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On different integrable systems sharing the same nondynamical r-matrix

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In a recent paper [Zhijun Qiao and Ruguang Zhou, Phys. Lett. A **235**, 35 (1997)], the amazing fact was reported that a discrete and a continuous integrable system share the same r-matrix with the interesting property of being nondynamical. Now, we present three further pairs of different continuous integrable systems sharing the same r-matrix again being nondynamical. The first pair is the finite-dimensional constrained system (FDCS) of the famous AKNS hierarchy and the Dirac hierarchy; the second pair is the FDCS of the well-known geodesic flows on the ellipsoid and the Heisenberg spin chain hierarchy; and the third pair is the FDCS of one hierarchy studied by Xianguo Geng [Phys. Lett. A **162**, 375 (1992)] and another hierarchy proposed by Zhijun Qiao [Phys. Lett. A **192**, 316 (1994)]. All those FDCSs possess Lax representations and from the viewpoint of r-matrix can be shown to be completely integrable in Liouville's sense. © *1998 American Institute of Physics.* [S0022-2488(98)02506-7]

I. INTRODUCTION

The well-known nonlinearization method[1] is a powerful tool to generate completely integrable finite-dimensional Hamiltonian systems (CIFDHSs) from the eigenvalue problems or Lax pairs of nonlinear evolution equations (NLEEs). With the help of this method many new CIFDHSs have been successively found, and solutions to NLEEs have been given by involutive or parametric representations.[2–12] The r-matrix[13] method is quite an effective approach to the Lie–Poisson structure in $1+1$-dimensional space,[14,15] providing fundamental commutator relations in the quantum inversing scattering.[16] Recently, the study of CIFDHSs admitting an r-matrix has already received attention.[17–22] It has been shown that r-matrices play a very important part in proving the integrability of CIFDHSs. Besides, the Lax matrix associated with the r-matrix of a CIFDHS can be applied for obtaining algebraic–geometric solutions of soliton equations or NLEEs.[23]

In a recent paper,[24] we reported an interesting and surprising result: a discrete and a continuous integrable system possess the same nondynamical r-matrix being independent from the dynamical variables p,q. The question of whether there are other pairs of continuous or discrete finite-dimensional integrable systems sharing the same r-matrix which should be nondynamical arises. In the present paper, on the basic idea of Ref. 24 we have considered a wide class of 2×2 spectral problems and finally succeeded in finding three other pairs of different continuous integrable systems with the common nondynamical r-matrix. The first pair is formed by the finite-dimensional constrained system (FDCS) of the famous AKNS and Dirac hierarchies; the second pair is formed by the FDCS of the well-known geodesic flow on the ellipsoid and the Heisenberg spin chain hierarchy; while the third pair consists of the FDCS of a hierarchy studied by Xianguo Geng[7] and a hierarchy proposed by Zhijun Qiao.[11] All of these FDCSs possess Lax representations, and by using the r-matrix approach can be shown to be completely integrable in Liouville's sense.

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Before displaying these results, let us first introduce some basic symbols and notations. Let $(R^{2N},dp\wedge dq)$ stand for the standard symplectic structure in the Euclidic space $R^{2N}=\{(p,q)|p=(p_1,\dots,p_N),\ q=(q_1,\dots,q_N)\}$, where p_i,q_i $(i=1,...,N)$ are N pairs of canonical coordinates. The standard inner product in R^N will be denoted by $\langle\cdot,\cdot\rangle$ and the Poisson bracket of two Hamiltonian functions F,G is given through[25]

$$\{F,G\}=\sum_{i=1}^{N}\left(\frac{\partial F}{\partial q_i}\frac{\partial G}{\partial p_i}-\frac{\partial F}{\partial p_i}\frac{\partial G}{\partial q_i}\right)=\left\langle\frac{\partial F}{\partial q},\frac{\partial G}{\partial p}\right\rangle-\left\langle\frac{\partial F}{\partial p},\frac{\partial G}{\partial q}\right\rangle.$$

With N arbitrary distinct constants $\lambda_1,\dots,\lambda_N$ we form the diagonal matrix $\Lambda=\mathrm{diag}(\lambda_1,...,\lambda_N)$, while λ, μ are used for two different spectral parameters. By $C^\infty(R)$ we denote the set of all functions on real field R being infinitely many times differentiable. Finally, x stands for the spatial continuous variable.

