On the Restricted Toda and c-KdV Flows of Neumann Type*

ZHOU RuGuang[1] and QIAO ZhiJun[2,3]

[1]Department of Mathematics, Xuzhou Normal University, Xuzhou 221009, China

[2]Institute of Mathematics, Fudan University, Shanghai 200433, China

[3]Fachbereich 17, Mathematik, Universitaet-Gh Kassel, 34109 Kassel, Germany

Abstract *It is proven that on a symplectic submanifold the restricted c-KdV flow is just the interpolating Hamiltonian flow of invariant for the restricted Toda flow, which is an integrable symplectic map of Neumann type. They share the common Lax matrix, dynamical r-matrix and system of involutive conserved integrals. Furthermore, the procedure of separation of variables is considered for the restricted c-KdV flow of Neumann type.*

PACS numbers: 02.90.p, 03.40.t
Key words: restricted Toda flow, restricted c-KdV flow, r-matrix, Lax matrix

I. Introduction

It is well known that the nonlinearization method of Lax pair is applied to generate finite dimensional continuous integrable systems as well as discrete integrable systems.[1−7] In continuous case we can obtain finite dimensional integrable Hamiltonian systems, while in discrete case the integrable symplectic maps. For example, in continuous case two kinds of nonlinearized c-KdV flows (Bargmann and Neumann types) were proven to be completely integrable finite dimensional Hamiltonian systems by Cao and Geng.[4] And in discrete case two kinds of integrable symplectic maps (discrete Bargmann and Neumann types) were studied by Ragnisco, Cao and Wu.[6,7] Whether finite dimensional integrable Hamiltonian systems or integrable symplectic maps, all of them possess sufficiently many involutive conserved integrals. In general, an integrable symplectic map has a set of continuous involutive conserved integrals.[7,8]

In Ref. [9], we reported an interesting and amazing fact: the discrete and continuous integrable systems share the same Lax matrix and r-matrix with the good property of being nondynamical. Recently, Qiao and Strampp have found three further pairs of different continuous integrable systems sharing the common r-matrix again being nondynamical.[10] To our knowledge, these are so far the only four examples of pairs of different finite dimensional integrable systems possessing the above property. Then the question arises whether or not these pairs can be restricted into a symplectic submanifold to retain the above property. We will give a sure reply in this paper.

Applying the previous idea[11] to a symplectic submanifold, we shall show that on this symplectic submanifold the restricted c-KdV flow[4] is just the interpolating Hamiltonian flow of invariant of the restricted Toda flow being an integrable symplectic map of Neumann type. They share the common Lax matrix, dynamical r-matrix and system of involutive conserved integrals. The whole paper is organized as follows. In the next section, beginning with a Lax

matrix endowed with two different auxiliary matrices, we present another Lax representation for the restricted Toda flow (a discrete system of Neumann type) ever considered in Ref. [6] and the Lax representation for the restricted c-KdV flow (a continuous system of Neumann type). In Sec. III, introducing Dirac bracket on a symplectic submanifold, we inlay the restricted c-KdV flow in it. Then we give the common r-matrix (dynamical) of the restricted c-KdV flow and the restricted Toda flow under this Dirac bracket, which guarantees the integrability of them. Section IV is devoted to describing the separability of variables for the restricted c-KdV flow. Some remarks are given in the last section.

Before displaying our main results, let us give some basic symbols and notations. Let $dp \wedge dq = \sum_{j=1}^{N} dp_j \wedge dq_j$ be a standard symplectic structure in the Euclid space $R^{2N} = \{(p,q)|p = (p_1, \cdots, p_N), q = (q_1, \cdots, q_N)\}$, $\lambda_1, \cdots, \lambda_N$ be N arbitrary given distinct parameters, λ and μ be two different spectral parameters, and $\langle , \rangle$ stands for the standard inner-product in the Euclid space R^N. Denote Λ by $\Lambda = \text{diag}(\lambda_1, \cdots, \lambda_N)$. On symplectic manifold $(R^{2N}, dp \wedge dq)$ the Poisson bracket of two Hamiltonian functions F, G is defined by

$$\{F, G\} = \sum_{i=1}^{N} \Big(\frac{\partial F}{\partial q_i}\frac{\partial G}{\partial p_i} - \frac{\partial F}{\partial p_i}\frac{\partial G}{\partial q_i}\Big). \tag{1}$$

