P} \widetilde{\Omega}^{(2)}_{P_0,3} \underset{\zeta\to 0}{=} -\zeta^{-2} + \tilde{e}^{(2)}_3(Q_0) + O(\zeta), \quad \text{as } P \to P_0, \tag{6.11}$$

where $\tilde{e}^{(2)}_3(Q_0)$ is a constant that arises from evaluating the integral at its lowers limits Q_0. Combining (6.3), (6.4), (6.10), and (6.11) yields

$$\int_{t_{0,p}}^{t_p} I_s(P,x,\tau) d\tau \underset{\zeta\to 0}{=} (t_p - t_{0,p}) \int_{Q_0}^{P} \widetilde{\Omega}^{(2)}_{P_\infty,s+1} + O(\zeta), \quad \text{as } P \to P_\infty, \tag{6.12}$$

and

$$\begin{aligned}
\int_{t_{0,0}}^{t_0} I_{-2}(P,x,\tau) d\tau \underset{\zeta\to 0}{=} \int_{t_{0,0}}^{t_0} \left(u(x,\tau) m^{\frac{1}{3}}(x,\tau) \left(\tilde{e}^{(2)}_3(Q_0) - \int_{Q_0}^{P} \widetilde{\Omega}^{(2)}_{P_0,3} \right) \right) d\tau \\
+ O(\zeta), \quad \text{as } P \to P_0.
\end{aligned} \tag{6.13}$$

Given these preparations, the theta function representation of $\psi_2(P, x, x_0, t_p, t_{0,p})$ reads as follows.

Theorem 6.3 *Assume that the curve $\mathcal{K}_{r-1}$ is nonsingular. Let $P = (\tilde{z}, y) \in \mathcal{K}_{r-1} \backslash \{P_\infty, P_0\}$ and let $(x, t_p), (x_0, t_{0,p}) \in \Omega_\mu$, where $\Omega_\mu \subseteq \mathbb{C}^2$ is open and connected. Suppose that $\mathcal{D}_{\underline{\hat{\mu}}(x,t_p)}$, or equivalently $\mathcal{D}_{\underline{\hat{\nu}}(x,t_p)}$, is nonspecial for $(x, t_p) \in \Omega_\mu$. Then for $p = 0$*

$$\begin{aligned}
&\psi_2(P, x, x_0, t_0, t_{0,0}) \\
&\quad = \frac{\theta\left(\underline{\tilde{z}}(P, \underline{\hat{\mu}}(x, t_0))\right)\theta(\underline{\tilde{z}}(P_0, \underline{\hat{\mu}}(x_0, t_{0,0})))}{\theta(\underline{\tilde{z}}(P_0, \underline{\hat{\mu}}(x, t_0)))\theta(\underline{\tilde{z}}(P, \underline{\hat{\mu}}(x_0, t_{0,0})))} \\
&\qquad \times \exp\left(\int_{x_0}^{x} 2m^{1/3}(x', t_0)\mathrm{d}x' \left(\int_{Q_0}^{P} \omega_{P_0,3}^{(2)} - e_3^{(2)}(Q_0)\right)\right) \\
&\qquad \times \exp\left(\int_{t_{0,0}}^{t_0} u(x_0, \tau)m^{1/3}(x_0, \tau)\left(\tilde{e}_3^{(2)}(Q_0) - \int_{Q_0}^{P} \widetilde{\Omega}_{P_0,3}^{(2)}\right) d\tau\right),
\end{aligned} \tag{6.14}$$

and for $p > 0$

$$\begin{aligned}
\psi_2(P, x, x_0, t_p, t_{0,p}) &= \frac{\theta\left(\underline{\tilde{z}}(P, \underline{\hat{\mu}}(x, t_p))\right)\theta(\underline{\tilde{z}}(P_0, \underline{\hat{\mu}}(x_0, t_{0,p})))}{\theta(\underline{\tilde{z}}(P_0, \underline{\hat{\mu}}(x, t_p)))\theta(\underline{\tilde{z}}(P, \underline{\hat{\mu}}(x_0, t_{0,p})))} \\
&\quad \times \exp\left(\int_{x_0}^{x} 2m^{1/3}(x', t_p)\mathrm{d}x' \left(\int_{Q_0}^{P} \omega_{P_0,3}^{(2)} - e_3^{(2)}(Q_0)\right)\right) \\
&\quad \times \exp\left((t_p - t_{0,p}) \int_{Q_0}^{P} \widetilde{\Omega}_{P_\infty,s+1}^{(2)}\right).
\end{aligned} \tag{6.15}$$

The straightening out of the DB flows by the Abel map is contained in our next result.

Theorem 6.4 *Assume that the curve $\mathcal{K}_{r-1}$ is nonsingular, and let $(x, t_p), (x_0, t_{0,p}) \in \mathbb{C}^2$. Then for $p > 0$,*

$$\underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\mu}}(x,t_p)}) = \underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\mu}}(x_0,t_{0,p})}) - \left(\int_{x_0}^{x} 2m^{\frac{1}{3}}(x', t_p)\mathrm{d}x'\right)\underline{\hat{U}}_3^{(2)} - \underline{\widetilde{U}}_{s+1}^{(2)}(t_p - t_{0,p}), \tag{6.16}$$

$$\underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\nu}}(x,t_p)}) = \underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\nu}}(x_0,t_{0,p})}) - \left(\int_{x_0}^{x} 2m^{\frac{1}{3}}(x', t_p)\mathrm{d}x'\right)\underline{\hat{U}}_3^{(2)} - \underline{\widetilde{U}}_{s+1}^{(2)}(t_p - t_{0,p}), \tag{6.17}$$

and for $p = 0$,

$$\begin{aligned}
\underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\mu}}(x,t_0)}) &= \underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\mu}}(x_0,t_{0,0})}) - \left(\int_{x_0}^{x} 2m^{\frac{1}{3}}(x', t_0)\mathrm{d}x'\right)\underline{\hat{U}}_3^{(2)} \\
&\quad + \left(\int_{t_{0,0}}^{t_0} 2u(x_0, \tau)m^{\frac{1}{3}}(x_0, \tau)\mathrm{d}\tau\right)\underline{\hat{U}}_3^{(2)},
\end{aligned} \tag{6.18}$$

$$\underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\nu}}(x,t_0)}) = \underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\nu}}(x_0,t_{0,0})}) - \left(\int_{x_0}^{x} 2m^{\frac{1}{3}}(x',t_0)\mathrm{d}x'\right)\underline{\hat{U}}_3^{(2)} + \left(\int_{t_{0,0}}^{t_0} 2u(x_0,\tau)m^{\frac{1}{3}}(x_0,\tau)\mathrm{d}\tau\right)\underline{\hat{U}}_3^{(2)}. \tag{6.19}$$

Proof The proof is analogous to the Boussinesq case in Dickson et al. (1999b). □

Our main result, the theta function representation of time-dependent algebro-geometric solutions for the DB hierarchy, now quickly follows from the materials prepared above.

Theorem 6.5 *Assume that u satisfies the p-th DB equation (2.14), that is,* $DB_p(u) = m_{t_p} - X_p = 0$*, and the curve* $\mathcal{K}_{r-1}$ *is nonsingular. Let* $(x,t_p) \in \Omega_\mu$*, where* $\Omega_\mu \subseteq \mathbb{C}^2$ *is open and connected. Suppose also that* $\mathcal{D}_{\underline{\hat{\mu}}(x,t_p)}$*, or equivalently* $\mathcal{D}_{\underline{\hat{\nu}}(x,t_p)}$*, is nonspecial for* $(x,t_p) \in \Omega_\mu$*. Then*

$$u(x,t_p) = u(x_0,t_{0,p})\frac{\theta(\underline{\tilde{z}}(P_0,\underline{\hat{\mu}}(x_0,t_{0,p})))\theta(\underline{\tilde{z}}(P_\infty,\underline{\hat{\mu}}(x,t_p)))}{\theta(\underline{\tilde{z}}(P_\infty,\underline{\hat{\mu}}(x_0,t_{0,p})))\theta(\underline{\tilde{z}}(P_0,\underline{\hat{\mu}}(x,t_p)))}. \tag{6.20}$$

Proof The proof is analogous to the stationary case in Theorem 4.8. □

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