II. THE CONSTRAINED AKNS AND DIRAC (D) SYSTEM

We consider the following two 2×2 traceless Lax matrices:

$$L^{\mathrm{AKNS}}=L^{\mathrm{AKNS}}(\lambda)=\begin{pmatrix}1&0\\0&-1\end{pmatrix}+\sum_{j=1}^{N}\frac{1}{\lambda-\lambda_j}\begin{pmatrix}p_jq_j&-q_j^2\\p_j^2&-p_jq_j\end{pmatrix}\equiv\begin{pmatrix}A_{\mathrm{AKNS}}(\lambda)&B_{\mathrm{AKNS}}(\lambda)\\C_{\mathrm{AKNS}}(\lambda)&-A_{\mathrm{AKNS}}(\lambda)\end{pmatrix},\tag{1}$$

$$L^D=L^D(\lambda)=\begin{pmatrix}0&1\\-1&0\end{pmatrix}+\sum_{j=1}^{N}\frac{1}{\lambda-\lambda_j}\begin{pmatrix}p_jq_j&-q_j^2\\p_j^2&-p_jq_j\end{pmatrix}\equiv\begin{pmatrix}A_D(\lambda)&B_D(\lambda)\\C_D(\lambda)&-A_D(\lambda)\end{pmatrix}.\tag{2}$$

Let us choose two 2×2 matrices M_{AKNS} and M_D:

$$M_{\mathrm{AKNS}}=\begin{pmatrix}\lambda&-\langle q,q\rangle\\\langle p,p\rangle&-\lambda\end{pmatrix},\tag{3}$$

$$M_D=\begin{pmatrix}\langle p,q\rangle&\lambda+\frac12(\langle p,p\rangle-\langle q,q\rangle)\\-\lambda+\frac12(\langle p,p\rangle-\langle q,q\rangle)&-\langle p,q\rangle\end{pmatrix}.\tag{4}$$

Then we have the following.

Theorem 1: The Lax equations

$$L_x^{\mathrm{AKNS}}=[M_{\mathrm{AKNS}},L^{\mathrm{AKNS}}],\tag{5}$$

$$L_x^D=[M_D,L^D],\tag{6}$$

respectively describe the following finite-dimensional Hamiltonian systems (FDHSs) (H_{AKNS}) and (H_D),

$$(H_{\mathrm{AKNS}}):\begin{cases}q_x=-\langle q,q\rangle p+\Lambda q=\dfrac{\partial H_{\mathrm{AKNS}}}{\partial p},\\[2mm] p_x=\langle p,p\rangle q-\Lambda p=-\dfrac{\partial H_{\mathrm{AKNS}}}{\partial q};\end{cases}\tag{7}$$

$$(H_D):\begin{cases}q_x=\langle p,q\rangle q+\frac12(\langle p,p\rangle-\langle q,q\rangle)p+\Lambda p=\dfrac{\partial H_D}{\partial p},\\[2mm] p_x=-\langle p,q\rangle p-\frac12(\langle p,p\rangle-\langle q,q\rangle)q-\Lambda q=-\dfrac{\partial H_D}{\partial q},\end{cases}\tag{8}$$

with Hamiltonian functions

$$H_{\mathrm{AKNS}}=\langle\Lambda q,p\rangle-\tfrac{1}{2}\langle p,p\rangle\langle q,q\rangle, \tag{9}$$

$$H_D=\tfrac{1}{2}(\langle\Lambda p,p\rangle+\langle\Lambda q,q\rangle)+\tfrac{1}{2}(\langle p,q\rangle^2-\langle q,q\rangle\langle p,p\rangle)+\tfrac{1}{8}(\langle p,p\rangle+\langle q,q\rangle)^2. \tag{10}$$

Proof: By direct calculation.