II. Lax Representations of the Restricted Toda and c-KdV Flows

In the light of thought in Ref. [11], we first introduce the following 2×2 traceless matrix (the Lax matrix)

$$L = L(\lambda, p, q) = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} + \sum_{j=1}^{N} \frac{1}{\lambda - \lambda_j} \begin{pmatrix} p_j q_j & -p_j^2 \\ q_j^2 & -p_j q_j \end{pmatrix} \overset{\text{def}}{=\!=} \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & -A(\lambda) \end{pmatrix}. \tag{2}$$

When viewing the variables q and p as the functions of continuous variable x, we give an auxiliary matrix U as

$$U = \begin{pmatrix} -\frac{1}{2}\lambda + \frac{1}{2}(\langle \Lambda q, q\rangle - 1) & \langle p, p\rangle \\ -1 & \frac{1}{2}\lambda - \frac{1}{2}(\langle \Lambda q, q\rangle - 1) \end{pmatrix}. \tag{3}$$

After a direct calculation, we have the following theorem.

Theorem 1. The continuous Lax equation

$$L_x = [U, L] \equiv UL - LU, \qquad L_x = \partial L/\partial x, \tag{4}$$

is equivalent to a continuous Neumann type of finite dimensional Hamiltonian system (called the restricted c-KdV flow)

$$p_{j,x} = -\frac{1}{2}\lambda_j p_j + \frac{1}{2}(\langle \Lambda q, q\rangle - 1)p_j + \langle p, p\rangle q_j,$$

$$q_{j,x} = -p_j + \frac{1}{2}\lambda_j q_j - \frac{1}{2}(\langle \Lambda q, q\rangle - 1)q_j, \qquad j = 1, 2, \cdots, N,$$

$$\langle q, q\rangle = 1, \qquad \langle q, p\rangle = \frac{1}{2}. \tag{5}$$

Set

$$u = \langle \Lambda q, q\rangle - 1, \qquad v = \langle p, p\rangle, \tag{6}$$

then equation (5) reads

$$p_{jx} = -\frac{1}{2}\lambda_j p_j + \frac{1}{2}up_j + vq_j, \qquad q_{jx} = -p_j + \frac{1}{2}\lambda_j q_j - \frac{1}{2}uq_j, \qquad j = 1, 2, \cdots, N,$$

$$\langle q, q\rangle = 1, \qquad \langle q, p\rangle = \frac{1}{2}, \tag{7}$$

which is the c-KdV spectral problem[12]

$$\Psi_x = \begin{pmatrix} -\frac{1}{2}\lambda + \frac{1}{2}u & v \\ -1 & \frac{1}{2}\lambda - \frac{1}{2}u \end{pmatrix} \Psi \tag{8}$$

with the above two constraints (6), $\lambda = \lambda_j$, $\Psi = (p_j, q_j)^T$. Simultaneously, the potentials u, v determined by Eq. (6) exactly give the Neumann type of constraint[4] $G_{-1} = (\frac{1}{2}, 1)^T = \sum_{j=1}^{N} (\delta\lambda_j/\delta u, \delta\lambda_j/\delta v)^T$ of the c-KdV spectral problem (8).

Now, we turn to the Lax matrix (2). After introducing another auxiliary matrix $\bar{U}$,

$$\bar{U} = \begin{pmatrix} 0 & a \\ -1/a & (\lambda - b)/a \end{pmatrix} \tag{9}$$

with $a^2 = \langle \Lambda q - p, \Lambda q - p \rangle + \langle \Lambda q, q \rangle - \langle \Lambda q, q \rangle^2$ and $b = \langle \Lambda q, q \rangle - 1$, we have the following theorem through a lengthy but direct calculation.

Theorem 2. The discrete Lax equation

$$L'\bar{U} = \bar{U}L, \qquad L' = L(\lambda, p', q') \tag{10}$$

is equivalent to a discrete Neumann type of finite dimensional symplectic map $\mathcal{H}$: $(p, q)^T \rightarrow (p', q')^T$ from R^{2N} to R^{2N},

$$p'_j = aq_j, \quad q'_j = a^{-1}(\lambda_j q_j - p_j - bq_j), \quad j = 1, 2, \cdots, N, \quad \langle q, q \rangle = 1, \quad \langle q, p \rangle = \tfrac{1}{2}, \tag{11}$$

which is called the restricted Toda flow.

Remark 1. When we understand the above two matrices L' and $\bar{U}$ in the sense $L' \rightarrow L_{n+1}$, $\bar{U} \rightarrow \bar{U}_n$ (i.e., $q \rightarrow q_n$, $p \rightarrow p_n$, $a \rightarrow a_n$, $b \rightarrow b_n$, here n is the discrete variable), then the restricted Toda flow (11) on the symplectic submanifold $M = \{(q, p) \in R^{2N} | \langle q, q \rangle = 1$, $\langle q, p \rangle = \tfrac{1}{2}\}$ is nothing but the discrete Neumann system studied by Ragnisco.[6]

Remark 2. For the restricted Toda flow (11), two other Lax representations were presented in Refs [6] and [13]. Comparing with them, our Lax matrix (2) is much like that in Ref. [13], but the auxiliary matrix $\bar{U}$ is evidently different. Due to the choice of our matrices (2) and $\bar{U}$, from the next section we can see that the procedure of r-matrix is simple.

III. Dynamical r-Matrix and Integrability

In the last section, starting from the Lax matrix (2), we obtained a discrete finite dimensional symplectic map (11) and a continuous finite dimensional Hamiltonian system (5) through introducing two different auxiliary matrices. Since their Lax matrices are the same, they should have a common r-matrix. In this section, the dynamical common r-matrix is presented, and with the aid of this r-matrix, equations (5) and (11) are shown to be completely integrable in Liouville's sense.

To see this, on the symplectic submanifold in R^{2N}, $M = \{(q, p) \in R^{2N} | F \equiv \langle q, q \rangle - 1 = 0$, $G \equiv \langle q, p \rangle - 1/2 = 0\}$, we introduce a Dirac bracket

$$\{f, g\}_D = \{f, g\} + \tfrac{1}{2}(\{f, F\}\{G, g\} - \{f, G\}\{F, g\}), \tag{12}$$

which can be easily proven to be a Poisson bracket. With Eq. (12), a direct calculation yields

Proposition 1. The restricted c-KdV flow (5) can cast the Hamiltonian canonical equation in the Dirac bracket

$$q_{j,x} = \{q_j, H\}_D, \quad p_{j,x} = \{p_j, H\}_D, \quad j = 1, 2, \cdots, N$$

with a Hamiltonian

$$H = \tfrac{1}{2}\langle \Lambda p, q \rangle - \tfrac{1}{2}\langle p, p \rangle. \tag{13}$$

Proposition 2. Let $A(\lambda)$, $B(\lambda)$, $C(\lambda)$ be defined in Eq. (2). Then

$$\{A(\lambda), A(\mu)\}_D = \{C(\lambda), C(\mu)\}_D = 0,$$

$$\{B(\lambda), B(\mu)\}_D = 2[B(\lambda) - B(\mu)] + 4[A(\lambda)B(\mu) - A(\mu)B(\lambda)],$$

$$\{A(\lambda), B(\mu)\}_D = \frac{2}{\mu - \lambda}[B(\lambda) - B(\mu)] - 2C(\lambda)B(\mu),$$

$$\{A(\lambda), C(\mu)\}_D = \frac{2}{\mu - \lambda}[C(\mu) - C(\lambda)] + 2C(\lambda)C(\mu),$$

$$\{B(\lambda), C(\mu)\}_D = \frac{4}{\mu - \lambda}[A(\lambda) - A(\mu)] + 2C(\mu) - 4A(\lambda)C(\mu). \tag{14}$$

Let $L_1(\lambda) = L(\lambda) \otimes I$ and $L_2(\mu) = I \otimes L(\mu)$, here I is the 2×2 unit matrix. Then from the above proposition we have

Theorem 3. The Lax matrix L defined by Eq. (2) satisfies the fundamental Poisson bracket

$$\{L(\lambda) \overset{\otimes}{,} L(\mu)\}_D = [r_{12}(\lambda, \mu), L_1(\lambda)] - [r_{21}(\lambda, \mu), L_2(\mu)] \tag{15}$$

with a dynamical r-matrix $r_{12}(\lambda, \mu) = -[2/(\mu - \lambda)P + S_{12}(\lambda, \mu), \quad r_{21}(\lambda, \mu) = Pr_{12}(\mu, \lambda)P$, where

$$S_{12} = (E_{11} - E_{22}) \otimes E_{12} + E_{11} \otimes \begin{pmatrix} C(\mu) & 0 \\ 0 & 0 \end{pmatrix} + E_{12} \otimes \begin{pmatrix} 0 & -B(\mu) \\ C(\mu) & 0 \end{pmatrix} + E_{22} \otimes \begin{pmatrix} 0 & 2A(\mu) \\ 0 & C(\mu) \end{pmatrix},$$

E_{ij} is a special 2×2 matrix with i-th row and j-th column element $e_{ij} = 1$, other elements 0; $P = \frac{1}{2}(I + \sum_{j=1}^{3} \sigma_j \otimes \sigma_j)$ is a permutation matrix; σ_j $(j = 1, 2, 3)$ are Pauli matrices.

By the r-matrix relation expression (15), we immediately obtain[14]

$$\{L^2(\lambda) \overset{\otimes}{,} L^2(\mu)\}_D = [\bar{r}_{12}(\lambda, \mu), L_1(\lambda)] - [\bar{r}_{21}(\mu, \lambda), L_2(\mu)],$$

$$\bar{r}_{ij}(\lambda, \mu) = \sum_{k=0}^{1} \sum_{l=0}^{1} L_1^{1-k}(\lambda) L_2^{1-l}(\mu) r_{ij}(\lambda, \mu) L_1^k(\lambda) L_2^l(\mu), \qquad ij = 12, \ 21. \tag{16}$$

Then, it follows from Eq. (16) that

$$4\{\operatorname{Tr} L^2(\lambda), \operatorname{Tr} L^2(\mu)\}_D = \operatorname{Tr}\{L^2(\lambda) \overset{\otimes}{,} L^2(\mu)\}_D = 0. \tag{17}$$

Apparently, the Lax matrix (2) yields

$$\det L(\lambda) = -\frac{1}{2} \operatorname{Tr} L^2(\lambda) = \frac{1}{4} + \sum_{i=1}^{N} \frac{E_i}{\lambda - \lambda_i},$$

$$E_i = E_i(p, q) = p_i q_i + \sum_{j \neq i, j=1}^{N} \frac{(q_i p_j - p_i q_j)^2}{\lambda_j - \lambda_i}, \qquad i = 1, \cdots, N. \tag{18}$$

Substituting Eq. (18) into Eq. (17), we get $\{E_i, E_j\}_D = 0$, $i, j, = 1, \cdots, N$. For the restricted Toda flow (11) on the symplectic submanifold M, we have $E_i(p', q') = E_i(p, q)$ as well as $\sum_{i=1}^{n} E_i = \langle p, q \rangle = \frac{1}{2}$ from the discrete Lax equation (10). Therefore, among $E_1, E_2, \cdots, E_N$ only $E_1, E_2, \cdots, E_{N-1}$ are independent of M. Thus, we obtain

Proposition 3. The restricted Toda flow $\mathcal{H}$ is completely integrable, and its independent and invariant $(N - 1)$-involutive systems are $\{E_i\}_{i=1}^{N-1}$.

For the restricted c-KdV flow on M, we have $H = \frac{1}{2} \sum_{j=1}^{N} \lambda_j E_j - \frac{1}{8}$, which implies $\{H, E_j\}_D = 0$, $j = 1, 2, \cdots, N$. Therefore, we get

Proposition 4. The restricted c-KdV flow (H) (5) is completely integrable, and its independent $(N - 1)$-involutive systems are $\{E_k\}_{k=1}^{N-1}$, too.

Remark 3. As shown above and in Ref. [9], the restricted (i.e. Neumann type) Toda flow and restricted c-KdV flow, and the constrained (i.e. Bargmann type) Toda flow and the constrained c-KdV flow share the completely same involutive conserved integrals, respectively. Thus, we say that both the restricted and constrained finite dimensional integrable c-KdV flows are the interpolating Hamiltonian flow of invariant of the corresponding Toda integrable symplectic map.

IV. Separability of Variables for the Restricted c-KdV Flow

The separation of variables for the restricted Toda flow was studied in Ref. [6]. In this section, we consider the separation of variables of the restricted c-KdV flow on M.

(14)

om the

bracket
(15)

$(\mu, \lambda)P,$

$A(\mu)$)
$C(\mu)$) ,

ments 0;

(16)

(17)

(18)

For the
$E_i(p, q)$
e, among

ependent

h implies

its inde-

a flow and
nstrained
Thus, we
vs are the
ymplectic

]. In this

As usual,[15] let us first introduce new coordinates on the $2N - 2$ dimensional symplectic submanifold M in R^{2N}. Let $u_1, \cdots, u_{N-1}$ be the $N - 1$ zero points of $C(\lambda)$ and $v_1, \cdots, v_{N-1}$ be the half values of $A(\lambda)$ on these points, i.e.,

$$\sum_{j=1}^{N} \frac{q_j^2}{\lambda - \lambda_j} = \frac{\prod_{k=1}^{N-1}(\lambda - u_k)}{\prod_{j=1}^{N}(\lambda - \lambda_j)}, \qquad v_k = \frac{1}{2}A(u_k), \qquad k = 1, 2, \cdots, N - 1. \tag{19}$$

For these new $N - 1$ pairs of variables u_k, v_k, it is easy to show[16]

Proposition 5. New coordinates $\{u_k, v_k\}_{k=1}^{N-1}$ are canonically conjugated on the symplectic submanifold M in R^{2N}, i.e.,

$$\{u_j, u_k\}_D = \{v_j, v_k\}_D = 0, \qquad \{u_j, v_k\}_D = \delta_{jk}, \qquad j, k = 1, 2, \cdots, N - 1. \tag{20}$$

Write

$$\det L(\lambda) = P(\lambda)/K(\lambda), \tag{21}$$

where $K(\lambda) = 4\prod_{k=1}^{N}(\lambda - \lambda_k) = 4\sum_{j=1}^{N}(-1)^j \alpha_j \lambda^{N-j}$, and $P(\lambda) = \lambda^N + P_{N-1}\lambda^{N-1} + \cdots + P_0$ is a polynomial of order N in terms of λ. Equation (21) yields

$$P_{N-1} = -\alpha_1 + 2, \qquad P_{N-2} = \alpha_2 - 2\alpha_1 + 8H + 1. \tag{22}$$

Generally, each P_k is consisted of involutive functions E_j and constants λ_j, thus P_k and P_j are in involution. For our Hamiltonian function H, we have $H = \frac{1}{8}(P_{N-2} - 1 + 2\alpha_1 - \alpha_2)$.

In the following, motivated by Refs [17] and [18] we consider the separation of variables for the Hamiltonian $P(\lambda)$. From Eq. (21), we know

$$P(u_k) = -4v_k^2 K(u_k), \qquad k = 1, \cdots, N - 1. \tag{23}$$

Using the Lagrange interpolation, $P(\lambda)$ can be expressed as

$$P(\lambda) = Q(\lambda)\Big(\lambda + 2 - \alpha_1 + \sum_{k=1}^{N-1} u_k\Big) - 4\sum_{k=1}^{N-1} \frac{Q(\lambda)K(u_k)}{(\lambda - u_k)Q'(u_k)}v_k^2, \qquad Q(\lambda) = \prod_{k=1}^{N-1}(\lambda - u_k). \tag{24}$$

Replace v_k by $\partial S/\partial u_k$ and interpret the coefficients of $P(\lambda)$, $P_0, P_1, \cdots, P_{N-2}$, as the integral constants. The Hamilton–Jacobi equation follows

$$Q(\lambda)\Big(\lambda + 2 - \alpha_1 + \sum_{k=1}^{N-1} u_k\Big) - 4\sum_{k=1}^{N-1} \frac{Q(\lambda)K(u_k)}{(\lambda - u_k)Q'(u_k)}\Big(\frac{\partial S}{\partial u_k}\Big)^2 = P(\lambda). \tag{25}$$

We wish to look for the action function S with the form

$$S(u_1, \cdots, u_{N-1}) = \sum_{k=1}^{N-1} s_k(u_k). \tag{26}$$

Inserting Eq. (26) into Eq. (25), dividing the two sides by $Q(\lambda)$ and taking residue of Eq. (25) at $\lambda = u_k$, we get $4(\partial s_k/\partial u_k)^2 = -P(u_k)/K(u_k)$, $k = 1, \cdots, N - 1$. Finally, we have

$$S = \frac{1}{2}\sum_{k=1}^{N-1} \int_0^{u_k} \sqrt{-\frac{P(\lambda)}{K(\lambda)}}\, d\lambda. \tag{27}$$

Therefore the linearized coordinates are

$$Q_j = \frac{\partial S}{\partial P_j} = \frac{1}{4}\sum_{k=1}^{N-1} \int_0^{u_k} \frac{\lambda^j}{\sqrt{-K(\lambda)P(\lambda)}}\, d\lambda, \qquad j = 0, \cdots, N - 2. \tag{28}$$

Thus, on the symplectic submanifold $(M, \sum_{j=0}^{N-2} dQ_j \wedge dP_j)$ the phase flow generated by the Hamiltonian function H is

$$P_j = P_j^0, \qquad Q_j = Q_j^0 + \frac{1}{8}\delta_{j,N-2}x, \qquad j = 0, 1, \cdots, N - 2, \tag{29}$$

where P_j^0 and Q_j^0 are $2N - 2$ constants. This is the linearized equation of the restricted c-KdV flow on M. In a further procedure we can obtain the algebraic-geometric solution of c-KdV equation using the method described in Refs [11] and [16], which is omitted here.

V. Conclusion

Before starting our conclusions, it seems necessary to re-stress the two "technical terms" usually used in the theory of integrable systems in order to avoid confusion: one is "constrained flow", which means the finite dimensional Hamiltonian flow or symplectic map in R^{2N} under the Bargmann type of constraint; the other "restricted flow" means the finite dimensional Hamiltonian flow or symplectic map on some symplectic submanifold of R^{2N} under the Neumann type of constraint. In the future, we shall insist on this principle.

In the present paper, we apply the idea proposed in Ref. [11] to some submanifold in R^{2N} determined to be symplectic by the Neumann type of constraint. By making use of the Dirac bracket (i.e. Poisson bracket on a symplectic submanifold), it results in the two restricted flows (one the discrete Toda symplectic map, the other the continuous c-KdV Hamiltonian system) sharing the same r-matrix being dynamical instead of nondynamical. This point is innately different from that in Refs [9] and [10].

Besides the pair of constrained Toda flow and c-KdV flow presented in Ref. [9], recently other three further pairs[19] of different constrained flows are found to possess the common r-matrix with an interesting property being nondynamical instead of dynamical. All those results were given under the Bargmann type of constraint in the continuous cases. As for the Neumann type of constraint, we first time discuss here the two restricted flows owning the same r-matrix being dynamical on a symplectic submanifold.

Since the constrained or restricted discrete Toda and continuous c-KdV flows have the same Lax matrix, r-matrix and involutive conserved integrals, we can come out this result: can the discrete Toda symplectic map become an exact discretization of the continuous c-KdV flow? This problem is still open. Additionally, because of the examples in Ref. [9] and this paper, we would like to give a further conjecture: whether can any finite dimensional continuous Hamiltonian flow correspond to a finite dimensional discrete symplectic map such that they share a common Lax matrix? If this is OK, then the discrete integrable systems will be greatly enlarged.

References

[1] C.W. CAO, Chin. Quart. J. Math. **3** (1988) 90.
[2] C.W. CAO, Sci. Sin. **A7** (1989) 701.
[3] Z.J. QIAO, Master Thesis, Zhengzhou University, China (1989).
[4] C.W. CAO and X.G. GENG, J. Phys. **A23** (1990) 4117.
[5] M. Antonowicz and S. Rauch-Wojciechowski, Phys. Lett. **A147** (1990) 455.
[6] O. Ragnisco, Phys. Lett. **A198** (1995) 295.
[7] O. Ragnisco, C.W. Cao and Y.T. Wu, J. Phys. **A28** (1995) 573.
[8] Z.J. QIAO, "Non-Dynamical r-Matrix and Algebraic-Geometric Solution for Discrete System", Research Report #91 (1997), School of Mathematical Sciences, Peking University, China.
[9] Z.J. QIAO and R.G. ZHOU, Phys. Lett. **A235** (1997) 35.
[10] Z.J. Qiao and W. Strampp, J. Math. Phys. **39** (1998) 3271.
[11] Z.J. QIAO, Ph. D. Thesis, Fudan University (1997).
[12] D. Levi, A. Sym and S. Rauch-Wojciechowski, J. Phys. **A16** (1983) 2423.
[13] O. Ragnisco and Y.B. Suris, *Algebraic Aspects of Integrable Systems*, eds A.S. Fokas and I.M. Gelfand, Birkäuser, Bosto (1997) p. 285.
[14] O. Babelon and C.M. Viallet, