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Peakon, pseudo-peakon, and cuspon solutions for two generalized Camassa-Holm equations

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In this paper, we study peakon, cuspon, and pseudo-peakon solutions for two generalized Camassa-Holm equations. Based on the method of dynamical systems, the two generalized Camassa-Holm equations are shown to have the parametric representations of the solitary wave solutions such as peakon, cuspon, pseudo-peakons, and periodic cusp solutions. In particular, the pseudo-peakon solution is for the first time proposed in our paper. Moreover, when a traveling system has a singular straight line and a heteroclinic loop, under some parameter conditions, there must be peaked solitary wave solutions appearing. [http://dx.doi.org/10.1063/1.4835395]

I. INTRODUCTION

In recent years, nonlinear wave equations with non-smooth solitary wave solutions, such as peaked solitons (peakons) and cusped solitons (cuspons), attract much attention in the literature. Peakon was first proposed by Cammasa and Holm,[1,2] and thereafter other peakon equations were developed (see Degasperis and Procesi,[3] Degasperis, Holm, and Hone,[4] Qiao,[5,6] Li and Dai,[7] Novikov,[8] and cited references therein). Peakons are the so-called peaked solitons, i.e., solitons with discontinuous first order derivative at the peak point. Usually, the profile of a wave function is called peakon if at a continuous point its left and right derivatives are finite and have different sign.[9] But if its both left and right derivatives are positive and negative infinities, then the wave profile is called cuspon.

In this paper, we shall show that there exists pseudo-peakon solution for nonlinear wave equations. The so called "pseudo-peakon" means that the wave profile looks like peakon, but the solution still has continuous first order derivative. By using the dynamical system approach, it has theoretically been proved that there exists at least one singular straight line in the traveling wave system corresponding to some nonlinear wave equation such that the traveling wave solutions have peaked profiles and lose their smoothness. In fact, the existence of a singular straight line leads to a dynamical behavior with two scale variables. For a singular nonlinear traveling wave system of the first class, the following two results hold (see Li and Dai,[7] Li and Chen,[10] and more recently Li[11]).

Theorem A (The rapid-jump property of the derivative near the singular straight line). *Suppose that in a left (or right) neighborhood of a singular straight line there exists a family of periodic orbits. Then, along a segment of every orbit near the straight line, the derivative of the wave function jumps down rapidly on a very short time interval.*

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Theorem B (Existence of finite time interval of solution with respect to wave variable in the positive or negative direction). *For a singular nonlinear traveling wave system of the first class with possible change of the wave variable, if an orbit transversely intersects with a singular straight line at a point or it approaches a singular straight line, but the derivative tends to infinity, then it only takes a finite time interval to make moved point of the orbit arrive on the singular straight line.*

In order to understand rigorously the occurrence of "peaked" traveling wave solutions and the change of wave profiles, we hope to obtain exact parametric representations of traveling wave solutions for a given nonlinear partial differential system. Using exact solution formulas, we can see the change of wave profiles. In this paper, we take the following two nonlinear wave equations as examples to achieve this goal.

1. The generalized Camassa-Holm (CH) equation with real parameters k, α:

$$u_t + ku_x - u_{xxt} + \alpha u u_x = 2u_x u_{xx} + u u_{xxx}. \tag{1}$$

Equation (1) with $\alpha = 3$ is exactly the standard CH equation[1] as a shallow water model.

2. The two-component Camassa-Holm system with real parameters $k, \alpha, e_0 = \pm 1$ (see Olver and Rosenau,[12] Chen, Liu, and Zhang,[13] and Chen, Liu, and Qiao[14]):

$$\begin{aligned} &m_t + \sigma u m_x - A u_{xx} + 2\sigma m u_x + 3(1-\sigma) u u_x + e_0 \rho \rho_x = 0, \\ &\rho_t + (\rho u)_x = 0, \end{aligned} \tag{2}$$

where $m = u - \alpha^2 u_{xx} - \frac{k}{2}$.

The corresponding traveling wave systems of equations (1) and (2) have one and two singular straight lines, respectively (see Secs. II and III). Under some particular parameter conditions, there exists at least one family of periodic orbits such that the boundary curves of the period annulus are a homoclinic orbit or a heteroclinic loop (see the phase portraits in Secs. II and III). Applying the classical analysis method, we can obtain the parametric representations for these boundary curves. When we take these homoclinic orbits and heteroclinic loops into account as the limit curves of period annulus, these exact parametric representations provide very good understanding for the occurrence of peaked traveling wave solutions. Namely, the homoclinic curve gives rise to a solitary peakon-like wave solution (called pseudo-peakon), while the curve triangle (heteroclinic loop) gives rise to a solitary wave solution with some peak (peakon).

How to classify traveling wave solutions seems kind of interest for a given traveling system. Lenells[15,16] studied traveling wave solutions for some nonlinear shallow water wave models admitting smooth, peaked, and cusped solutions, as well as stumpons. Qiao and Zhang investigated all possible single soliton solutions of the CH equation through the procedure of functional analysis.[17] Our method is different from those two approaches.[15–17] Adopting the method of dynamical systems with **Theorem A** and **Theorem B**, we can obtain dynamical behavior of all traveling wave solutions to integrable PDE models. Therefore, we know which orbit gives rise to what wave profiles and how the wave profiles are changed depending on the parameters. In addition, applying the first integrals of the integrable traveling wave systems, we are able to get some explicit solutions (see Li *et al.*,[18,19] and Li[11]).

This paper is organized as follows. In Secs. II and III, we discuss the exact solutions of equation (1) and equation (2), respectively.

II. PEAKON, PSEUDO-PEAKONS, CUSPON, AND PERIODIC CUSP WAVE SOLUTIONS OF EQUATION (1)

Let $u(x, t) = \phi(x - ct) = \phi(\xi)$, where c is the wave speed. Substituting it into (1) and integrating once, we have

$$(\phi - c)\phi'' = -\frac{1}{2}(\phi')^2 + \frac{1}{2}\alpha\phi^2 + (k - c)\phi - g,$$

where g is an integral constant and "$'$" is the derivative with respect to ξ. The above equation with $g = 0$ is equivalent to the following differential system

$$\frac{d\phi}{d\xi} = y, \qquad \frac{dy}{d\xi} = \frac{-y^2 + 2(k-c)\phi + \alpha\phi^2}{2(\phi - c)}, \tag{3}$$

which has the following first integral:

$$H(\phi, y) = (\phi - c)y^2 - [(k-c)\phi^2 + \frac{1}{3}\alpha\phi^3] = h. \tag{4}$$

System (3) has two critical points $E_0(0, 0)$ and $E_1\left(\frac{2(c-k)}{\alpha}, 0\right)$ in the ϕ-axis. Making the transformation $d\xi = (\phi - c)d\zeta$, $\phi \neq c$, system (3) becomes its regular associated system[10]

$$\frac{d\phi}{d\zeta} = 2y(\phi - c), \qquad \frac{dy}{d\zeta} = -y^2 + 2(k-c)\phi + \alpha\phi^2. \tag{5}$$

Apparently, the straight line $\phi = c$ is a solution of system (5). On this straight line, system (5) has two critical points $S_{1,2}(c, \pm\sqrt{Y_0})$ when $Y_0 = \alpha c^2 + 2(k - c)c > 0$. By the results[10] and **Theorem A** and **Theorem B** in Sec. I, we know that the dynamics of systems (3) and (5) are different in the neighborhood of the straight line $\phi = c$. Specially, the variable "ζ" is a fast variable while the variable "ξ" is a slow variable in the sense of the geometric singular perturbation theory.

Let $h_0 = H(0, 0) = 0$, $h_1 = H\left(\frac{2(c-k)}{\alpha}, 0\right) = \frac{4(c-k)^3}{3\alpha^2}$, $h_s = H(c, \pm\sqrt{Y_0}) = c^2\left[(c-k) - \frac{1}{3}\alpha c\right]$. Under the parameter condition of $c > 0$, $k < c$, system (5) has the phase portraits shown below in Figs. 1(a)–1(c).

Let us first consider the parametric representations of the bounded orbits given by Fig. 1(a).

(i) For the family of periodic orbits of Eq. (3) defined by $H(\phi, y) = h, h \in (0, h_1)$ in (4), we see that $y^2 = \frac{\alpha(-\frac{3h}{\alpha} + \frac{3}{\alpha}(c-k)\phi^2 - \phi^3)}{3(c-\phi)} = \frac{\alpha(r_1-\phi)(\phi-r_2)(\phi-r_3)}{3(c-\phi)}$. Thus, from the first equation of system (3) we have

$$\xi = \sqrt{\frac{3}{\alpha}}\int_{\phi}^{r_1} \frac{(c-\phi)d\phi}{\sqrt{(c-\phi)(r_1-\phi)(\phi-r_2)(\phi-r_3)}},$$

which leads to the parametric representation

$$\begin{aligned} \phi(\chi) &= r_1 - \frac{\alpha_0^2(c-r_1)\mathrm{sn}^2(\chi,k)}{1-\alpha_0^2\mathrm{sn}^2(\chi,k)}, \\ \xi(\chi) &= \frac{2(c-r_1)\sqrt{\alpha}}{\sqrt{3(c-r_2)(r_1-r_3)}}\Pi(\arcsin(\mathrm{sn}(\chi,k)), \alpha_0^2, k), \end{aligned} \tag{6}$$

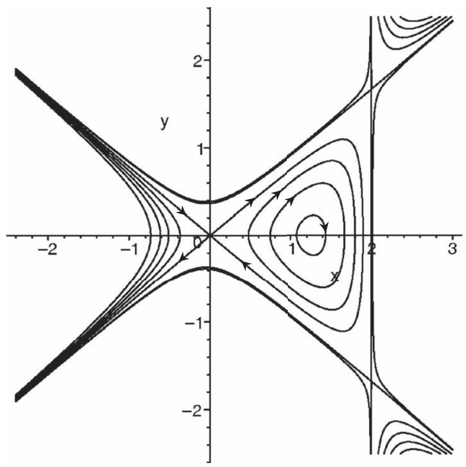
where $k^2 = \frac{(r_1-r_2)(c-r_3)}{(c-r_2)(r_1-r_3)}$, $\alpha_0^2 = \frac{r_1-r_2}{c-r_2}$, $\Pi(\cdot, \alpha^2, k)$ is the elliptic integral of the third kind, $\mathrm{sn}(u, k)$, $\mathrm{cn}(u, k)$, $\mathrm{dn}(u, k)$ are the Jacobian elliptic functions (see Byrd and Fridman[20]).

We notice from Fig. 1(a) that when parameter α is very close to $\frac{3(c-k)}{c}$, we have $0 < |h_s|$ and h_s is arbitrarily small. It implies that a segment of the homoclinic orbit defined by a branch of the level curve $H(\phi, y) = 0$ completely lies in a left neighborhood of the singular straight line $\phi = c$. By using Theorem A, the state coordinate y of the points in this segment rapidly jumps (following ξ varies) from a positive number to negative number, such that this homoclinic orbit gives rise to a solitary cusp wave.

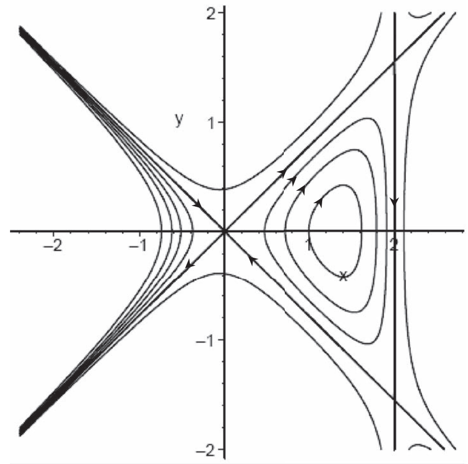
In addition, from (6), one can easily see that

$$\frac{d\phi}{d\xi} = \frac{\sqrt{3}\alpha_0^2\mathrm{sn}(\chi,k)\mathrm{cn}(\chi,k)\mathrm{dn}(\chi,k)}{\sqrt{\alpha}(1-\alpha_0^2\mathrm{sn}^2(\chi,k))}. \tag{7}$$

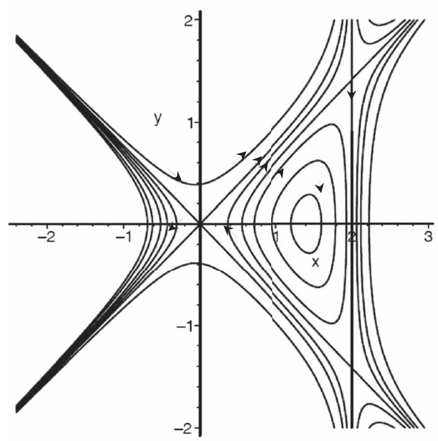
When $|h - h_s| \ll 1$, i.e., $|1 - k| \ll 1$ ($\ll 1$ means very close to 1), the graph of $\frac{d\phi}{d\xi}$ is shown in Fig. 2(a). Clearly, when $\chi = 4nK(k)$, $n = 0, \pm 1, \pm 2, \cdots$, $\frac{d\phi}{d\xi} = 0$, where $K(k)$ is the complete elliptic integral of the first kind with the modulo k. By Theorem A, when χ passes through $4nK(k)$, its sign changes from $+$ to $-$ and its value jumps rapidly from a positive maximum to a negative maximum. This fact implies that the wave profile of $\phi(\xi)$, determined by the periodic orbit closing to the homoclinic orbit, is a smooth periodic cusp wave (see Fig. 2(b). Finally, when $h \to 0, k \to 1$,



(a) alpha > 3(c−k)/c



(b) alpha = 3(c−k)/c



(c) 2(c−k)/c < alpha < 3(c−k)/c

FIG. 1. The phase portraits of (5) for $c > 0$, $k < c$.

the period of the periodic cusp wave solutions tends to ∞, thus, the homoclinic orbit in Fig. 1(a) gives rise to a smooth solitary cusp wave solution of (3) (see Fig. 2(c)). Because for $\phi(\xi)$ given by (8), we have $\frac{d\phi}{d\xi}|_{\xi=0} = 0$, therefore, the solitary cusp wave solution defined by (8) is not a peakon, it is a pseudo-peakon.

(ii) For the homoclinic orbit defined by $H(\phi, y) = 0$, let $P_M(\phi_M, 0)$ be the intersection point of the homoclinic orbit with the ϕ-axis. We know that $\phi_M = \frac{3(c-k)}{\alpha}$. When $h = h_0 = 0$, Eq. (4) becomes $y = \pm\sqrt{\frac{\alpha\phi^2(\phi_M-\phi)}{3(c-\phi)}}$. Using the first equation of system (3) and taking initial value $\phi(0) = \phi_M$, we obtain the parametric representation of the homoclinic orbit to the critical point $E_0(0, 0)$ defined by $H(\phi, y) = 0$ as follows:

$$\begin{aligned}
&\phi(\chi) = \frac{2c\phi_M}{(c-\phi_M)\cosh\left(\sqrt{c\phi_M}\,\chi\right)+(c+\phi_M)},\\
&\xi(\chi) = \sqrt{\frac{3}{\alpha}}\left[c\chi \mp \left(\ln\left|\sqrt{(\phi_M-\phi)(c-\phi)}+\phi-\tfrac{1}{2}(c+\phi_M)\right| - \ln\left(\tfrac{1}{2}(c-\phi_M)\right)\right)\right],\\
&\qquad \text{for } \chi \in (-\infty, 0] \text{ and for } \chi \in [0, \infty), \text{ respectively.}
\end{aligned} \tag{8}$$

In a short, we have the following conclusion.