Through setting

$$u=-\langle q,q\rangle,\quad v=\langle p,p\rangle, \tag{11}$$

Eq. (7) becomes

$$\begin{aligned} q_{jx}&=\lambda_j q_j+up_j,\quad j=1,...,N,\\ p_{jx}&=vq_j-\lambda_j p_j,\quad j=1,...,N, \end{aligned} \tag{12}$$

which is nothing other than the well-known Zakharov–Shabat–AKNS spectral problem (ZS-AKNSSP)[26]

$$y_x=\begin{pmatrix}\lambda & u\\ v & -\lambda\end{pmatrix}y \tag{13}$$

where $\lambda=\lambda_j$, $y=(q_j,p_j)^T$. The potentials u,v defined by Eq. (11) are correspond exactly to the Bargmann–Garnier constraint[1(b)]

$$G_0=(v,u)^T=\sum_{j=1}^{N}\left(\frac{\delta\lambda_j}{\delta u},\frac{\delta\lambda_j}{\delta v}\right)^T \tag{14}$$

of ZS-AKNSSP (13), where $\delta\lambda_j/\delta u$, $\delta\lambda_j/\delta v$ are the two spectral gradients of spectral parameters λ_j with respect to the potentials u and v. Therefore Eq. (7) coincides with the constrained AKNS system (c-AKNSS).

Similarly, after setting

$$u=-\tfrac{1}{2}(\langle p,p\rangle-\langle q,q\rangle),\quad v=-\langle p,q\rangle, \tag{15}$$

it is easily seen that Eq. (8) becomes the Dirac spectral problem (DSP)[27]

$$y_x=\begin{pmatrix}-v & \lambda-u\\ -\lambda-u & v\end{pmatrix}y \tag{16}$$

with $\lambda=\lambda_j$, $y=(q_j,p_j)^T$. Thus, Eq. (8) is the constrained Dirac system (c-DS).

Let us return to the Lax matrix (1) and (2). By a simple calculation we obtain the following.

Proposition 1: For $J=$AKNS and $J=D$ the following holds:

$$\begin{aligned} &\{A_J(\lambda),A_J(\mu)\}=\{B_J(\lambda),B_J(\mu)\}=\{C_J(\lambda),C_J(\mu)\}=0,\\ &\{A_J(\lambda),B_J(\mu)\}=\frac{2}{\mu-\lambda}(B_J(\mu)-B_J(\lambda)),\\ &\{A_J(\lambda),C_J(\mu)\}=\frac{2}{\mu-\lambda}(C_J(\lambda)-C_J(\mu)),\\ &\{B_J(\lambda),C_J(\mu)\}=\frac{4}{\mu-\lambda}(A_J(\mu)-A_J(\lambda)). \end{aligned} \tag{17}$$

Set $L_1^J(\lambda)=L^J(\lambda)\otimes I$, $L_2^J(\mu)=I\otimes L^J(\mu)$, where I is the 2×2 unit matrix. Then the above proposition yields the following.

Theorem 2: The Lax matrices $L^J(\lambda)$ $(J=\mathrm{AKNS},D)$ defined by Eqs. (1) and (2) satisfy the fundamental Poisson bracket

$$\{L_1^J(\lambda)\overset{\otimes}{,}L_2^J(\mu)\}=[r_{12}(\lambda,\mu),L_1^J(\lambda)]-[r_{21}(\mu,\lambda),L_2^J(\mu)]. \tag{18}$$

Here $\{L_1^J(\lambda)\overset{\otimes}{,}L_2^J(\mu)\}$ is a 4×4 matrix,[14] $[\cdot,\cdot]$ is the usual commutator of the matrix, and the r-matrices $r_{12}(\lambda,\mu)$, $r_{21}(\mu,\lambda)$ are exactly given by the following standard r-matrices:

$$r_{12}(\lambda,\mu)=\frac{2}{\mu-\lambda}P,\quad r_{21}(\mu,\lambda)=Pr_{12}(\mu,\lambda)P, \tag{19}$$

$$P=\begin{pmatrix}1&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&1\end{pmatrix}=\frac{1}{2}\left(I+\sum_{i=1}^{3}\sigma_i\otimes\sigma_i\right) \tag{20}$$

(here σ_i are the Pauli matrices).

So, the c-AKNSS (7) and c-DS (8) share the same standard r-matrix (19), which is obviously nondynamical.

Remark 1: In fact, since the r-matrix relation (18) is concerned only with the commutator, the r-matrix $r_{12}(\lambda,\mu)$ satisfying (18) in the case of the c-AKNSS (7) and c-DS (8) can be also chosen as

$$r_{12}(\lambda,\mu)=\frac{2}{\mu-\lambda}P+I\otimes\tilde{S},\quad \tilde{S}=\begin{pmatrix}a&b\\c&d\end{pmatrix}, \tag{21}$$

where the elements a,b,c,d can be arbitrary functions $a(\lambda,\mu,p,q)$, $b(\lambda,\mu,p,q)$, $c(\lambda,\mu,p,q)$, $d(\lambda,\mu,p,q)\in C^\infty(R)$ with respect to the spectral parameters λ, μ and the dynamical variables p,q. This shows that for a given Lax matrix, the associated r-matrix is not uniquely defined (there are even infinitely many r-matrices possible). Here we give the simplest case: $a=b=c=d=0$, i.e., the standard r-matrix (19).

III. THE CONSTRAINED HARRY–DYM (HD) AND HEISENBERG SPIN CHAIN (HSC) SYSTEM

The constrained Harry–Dym (HD) system describes the geodesic flow on an ellipsoid and shares the same r-matrix with the constrained Heisenberg spin chain (HSC).

To see this, we follow the process given in Sec. II, considering the following Lax matrices:

$$\begin{aligned}L^{\mathrm{HD}}=L^{\mathrm{HD}}(\lambda)&=\begin{pmatrix}-\langle p,q\rangle\lambda^{-1}&\lambda^{-2}+\langle q,q\rangle\lambda^{-1}\\-\langle p,p\rangle\lambda^{-1}&\langle p,q\rangle\lambda^{-1}\end{pmatrix}+\sum_{j=1}^{N}\frac{1}{\lambda-\lambda_j}\begin{pmatrix}p_jq_j&-q_j^2\\p_j^2&-p_jq_j\end{pmatrix}\\&\equiv\begin{pmatrix}A_{\mathrm{HD}}(\lambda)&B_{\mathrm{HD}}(\lambda)\\C_{\mathrm{HD}}(\lambda)&-A_{\mathrm{HD}}(\lambda)\end{pmatrix},\end{aligned} \tag{22}$$

$$\begin{aligned}L^{\mathrm{HSC}}=L^{\mathrm{HSC}}(\lambda)&=\begin{pmatrix}-\langle p,q\rangle\lambda^{-1}&\langle q,q\rangle\lambda^{-1}\\-\langle p,p\rangle\lambda^{-1}&\langle p,q\rangle\lambda^{-1}\end{pmatrix}+\sum_{j=1}^{N}\frac{1}{\lambda-\lambda_j}\begin{pmatrix}p_jq_j&-q_j^2\\p_j^2&-p_jq_j\end{pmatrix}\\&\equiv\begin{pmatrix}A_{\mathrm{HSC}}(\lambda)&B_{\mathrm{HSC}}(\lambda)\\C_{\mathrm{HSC}}(\lambda)&-A_{\mathrm{HSC}}(\lambda)\end{pmatrix}.\end{aligned} \tag{23}$$

Set

$$M_{\mathrm{HD}}=\begin{pmatrix} 0 & 1 \\ -\dfrac{\langle \Lambda p,p\rangle}{\langle \Lambda^2 q,q\rangle}\lambda & 0 \end{pmatrix}, \tag{24}$$

$$M_{\mathrm{HSC}}=\begin{pmatrix} -i\lambda\langle \Lambda p,q\rangle & i\lambda\langle \Lambda q,q\rangle \\ -i\lambda\langle \Lambda p,p\rangle & i\lambda\langle \Lambda p,q\rangle \end{pmatrix}, \quad i^2=-1. \tag{25}$$

Then a direct calculation leads to the following theorem.

Theorem 3: The Lax representations

$$L_x^{\mathrm{HD}}=[M_{\mathrm{HD}},L^{\mathrm{HD}}], \tag{26}$$

$$L_x^{\mathrm{HSC}}=[M_{\mathrm{HSC}},L^{\mathrm{HSC}}], \tag{27}$$

respectively give the following FDHS:

$$(H_{\mathrm{HD}}):\begin{cases} q_x=p=\left.\dfrac{\partial H_{\mathrm{HD}}}{\partial p}\right|_{TQ^{N-1}}, \\ p_x=-\dfrac{\langle \Lambda p,p\rangle}{\langle \Lambda^2 q,q\rangle}\Lambda q=-\left.\dfrac{\partial H_{\mathrm{HD}}}{\partial q}\right|_{TQ^{N-1}}, \\ \langle \Lambda q,q\rangle=1; \end{cases} \tag{28}$$

$$(H_{\mathrm{HSC}}):\begin{cases} q_x=i\langle \Lambda q,q\rangle\Lambda p-i\langle \Lambda p,q\rangle\Lambda q=\dfrac{\partial H_{\mathrm{HSC}}}{\partial p}, \\ p_x=i\langle \Lambda p,q\rangle\Lambda p-i\langle \Lambda p,p\rangle\Lambda q=-\dfrac{\partial H_{\mathrm{HSC}}}{\partial q}, \end{cases} \tag{29}$$

with the Hamiltonian functions

$$H_{\mathrm{HD}}=\frac{1}{2}\langle p,p\rangle-\frac{\langle \Lambda p,p\rangle}{2\langle \Lambda^2 q,q\rangle}(\langle \Lambda q,q\rangle-1), \tag{30}$$

$$H_{\mathrm{HSC}}=\tfrac{1}{2}i\langle \Lambda p,p\rangle\langle \Lambda q,q\rangle-\tfrac{1}{2}i\langle \Lambda p,q\rangle^2. \tag{31}$$

Here, in Eq. (28) TQ^{N-1} is a tangent bundle in R^{2N}:

$$TQ^{N-1}=\{(p,q)\in R^{2N}|F\equiv\langle \Lambda q,q\rangle-1=0,G\equiv\langle \Lambda p,q\rangle=0\}. \tag{32}$$

Obviously, Eq. (28) is equivalent to

$$q_{xx}+\frac{\langle \Lambda q_x,q_x\rangle}{\langle \Lambda^2 q,q\rangle}\Lambda q=0, \quad \langle \Lambda q,q\rangle=1, \tag{33}$$

which is nothing but the equation of the geodesic flow on the surface $\langle \Lambda q,q\rangle=1$ in R^N space.[3,22] On the other hand, by setting

$$u=\frac{\langle \Lambda q_x,q_x\rangle}{\langle \Lambda^2 q,q\rangle}, \tag{34}$$

Eq. (33) becomes the well-known Harry–Dym spectral problem

$$y_{xx}+\lambda uy=0 \tag{35}$$

with $\lambda=\lambda_j$, $y=q_j$, $j=1,...,N$. Simultaneously, Eq. (34) gives the constraint condition considered by Cao,[3] so that Eq. (28) coincides with the constrained HD system (c-HDS).

In addition, after setting

$$u=-\langle\Lambda q,q\rangle,\quad v=\langle\Lambda p,p\rangle,\quad w=-\langle\Lambda p,q\rangle,\tag{36}$$

we can see that Eq. (29) becomes the Heisenberg spin chain spectral problem[28]

$$y_x=\begin{pmatrix}-i\lambda w & -i\lambda u\\ -i\lambda v & i\lambda w\end{pmatrix}y,\quad i^2=-1,\tag{37}$$

with $\lambda=\lambda_j$, $y=(q_j,p_j)^T$. Thus, Eq. (29) reads as the constrained Heisenberg spin chain system (c-HSCS). The constraint defined by Eq. (36) has been studied and applied to obtaining involutive solutions for the Heisenberg spin chain equations in Ref. 29. However, the r-matrix was not given.

Now, we construct the r-matrix of c-HDS (28) and c-HSCS (29). Their Lax matrices (22) and (23) share all elements except one, namely

$$\begin{pmatrix}0 & \lambda^{-2}\\ 0 & 0\end{pmatrix}.$$

Since this element does not affect the calculations concerning the fundamental Poisson bracket $\{L_1^J(\lambda)\overset{\otimes}{,}L_2^J(\mu)\}$, one can readily deduce that the c-HDS and c-HSCS possess the same r-matrix.

Theorem 4: The Lax matrix $L^J(\lambda)$ ($J=$HD,HSC) defined by Eqs. (22) and (23) satisfies the fundamental Poisson bracket (18) with the common nondynamical r-matrix

$$r_{12}(\lambda,\mu)=\frac{2\lambda}{\mu(\mu-1)}P,\quad r_{21}(\mu,\lambda)=Pr_{12}(\mu,\lambda)P.\tag{38}$$

Apparently the r-matrix (38) solves the classical Yang–Baxter equation (CYBE)

$$[r_{ij},r_{ik}]+[r_{ij},r_{jk}]+[r_{kj},r_{ik}]=0,\quad i,j,k=1,2,3.\tag{39}$$

Remark 2: The r-matrix $r_{12}(\lambda,\mu)$ satisfying Eq. (18) in the case of c-HDS (28) and c-HSCS (29) can also be chosen as

$$r_{12}(\lambda,\mu)=\frac{2\lambda}{\mu(\mu-\lambda)}P+I\otimes\tilde{S}.\tag{40}$$

Evidently, Eq. (38) is the simplest case: $\tilde{S}=0$ of Eq. (40).

IV. THE CONSTRAINED *G* AND *Q* SYSTEM

In this section, we introduce the following Lax matrices:

$$\begin{aligned}L^G=L^G(\lambda)&=\begin{pmatrix}(\frac12+\langle p,q\rangle)\lambda^{-1} & \langle q,q\rangle\lambda^{-1}\\ 0 & -(\frac12+\langle p,q\rangle)\lambda^{-1}\end{pmatrix}+\sum_{j=1}^N\frac{1}{\lambda-\lambda_j}\begin{pmatrix}p_jq_j & -q_j^2\\ p_j^2 & -p_jq_j\end{pmatrix}\\ &\equiv\begin{pmatrix}A_G(\lambda) & B_G(\lambda)\\ C_G(\lambda) & -A_G(\lambda)\end{pmatrix},\end{aligned}\tag{41}$$

$$L^Q=L^Q(\lambda)=\begin{pmatrix}-\lambda^{-1} & \langle q,q\rangle\lambda^{-1}\\ 0 & \lambda^{-1}\end{pmatrix}+\sum_{j=1}^N\frac{1}{\lambda-\lambda_j}\begin{pmatrix}p_jq_j & -q_j^2\\ p_j^2 & -p_jq_j\end{pmatrix}\equiv\begin{pmatrix}A_Q(\lambda) & B_Q(\lambda)\\ C_Q(\lambda) & -A_Q(\lambda)\end{pmatrix}.\tag{42}$$

If we set

$$M_G=\begin{pmatrix} -\frac{1}{\alpha}\,\lambda & \frac{1}{\alpha}\,(\langle p,p\rangle-\langle q,q\rangle)-1 \\ \frac{1}{\alpha}\,(\langle p,p\rangle-\langle q,q\rangle+1)\lambda & \frac{1}{\alpha}\,\lambda \end{pmatrix}, \tag{43}$$

$$M_Q=\begin{pmatrix} \lambda+\frac{1}{2\beta^2}\,\langle \Lambda q,q\rangle\langle p,p\rangle & \frac{1}{\beta}\,\langle \Lambda q,q\rangle \\ -\frac{1}{\beta}\,\langle p,p\rangle\lambda & -\lambda-\frac{1}{2\beta^2}\,\langle \Lambda q,q\rangle\langle p,p\rangle \end{pmatrix}, \tag{44}$$

with

$$\alpha=\sqrt{(\langle p,p\rangle-\langle \Lambda q,q\rangle)^2-4\langle \Lambda q,p\rangle},\quad \beta=1-\langle p,q\rangle,$$

then, by a lengthy and straightforward calculation we obtain the following.

Theorem 5: The following Lax representations,

$$L_x^G=[M_G,L^G] \tag{45}$$

and

$$L_x^Q=[M_Q,L^Q], \tag{46}$$

where the first one is restricted to the surface $\Gamma=\{(p,q)\in R^{2N}|\langle p,q\rangle=0, \langle \Lambda q,q\rangle\langle p,p\rangle+\langle \Lambda q,p\rangle=0\}$ in the space R^{2N}, respectively produce the finite-dimensional systems

$$\begin{aligned} q_x&=\frac{1}{\alpha}\,(-\Lambda q+(\langle p,p\rangle-\langle \Lambda q,q\rangle)p)-p, \\ p_x&=\frac{1}{\alpha}\,(\Lambda p+(\langle p,p\rangle-\langle \Lambda q,q\rangle)\Lambda q)+\Lambda q, \end{aligned} \tag{47}$$

and

$$\begin{aligned} q_x&=\Lambda q+\frac{1}{\beta}\,\langle \Lambda q,q\rangle p+\frac{1}{2\beta^2}\,\langle p,p\rangle\langle \Lambda q,q\rangle q, \\ p_x&=-\Lambda p-\frac{1}{\beta}\,\langle p,p\rangle\Lambda q-\frac{1}{2\beta^2}\,\langle p,p\rangle\langle \Lambda q,q\rangle p. \end{aligned} \tag{48}$$

In Eqs. (47) and (48), respectively insert

$$\begin{aligned} u&=\frac{1}{\alpha}=\frac{1}{\sqrt{(\langle p,p\rangle-\langle \Lambda q,q\rangle)^2-4\langle \Lambda q,p\rangle}}, \\ v&=\frac{\langle p,p\rangle-\langle \Lambda q,q\rangle}{\alpha}=\frac{\langle p,p\rangle-\langle \Lambda q,q\rangle}{\sqrt{(\langle p,p\rangle-\langle \Lambda q,q\rangle)^2-4\langle \Lambda q,p\rangle}}; \end{aligned} \tag{49}$$

and

$$\begin{aligned} u&=\frac{\langle \Lambda q,q\rangle}{\beta}=\frac{\langle \Lambda q,q\rangle}{1-\langle q,p\rangle} \\ v&=\frac{-\langle p,p\rangle}{\beta}=-\frac{\langle p,p\rangle}{1-\langle q,p\rangle}. \end{aligned} \tag{50}$$

Then, Eqs. (47) and (48) turn out to become the spectral problem studied by Geng (simply called G-spectral problem),[7]

$$y_x=\begin{pmatrix} -\lambda u & v-1 \\ \lambda(v+1) & \lambda u \end{pmatrix} y, \tag{51}$$

with $\lambda=\lambda_j$, $y=(q_j,p_j)^T$, and the spectral problem proposed by Qiao (simply called Q-spectral problem),[11]

$$y_x=\begin{pmatrix} \lambda-\frac{1}{2}uv & u \\ \lambda v & -\lambda+\frac{1}{2}uv \end{pmatrix} y, \tag{52}$$

with $\lambda=\lambda_j$, $y=(q_j,p_j)^T$, respectively. So, Eqs. (47) and (48) are nothing but the constrained Geng system (c-GS) and constrained Qiao system (c-QS) under the constraint conditions (49) and (50). Since the Lax equation (45) gives the c-GS (47) on the surface Γ, the Lax matrix L^G should become

$$L_\Gamma^G=L_\Gamma^G(\lambda)=\begin{pmatrix} \frac{1}{2}\lambda^{-1} & \langle q,q\rangle\lambda^{-1} \\ 0 & -\frac{1}{2}\lambda^{-1} \end{pmatrix}+\sum_{j=1}^N \frac{1}{\lambda-\lambda_j}\begin{pmatrix} p_jq_j & -q_j^2 \\ p_j^2 & -p_jq_j \end{pmatrix}, \tag{53}$$

which is almost the same as L^Q. Hence, through calculating the fundamental Poisson bracket and commutator we have the following theorem.

Theorem 6: For the c-GS (47) and c-QS (48), their Lax matrices $L_\Gamma^G(\lambda)$ and $L^Q(\lambda)$ satisfy the fundamental Poisson bracket (18) with the same nondynamical r-matrix:

$$r_{12}(\lambda,\mu)=\frac{2}{\mu-\lambda}P-\frac{2}{\mu}S, \quad S=\begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&1&0&0\\0&0&0&0 \end{pmatrix}=\sigma_-\otimes\sigma^+. \tag{54}$$

We can easily show that Eq. (54) satisfies the CYBE (39).

Remark 3: The r-matrix $r_{12}(\lambda,\mu)$ in Theorem 6 can be also chosen as

$$r_{12}(\lambda,\mu)=\frac{2}{\mu-\lambda}P-\frac{2}{\mu}S+I\otimes\tilde{S}. \tag{55}$$

Equation (54) is the simplest case: $\tilde{S}=0$ of Eq. (55).

V. CONCLUSION

In this article, we present three pairs of different finite-dimensional constrained systems with common nondynamical r-matrices. Along the discrete Toda symplectic map and continuous constrained c-KdV system discovered in Ref. 24 these three pairs form (to the authors' knowledge), the only four examples of pairs of different finite-dimensional integrable systems possessing the above property. The question of whether or not there are any other pairs like them arises. It seems that this is not the case.

From Remarks 1–3, we see that the r-matrix $r_{12}(\lambda,\mu)$ satisfying the fundamental Poisson bracket (18) is composed of two parts, the first one being their main term, and the second one being the common term $I\otimes\tilde{S}$ of Eqs. (21), (40), and (55). Usually when proving the integrability of FDHS we choose their main term as the simplest nondynamical one in order to reduce the calculations.

Apparently, the r-matrix is not uniquely defined. In fact, there are infinitely many r-matrices since the elements a,b,c,d in the matrix $\tilde{S}$ of Eqs. (21), (40), and (55) can be arbitrarily chosen, and they may be constants as well as functions with respect to the spectral parameters λ, μ and the dynamical variables p,q. For a given Lax matrix $L^J(\lambda)$, our r-matrices formulas (21), (40), and (55) admit an infinite set of solutions $r_{12}(\lambda,\mu)$ of Eq. (18), which includes both the dynamical case [as $a=a(\lambda,\mu,p,q)$, $b=b(\lambda,\mu,p,q)$, $c=c(\lambda,\mu,p,q)$, $d=d(\lambda,\mu,p,q)\in C^{\infty}(R)$] and the nondynamical or constant case [as $a=a(\lambda,\mu)$, $b=b(\lambda,\mu)$, $c=c(\lambda,\mu)$, $d=d(\lambda,\mu)$, or $a,b,c,d=\text{const}$].

In analogy to the first author's thesis,[30] we can further discuss the involutive sets, integrability, separation of variables, and algebraic–geometric solutions for these constrained FDHSs by using the determinant of Lax matrix, r-matrix relation, Poisson bracket, and further modern algebraic–geometric tools.

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