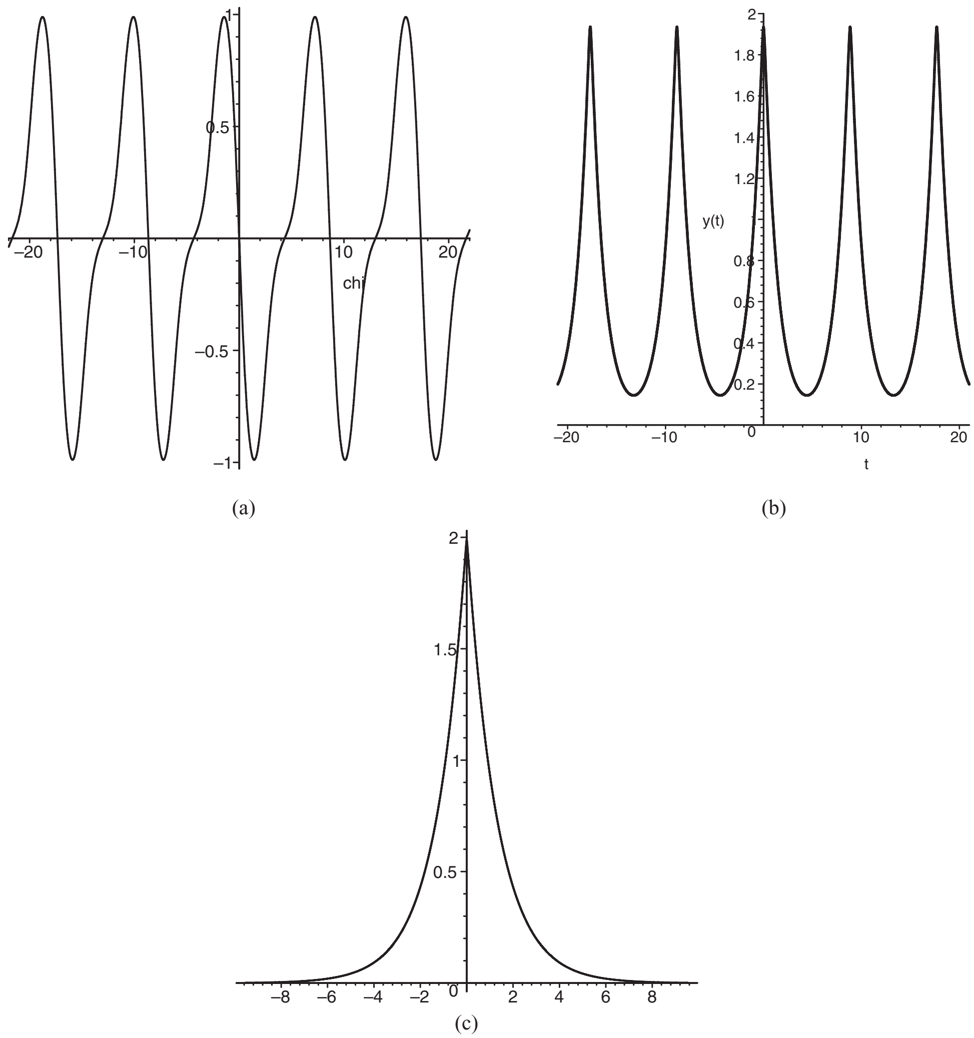


FIG. 2. The change of wave profiles of $\phi(\xi)$.

Theorem 1.

(1) When the parameter group (α, k, c) of system (3) satisfy the condition $\alpha > \frac{3(c-k)}{c}$ with $c > 0$, $k < c$, there exists a homoclinic orbit of system (5) given by a branch of the curves $H(\phi, y) = 0$. The homoclinic orbit has the exact parametric representation given by (8).

(2) When $\alpha - \frac{3(c-k)}{c} \ll 0$ ($\ll 0$ means very close to zero), i.e., $0 < |h_s| \ll 0$ (namely, $|h_s|$ is strictly positive and arbitrarily small), as a limit curve of a family of periodic orbits of system (3) defined by the closed branch of the curves $H(\phi, y) = h$, $h \in (h_1, 0))$ in Fig. 1(a), the homoclinic orbit gives rise to a smooth solitary cusp-like wave solution **(a pseudo-peakon)** of equation (1).

(3) When h varies from h_1 to $h_0 = 0$, periodic wave solutions of equation (1) determined by periodic orbits of system (3) will gradually become peaked periodic wave, and evolve from non-peaked periodic waves to the smooth periodic cusp-like waves and finally converge to a smooth solitary cusp-like wave **(a pseudo-peakon)**.

Second, let us consider the parametric representations of the orbits given by Fig. 1(b). Now, we have $h_0 = h_s = 0$.

(iii) For the family of periodic orbits of Eq. (3) defined by $H(\phi, y) = h, h \in (0, h_1)$ in (4), we have the same parametric representation as (6).

(iv) For two straight line orbits connecting the equilibrium points $(0, 0)$ and $(c, Y_\pm)$ of (4) defined by $H(\phi, y) = 0$, we have $y^2 = \frac{1}{3}\alpha\phi^2$. Thus, by **Theorem B** in Sec. I, we can take initial value as $\phi(0) = c$. Then, we have

$$u(x,t) = \phi(\xi) = ce^{\sqrt{\frac{\alpha}{3}}|\xi|}, \tag{9}$$

which is a real peakon solution to equation (1). Therefore, we have

Theorem 2.
(1) When the parameter group (α, k, c) of system (3) satisfy the condition $\alpha = \frac{3(c-k)}{c}$ with $c > 0$, $k < c$, there exists a heteroclinic loop of system (5) given by three branches of the curves $H(\phi, y) = 0$.
(2) As the limit curves of a family of periodic orbits of system (3), the curve triangle (i.e., heteroclinic loop) in Fig. 1(b) gives rise to a solitary peaked wave solution **(a peakon)** of equation (1), which has the exact parametric representation given by (9).
(3) When h varies from h_1 to 0, periodic wave solutions of equation (1) determined by periodic orbits of system (3) will gradually become peaked periodic wave, and evolve from non-peaked periodic waves to the smooth periodic cusp-like waves and finally converge to a solitary peaked wave solution **(a peakon)**.

Third, we discuss the exact solutions for the orbits shown in Fig. 1(c).
(v) For the family of periodic orbits of (3) defined by $H(\phi, y) = h, h \in (h_s, h_1)$ in (4), we see that $y^2 = \frac{\alpha(-\frac{3h}{\alpha}+\frac{3}{\alpha}(c-k)\phi^2-\phi^3)}{3(c-\phi)} = \frac{\alpha(r_1-\phi)(\phi-r_2)(\phi-r_3)}{3(c-\phi)}$. Thus, we have from the first equation of (3) that

$$\xi = \sqrt{\frac{3}{\alpha}}\int_{r_2}^{\phi}\frac{(c-\phi)d\phi}{\sqrt{(c-\phi)(r_1-\phi)(\phi-r_2)(\phi-r_3)}},$$

which implies the parametric representation

$$\begin{aligned}\phi(\chi) &= r_2 + \frac{(r_2-r_3)\mathrm{sn}^2(\chi,k)}{1-\alpha_1^2\mathrm{sn}^2(\chi,k)},\\ \xi(\chi) &= \sqrt{\frac{4\alpha}{3(c-r_2)(r_1-r_3)}}\left[(c-r_3)\chi-(r_2-r_3)\Pi(\arcsin(\mathrm{sn}(\chi,k)),\alpha_1^2,k)\right],\end{aligned} \tag{10}$$

where $k^2 = \frac{(r_1-r_2)(c-r_3)}{(c-r_2)(r_1-r_3)}$, $\alpha_1 = \frac{r_1-r_2}{r_1-r_3}$.
(vi) For the arch orbit defined by $H(\phi, y) = h_s$ in (4), we have $y^2 = \frac{\alpha}{3}(cm - m\phi + \phi^2)$, where $m = c - \frac{3}{\alpha}(c-k)$. Hence, we obtain the parametric representation of the smooth periodic cusp-like wave solution of (1) as follows:

$$\phi(\xi) = \frac{1}{2}\left[-\sqrt{m(m-4c)}\cosh\left(\sqrt{\frac{\alpha}{3}}\xi\right)+m\right], \quad 0 \le |\xi| \le \cosh^{-1}\left(\frac{m-2c}{\sqrt{m(m-4c)}}\right). \tag{11}$$

(vii) For the stable and unstable manifolds in the right phase plane of the critical point $E_0(0,0)$ defined by $H(\phi, y) = 0$ in (4), we have $y^2 = \frac{\alpha(\frac{3}{\alpha}(c-k)-\phi)\phi^2}{3(c-\phi)} \equiv \frac{\alpha(e_1-\phi)\phi^2}{3(c-\phi)}$.
On the basis of Theorem B in Sec. I, we can take initial value $\phi(0) = c$. Using the first equation of (3), we have the following cuspon solution of equation (1):

$$\begin{aligned}\phi(\xi) &= \frac{2e_1c}{(e_1-c)\cosh(\chi)+(e_1+c)}, \quad \chi \in (-\infty, 0] \text{ and } \chi \in [0,\infty), \text{ respectively},\\ \xi(\chi) &= \sqrt{\frac{3}{\alpha}}\left[\sqrt{\frac{c}{e_1}}\chi \mp \left(\ln\left|\sqrt{(e_1-\phi)(c-\phi)}+\phi-\frac{1}{2}(e_1+c)\right|-\ln\left(\frac{1}{2}(e_1-c)\right)\right)\right].\end{aligned} \tag{12}$$

According to (12), we may plot the graph of cuspon solution to equation (1) shown in Fig. 3.

Theorem 3. When the parameter group (α, k, c) of system (3) satisfy the condition $0 < \alpha < \frac{3(c-k)}{c}$ with $c > 0$, $k < c$, corresponding to the stable and unstable manifolds in the right phase plane of the critical point $E_0(0, 0)$ in Fig. 1(c) defined by $H(\phi, y) = 0$, equation (1) has a **cuspon** solution given by (12) (because its left and right derivatives equal to positive infinity and negative infinity, respectively).

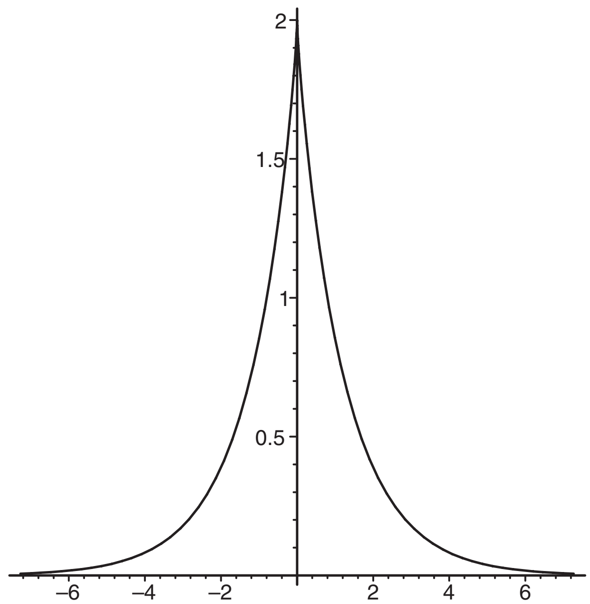


FIG. 3. The cuspon wave of equation (1).

III. PEAKON AND PSEUDO-PEAKON SOLUTIONS OF EQUATION (2)

Let $u(x, t) = \phi(x - ct) = \phi(\xi)$, $\rho(x, t) = v(x - ct) = v(\xi)$, where c is the wave speed. Then, the second equation of (2) becomes

$$-cv' + (v\phi)' = 0,$$

where "$\prime$" stands for the derivative with respect to ξ. Integrating this equation once and setting the integration constant as $B, B \neq 0$, it follows that

$$v(\xi) = \frac{B}{\phi - c}. \tag{13}$$

The first equation of (2) reads as

$$-c\phi''' = -(A + c)\phi' + 3\phi\phi' - \sigma\left[\frac{1}{2}(\phi')^2 + \phi\phi''\right]' + e_0 v v'.$$

Integrating this equation yields

$$(\sigma\phi - c)\phi'' = -\frac{1}{2}\sigma(\phi')^2 - (A + c)\phi + \frac{3}{2}\phi^2 + \frac{e_0 B^2}{2(\phi - c)^2} - \frac{1}{2}g, \tag{14}$$

where g is an integration constant. Equation (14) is equivalent to the following two-dimensional system:

$$\frac{d\phi}{d\xi} = y, \quad \frac{dy}{d\xi} = \frac{-\sigma y^2(\phi - c)^2 + (\phi - c)^2[3\phi^2 - 2(A + c)\phi - g] + e_0 B^2}{2(\phi - c)^2(\sigma\phi - c)}, \tag{15}$$

which admits the following first integral:

$$H(\phi, y) = y^2(\sigma\phi - c) - \phi^3 + (A + c)\phi^2 + g\phi + \frac{e_0 B^2}{(\phi - c)} = h. \tag{16}$$

Without loss of generality, the wave speed $c > 0$ is given. Then, system (15) is a four-parameter planar dynamical system with the parameter tuple (A, B, g, σ).

Assume $A > 0$. Imposing the transformation $d\xi = (\phi - c)^2(\sigma\phi - c)d\zeta$ for $\phi \neq c, \frac{c}{\sigma}$ on system (15) with $e_0 = \pm 1$, leads to the following regular system:

$$\frac{d\phi}{d\zeta} = y(\phi - c)^2(\sigma\phi - c), \quad \frac{dy}{d\zeta} = -\frac{1}{2}\sigma y^2(\phi - c)^2 + \frac{1}{2}[(\phi - c)^2(3\phi^2 - 2(A + c)\phi - g) + e_0 B^2]. \tag{17}$$

Apparently, two singular lines $\phi = c$ and $\phi = \frac{c}{\sigma}$ are two invariant straight line solutions of (17). Near these two straight lines, the variable "ζ" is a fast variable while the variable "ξ" is a slow variable in the sense of the geometric singular perturbation theory.

To see the equilibrium points of (17), let us mark and calculate the following

$$f(\phi) = (\phi - c)^2(3\phi^2 - 2(A + c)\phi - g) + e_0 B^2, \tag{18}$$

$$f'(\phi) = 2(\phi - c)[6\phi^2 - 3(A + 2c)\phi + c(A + c) - g], \tag{19}$$

$$f''(\phi) = 2(18\phi^2 - 6(A + 4c)\phi + c(4A + 7c) - 2g. \tag{20}$$

Apparently, $f'(\phi)$ has one zero at $\phi = \phi_{s1} = c$. When $\Delta = 9A^2 + 12Ac + 12c^2 + 24g > 0$, $f'(\phi)$ has two zeros at $\phi = \tilde{\phi}_{1,2} = \frac{1}{12}[3(A + 2c) \mp \sqrt{\Delta}]$. So, we have $f(c) = e_0 B^2$, $f'(c) = 0$, and $f''(c) = 2(c^2 - 2cA - g)$, $f(0) = e_0 B^2 - gc^2$.

In the ϕ-axis, the equilibrium points $E_j(\phi_j, 0)$ of (17) satisfy $f(\phi_j) = 0$. Geometrically, for a fixed $c > 0$, the real zeros ϕ_j ($j = 1, 2$ or $j = 1, 2, 3, 4$) of the function $f(\phi)$ can be determined by the intersection points of the quadratic curve $y = 3\phi^2 - 2(A + c)\phi - g$ and the hyperbola $y = -\frac{e_0 B^2}{(\phi - c)^2}$. Obviously, system (17) has at most 4 equilibrium points at $E_j(\phi_j, 0)$, $j = 1, 2, 3, 4$. On the straight line $\phi = c$, there is no equilibrium point of (17) if $B \neq 0$. On the straight line $\phi = \frac{c}{\sigma}$, there exist two equilibrium points $S_{\mp}\left(\frac{c}{\sigma}, \mp Y_s\right)$ of (17) with $Y_s = \sqrt{\frac{f(\frac{c}{\sigma})}{\sigma(\frac{c}{\sigma} - c)^2}}$, if $\sigma f(\frac{c}{\sigma}) > 0$.

Next, we assume that $e_0 = 1$. Let $h_i = H(\phi_i, 0)$ and $h_s = H\left(\frac{c}{\sigma}, \mp Y_s\right)$, where H is given by (16).

For a given wave speed $c > 0$, assume that one of the following two conditions holds:

(1) $g > 0, c < A + \sqrt{A^2 + g}$. For given A and g, $f(\tilde{\phi}_1) < 0$, $f(\tilde{\phi}_2) < 0$.

(2) $g < 0$, $A^2 + 4g > 0$, $A - \sqrt{A^2 + g} < c < A + \sqrt{A^2 + g}$. For given A and g, $f(\tilde{\phi}_1) < 0$, $f(\tilde{\phi}_2) < 0$.

Then, Eq. (17) has four simple equilibrium points $E_j(\phi_j, 0)$, $j = 1, 2, 3, 4$, satisfying $\phi_1 < \tilde{\phi}_1 < \phi_2 < c < \phi_3 < \tilde{\phi}_2 < \phi_4$. Notice that for every $j = 1, 2, 3, 4$, ϕ_j does not depend on the parameter σ.

Suppose that $\sigma < 1$. Then, we have the following different topological phase portraits of Eq. (15) shown in Figs. 4(a)–4(c).

Let us first consider exact solutions of the orbits shown in Fig. 4(a).

From (16), for a given integral constant h, we have

$$\begin{aligned} y^2 &= \frac{(\phi - c)[\phi^3 - (A + 2c)\phi^2 - g\phi + h] - e_0 B^2}{(\phi - c)(\sigma\phi - c)} \equiv \frac{G(\phi)}{(\phi - c)(\sigma\phi - c)} \\ &= \frac{\phi^4 - (A + 2c)\phi^3 + (c^2 + Ac - g)\phi^2 + (h + cg)\phi - (ch + e_0 B^2)}{(\phi - c)(\sigma\phi - c)}. \end{aligned} \tag{21}$$

In light of the first equation of (15) and taking integration on a branch of the invariant curve $H(\phi, y) = h$ with initial value $\phi(\xi_0) = \phi_0$, one can obtain

$$\xi - \xi_0 = \pm \int_{\phi_0}^{\phi} \sqrt{\frac{(\phi - c)(\sigma\phi - c)}{G(\phi)}} d\phi. \tag{22}$$

(i) The homoclinic orbit of system (15) to the saddle point $E_3(\phi_3, 0)$ is a closed branch of the level set $H(\phi, y) = h_3$, which is around the center $E_4(\phi_4, 0)$ in Fig. 4(a). In this case, function $G(\phi)$ in (21)

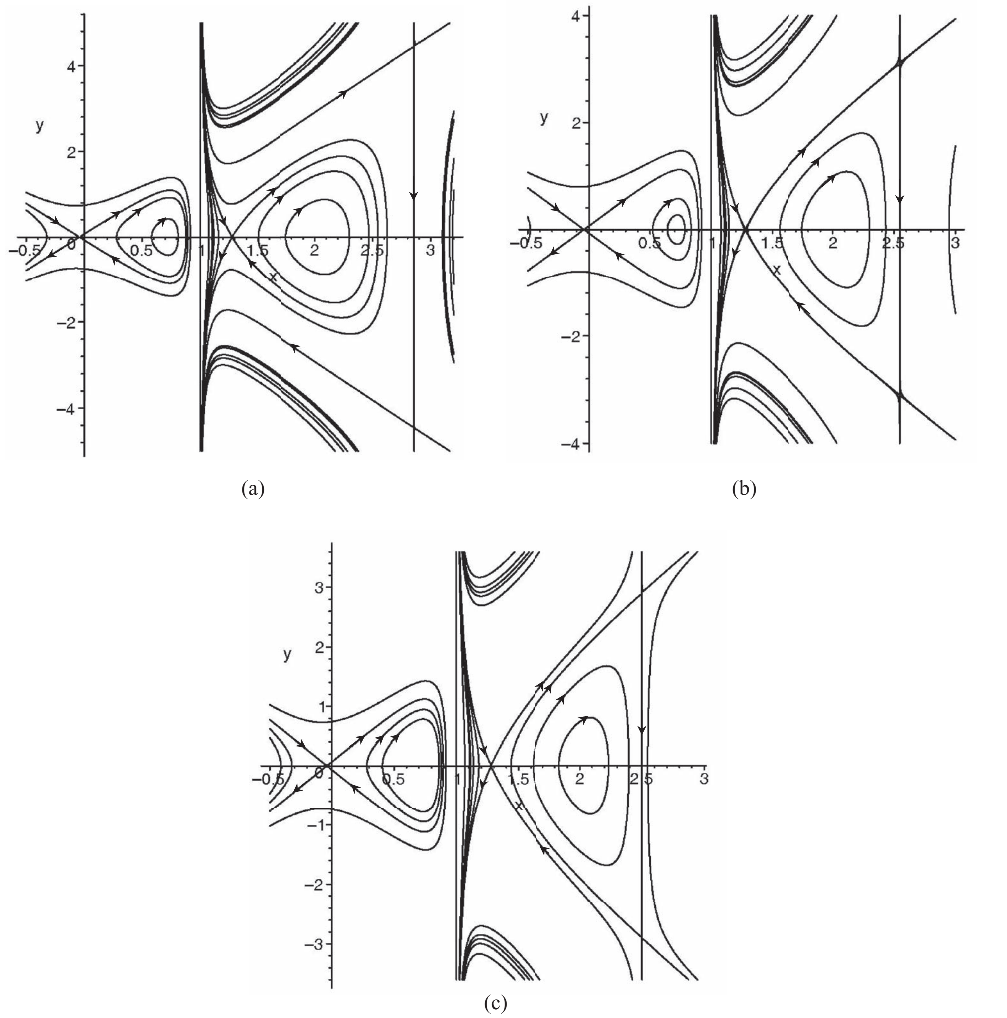


FIG. 4. The bifurcations of phase portraits of system (15) when $\sigma < 1$ and $\phi_4 < \frac{c}{\sigma}$.

can be written as $G(\phi) = (\phi_M - \phi)(\phi - \phi_3)^2(\phi - \phi_m)$, where $(\phi_M, 0)$ is the intersection point of the homoclinic orbit with the ϕ-axis. Thus, the right-hand side of (22) reads as

$$\sqrt{\sigma}\int_{\phi}^{\phi_M} \frac{\sqrt{(\frac{c}{\sigma}-\phi)(\phi-c)}d\phi}{(\phi-\phi_3)\sqrt{(\phi_M-\phi)(\phi-\phi_m)}} = \sqrt{\sigma}\int_{\phi}^{\phi_M}\left[\frac{\phi}{\sqrt{F_1(\phi)}} + \frac{A_{11}}{\sqrt{F_1(\phi)}} + \frac{A_{12}}{(\phi-\phi_3)\sqrt{F_1(\phi)}}\right], \tag{23}$$

where $F_1(\phi) = (\frac{c}{\sigma}-\phi)(\phi_M-\phi)(\phi-c)(\phi-\phi_m)$, $A_{11} = \phi_3 - c(\sigma^{-1}+1)$, $A_{12} = c^2\sigma^{-1} + \phi_3[\phi_3 - c(\sigma^{-1}+1)]$.

So, we have the following parametric representation of the solitary wave solution of (2):

$$\begin{aligned} \phi(\chi) &= \tfrac{\phi_M - \frac{c}{\sigma}\alpha_1^2\mathrm{sn}^2(\chi,k)}{1-\alpha_1^2\mathrm{sn}^2(\chi,k)}, \\ \xi(\chi) &= \tilde{g}\sqrt{\sigma}\Big[\pm\Big(A_{11} + \tfrac{c}{\sigma} + \tfrac{\sigma A_{12}}{(c-\sigma\phi_3)}\Big)\chi \\ &\qquad +(\phi_M - \tfrac{c}{\sigma})\Pi(\arcsin(\mathrm{sn}(\chi,k)), \alpha_1^2, k) \\ &\qquad + \tfrac{A_{12}(c-\sigma\phi_M)}{(\phi_M-\phi_3)(c-\sigma\phi_3)}\Pi(\arcsin(\mathrm{sn}(\chi,k)), \alpha_2^2, k)\Big], \end{aligned} \tag{24}$$

where $\tilde{g} = \frac{2}{\sqrt{(\frac{c}{\sigma}-c)(\phi_M-\phi_m)}}$, $\alpha_1^2 = \frac{\phi_M - c}{\frac{c}{\sigma}-c}$, $\alpha_2^2 = \frac{\alpha_1^2(\frac{c}{\sigma}-\phi_3)}{\phi_M-\phi_3}$, $k^2 = \frac{(\phi_M-c)(\frac{c}{\sigma}-\phi_m)}{(\frac{c}{\sigma}-c)(\phi_M-\phi_m)}$, $\Pi(\cdot, \alpha^2, k)$ is the elliptic integral of the third kind, $\mathrm{sn}(u, k)$ is the Jacobian elliptic function (see Byrd and Fridman[20]).

(ii) The homoclinic orbit of system (15) to saddle point $E_1(\phi_1, 0)$ is a closed branch of the level set $H(\phi, y) = h_1$, which is around the center $E_2(\phi_2, 0)$ in Figs. 4(a)–4(c). In this case, function $G(\phi)$ in (21) can be written as $G(\phi) = (\phi_L - \phi)(\phi_{M_1} - \phi)(\phi - \phi_1)^2$, where $(\phi_{M_1}, 0)$ is the intersection point of the homoclinic orbit with the ϕ-axis. Hence, the integral on the right side of (22) leads to

$$\sqrt{\sigma}\int_{\phi}^{\phi_{M_1}} \frac{\sqrt{(\frac{c}{\sigma}-\phi)(c-\phi)}d\phi}{(\phi-\phi_1)\sqrt{(\phi_L-\phi)(\phi_{M_1}-\phi)}} = \sqrt{\sigma}\int_{\phi}^{\phi_{M_1}}\left[\frac{\phi}{\sqrt{F_2(\phi)}} + \frac{A_{21}}{\sqrt{F_2(\phi)}} + \frac{A_{22}}{(\phi-\phi_1)\sqrt{F_2(\phi)}}\right], \tag{25}$$

where $F_2(\phi) = (\phi_L - \phi)(\frac{c}{\sigma} - \phi)(c - \phi)(\phi_{M_1} - \phi)$, $A_{21} = \phi_1 - c(\sigma^{-1} + 1)$, $A_{22} = c^2\sigma^{-1} + \phi_1[\phi_1 - c(\sigma^{-1} + 1)]$.

Therefore, we obtain the following parametric representation of solitary wave solution of (2):

$$\begin{aligned} \phi(\chi) &= \tfrac{(\phi_L-c)\phi_{M_1} - c(\phi_L-\phi_{M_1})\mathrm{sn}^2(\chi,k)}{(\phi_L-c)-(\phi_L-\phi_{M_1})\mathrm{sn}^2(\chi,k)},\\ \xi(\chi) &= \hat{g}\sqrt{\sigma}\Big[\left(A_{21} + \tfrac{c^2}{\sigma\phi_{M_1}} + \tfrac{A_{22}}{\phi_L-\phi_1}\right)\chi \\ &\qquad + \tfrac{c(\phi_{M_1}-c)}{\sigma\phi_{M_1}}\Pi(\arcsin(\mathrm{sn}(\chi,k)), \alpha_3^2, k) \\ &\qquad + \tfrac{A_{22}(\phi_L-\frac{c}{\sigma})}{(\frac{c}{\sigma}-\phi_1)(\phi_L-\phi_1)}\Pi(\arcsin(\mathrm{sn}(\chi,k)), \alpha_4^2, k)\Big], \end{aligned} \tag{26}$$

where $\hat{g} = \frac{2}{\sqrt{(\phi_L-c)(\frac{c}{\sigma}-\phi_{M_1})}}$, $\alpha_3^2 = \frac{\phi_L-\phi_{M_1}}{\phi_{M_1}-c}$, $\alpha_4^2 = \frac{(c-\phi_1)\alpha^2}{\phi_{M_1}-\phi_1}$, $k^2 = \frac{(\frac{c}{\sigma}-c)(\phi_L-\phi_{M_1})}{(\phi_L-c)(\frac{c}{\sigma}-\phi_{M_1})}$.

Second, we investigate exact parametric representations of the two heteroclinic orbits of (15) defined through $H(\phi, y) = h_3 = h_s$ in Fig. 4(b).

(iii) In this case, function $G(\phi)$ in (21) can be written as $G(\phi) = (\frac{c}{\sigma} - \phi)(\phi - \phi_3)^2(\phi - \phi_l)$.

Hence, taking integrals along the heteroclinic orbits E_3S_+ and E_3S_-, choosing initial value $\phi(0) = \frac{c}{\sigma}$, we arrive at

$$\pm\frac{\xi}{\sqrt{\sigma}} = \int_{\frac{c}{\sigma}}^{\phi}\frac{d\phi}{\sqrt{(\phi-c)(\phi-\phi_l)}} + (\phi_3 - c)\int_{\frac{c}{\sigma}}^{\phi}\frac{d\phi}{(\phi-\phi_3)\sqrt{(\phi-c)(\phi-\phi_l)}}. \tag{27}$$

Thus, we obtain a new peakon solution of (2) as follows:

$$\begin{aligned} \phi(\chi) &= \tfrac{B_0}{2}\left[e^{\chi} + \left(\tfrac{c-\phi_l}{2B_0}\right)^2 e^{-\chi} + \tfrac{c+\phi_l}{B_0}\right], \quad \chi \in (-\infty, 0] \\ \xi(\chi) &= \sqrt{\sigma}\left[\chi - \sqrt{\tfrac{\phi_3-c}{\phi_3-\phi_l}}\ln\left(\tfrac{\sqrt{X(\phi(\chi)-\phi_3)}+\sqrt{X(\phi_3)}}{\phi(\chi)-\phi_3} + \tfrac{2\phi_3-c-\phi_l}{2\sqrt{X(\phi_3)}}\right) + B_1\right] \end{aligned} \tag{28}$$

and

$$\begin{aligned} \phi(\chi) &= \tfrac{B_0}{2}\left[e^{-\chi} + \left(\tfrac{c-\phi_l}{2B_0}\right)^2 e^{\chi} + \tfrac{c+\phi_l}{B_0}\right], \quad \chi \in [0, \infty), \\ \xi(\chi) &= \sqrt{\sigma}\left[\chi + \sqrt{\tfrac{\phi_3-c}{\phi_3-\phi_l}}\ln\left(\tfrac{\sqrt{X(\phi(\chi)-\phi_3)}+\sqrt{X(\phi_3)}}{\phi(\chi)-\phi_3} + \tfrac{2\phi_3-c-\phi_l}{2\sqrt{X(\phi_3)}}\right) - B_1\right], \end{aligned} \tag{29}$$

where $X(\phi) = (\phi - c)(\phi - \phi_l)$, $B_0 = \sqrt{X(\frac{c}{\sigma})} + \frac{c}{\sigma} - \frac{1}{2}(c + \phi_l)$,
$B_1 = \sqrt{\frac{\phi_3-c}{\phi_3-\phi_l}}\ln\left(\frac{\sqrt{X(\frac{c}{\sigma}-\phi_3)}+\sqrt{X(\phi_3)}}{\frac{c}{\sigma}-\phi_3} + \frac{2\phi_3-c-\phi_l}{2\sqrt{X(\phi_3)}}\right)$.

Third, we consider exact cuspon solutions of (2) shown in Fig. 4(c).

(iv) The stable and unstable manifolds in the right phase plane of the critical point $E_3(\phi_3, 0)$, defined by $H(\phi, y) = h_3$ in (16), approach the singular straight line $\phi = \frac{c}{\sigma}$. The function $G(\phi)$ in (21) can be written as $G(\phi) = (\phi_L - \phi)(\phi - \phi_3)^2(\phi - \phi_l)$. On the basis of **Theorem B** in Sec. I, we can take initial value $\phi(0) = \frac{c}{\sigma}$. Thus, the right-hand side of (22) reads as

$$\sqrt{\sigma}\int_{\phi}^{\frac{c}{\sigma}} \frac{\sqrt{(\frac{c}{\sigma}-\phi)(\phi-c)}d\phi}{(\phi-\phi_3)\sqrt{(\phi_L-\phi)(\phi-\phi_l)}} = \sqrt{\sigma}\int_{\phi}^{\frac{c}{\sigma}}\left[\frac{\phi}{\sqrt{F_3(\phi)}} + \frac{A_{11}}{\sqrt{F_3(\phi)}} + \frac{A_{12}}{(\phi-\phi_3)\sqrt{F_3(\phi)}}\right], \quad (30)$$

where $F_3(\phi) = (\phi_L - \phi)(\frac{c}{\sigma} - \phi)(\phi - c)(\phi - \phi_l)$, $A_{11} = \phi_3 - c(\sigma^{-1} + 1)$, $A_{12} = c^2\sigma^{-1} + \phi_3[\phi_3 - c(\sigma^{-1} + 1)]$.

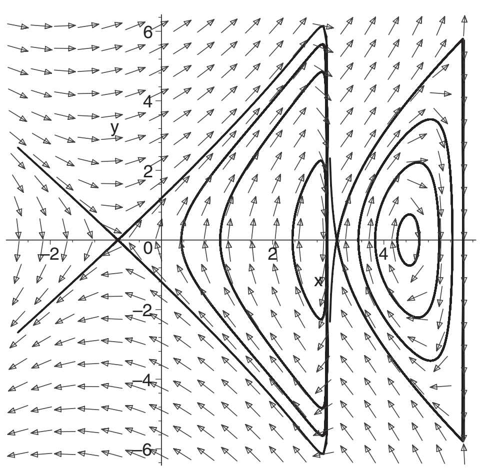
So, we have the following parametric representation of the cuspon solution to (2):

$$\begin{aligned}
\phi(\chi) &= \frac{\frac{c}{\sigma} - \phi_l\alpha_5^2\mathrm{sn}^2(\chi,k)}{1-\alpha_1^2\mathrm{sn}^2(\chi,k)},\\
\xi(\chi) &= \breve{g}\sqrt{\sigma}\left[\left(A_{11} + \phi_L + \frac{A_{12}}{(\phi_L-\phi_3)}\right)\chi\right.\\
&\qquad + (\tfrac{c}{\sigma} - \phi_L)\Pi(\arcsin(\mathrm{sn}(\chi,k)), \alpha_5^2, k)\\
&\qquad \left.+ \frac{A_{12}(\phi_L - \frac{c}{\sigma})}{(\phi_L-\phi_3)(\frac{c}{\sigma}-\phi_3)}\Pi(\arcsin(\mathrm{sn}(\chi,k)), \alpha_6^2, k)\right],
\end{aligned} \quad (31)$$

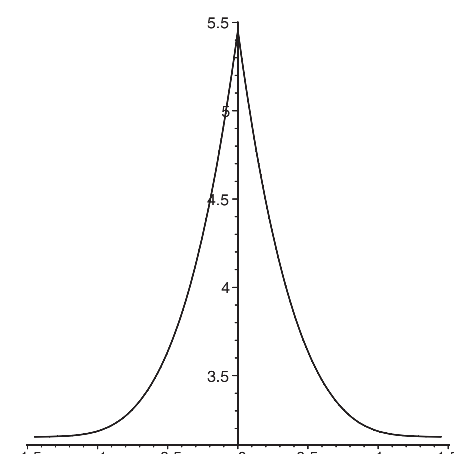
where $\breve{g} = \frac{2}{\sqrt{(\frac{c}{\sigma}-\phi_l)(\phi_L-c)}}$, $\alpha_5^2 = \frac{\frac{c}{\sigma}-c}{\phi_L-c}$, $\alpha_6^2 = \frac{\alpha_5^2(\phi_L-\phi_3)}{(\frac{c}{\sigma}-\phi_3)}$, $k^2 = \frac{(\phi_L-\phi_l)(\frac{c}{\sigma}-c)}{(\frac{c}{\sigma}-\phi_l)(\phi_L-c)}$.

As an example, we take the following parameter values: $A = 2.5$, $B = 0.6$, $g = 10.508762$, $\sigma = 0.55$, and $c = 3$. Then, we have $\phi_1 = -0.7849973408$, $\phi_2 = 2.856376463$, $\phi_3 = 3.153057853$, $\phi_4 = 4.442229691$, $\phi_s = 5.454545455$, $Y_s = 5.847639686$, and $h_1 = -4.471516330$, $h_2 = 49.07945187$, $h_3 = h_s = 58.81955436$, $h_4 = 67.80535766$. Under this parameter condition, the phase portrait of system (15) is shown in Fig. 5(a). Corresponding to the curve defined by $H(\phi, y) = h_3 = h_s$, we obtain a peaked solitary wave solution to equation (2) shown in Fig. 5(b).

Next, we take $A = 2.5$, $B = 0.6$, $g = 10.49$, $\sigma = 0.55$, and $c = 3$. Then, we have a similar phase portrait as Fig. 4(a). In this case, we know that $\phi_M = 5.45224874$, which is close to $\phi = \phi_{s1} = \frac{c}{\sigma} = 5.454545455$. In addition, $\phi_{M_1} = 2.993827677$ is close to $\phi = \phi_{s2} = c = 3$.



(a)



(b)

FIG. 5. A peaked solitary wave solution defined by formulas (28) and (29).

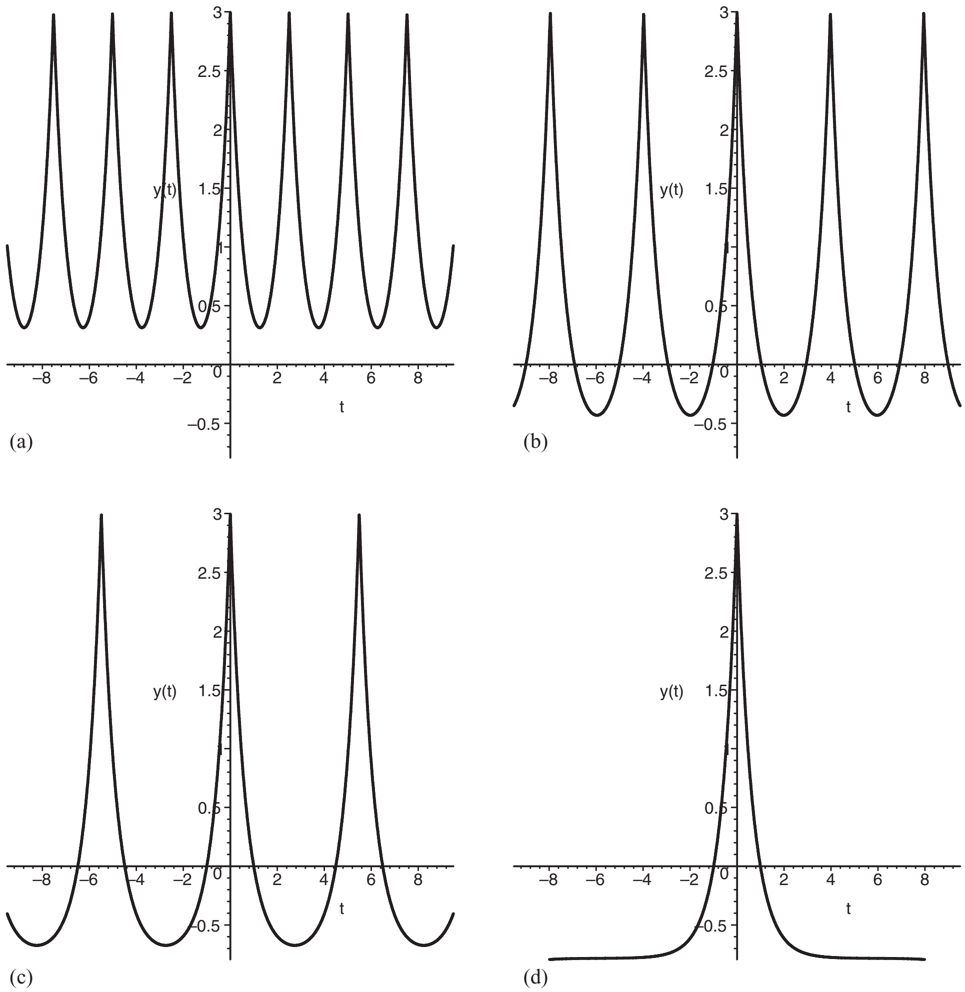


FIG. 6. The change of wave profiles of $\phi(\xi)$.

By taking initial values $\phi(0) = \phi_{M_1} - 0.001$, $\phi(0) = \phi_{M_1} - 0.0001$, $\phi(0) = \phi_{M_1} - 0.00001$, $\phi(0) = \phi_{M_1}$, respectively, we obtain three profiles of periodic cusped solutions and a pseudo-peakon solution of equation (2) shown in Figs. 6(a)–6(d).

By taking other initial values $\phi(0) = \phi_M - 0.1$, $\phi(0) = \phi_M - 0.001$, $\phi(0) = \phi_M - 0.00001$, $\phi(0) = \phi_M$, respectively, we may obtain three profiles of periodic cusped solutions and a pseudo-peakon solution of equation (2) shown in Figs. 7(a)–7(d).

In a summary, we obtain the following results.

Theorem 4. Suppose that the travelling wave system (15) of equations (2) satisfies the parameter condition $\sigma < 0$, $g > 0$, $c < A + \sqrt{A^2 + g}$ and for given A and g, $f(\tilde{\phi}_1) < 0$, $f(\tilde{\phi}_2) < 0$. Then, we have the following results:

(1) If the parameter group (A, B, σ, g, c) make the numbers ϕ_M and ϕ_{M_1} very close to $\phi = \frac{c}{\sigma}$ and $\phi = c$, respectively, then, corresponding to two homoclinic orbits of system (17) defined by $H(\phi, y) = h_3$ and $H(\phi, y) = h_1$ in (16), respectively, the formulas (24) and (26) give rise to two **pseudo-peakon solutions** of equation (2).

(2) Corresponding to the heteroclinic loop of system (17) defined by $H(\phi, y) = h_s$ in (16), formulas (28) and (29) gives rise to **a peakon solution** of equation (2).

(3) Corresponding to the stable and unstable manifolds in the right phase plane of the critical point $E_3(\phi_3, 0)$, defined by $H(\phi, y) = h_3$ in (16), formulas (31) gives rise to **a cuspon solution** of equation (2).

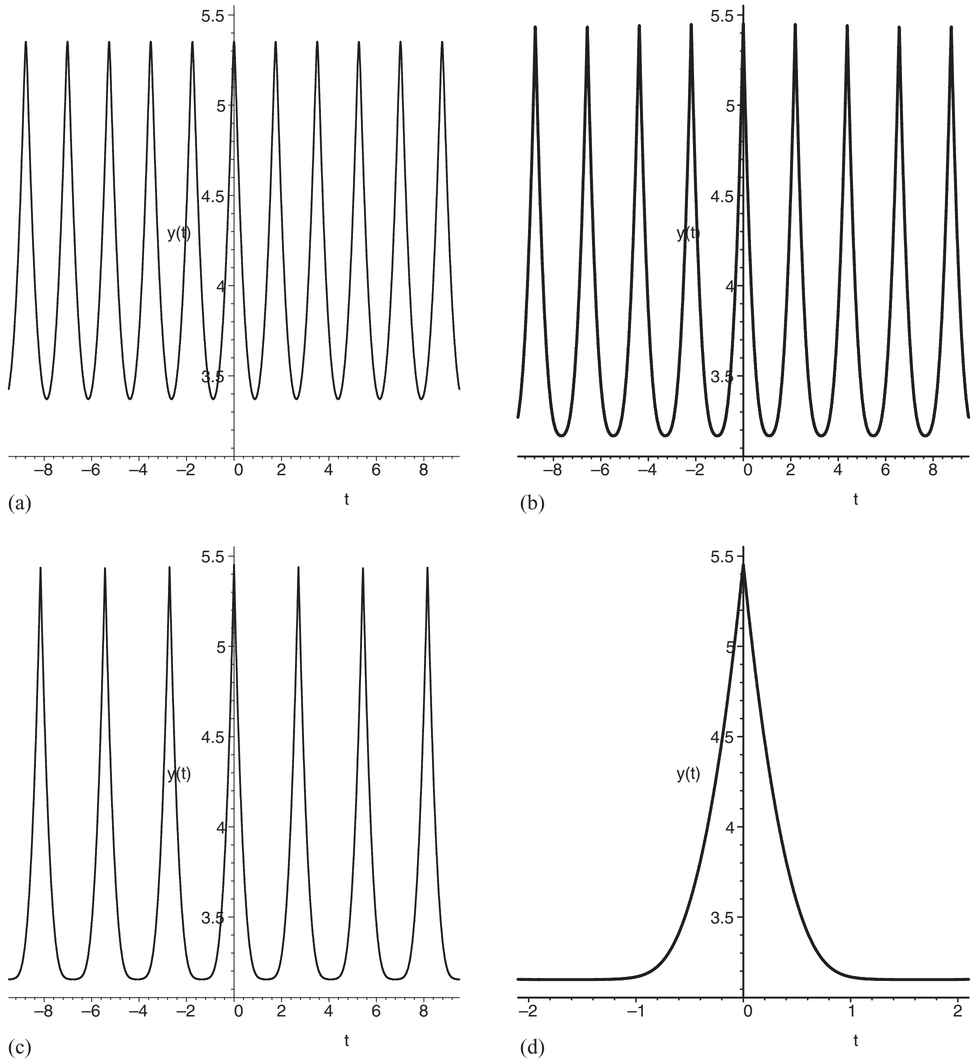


FIG. 7. The change of wave profiles of $\phi(\xi)$.

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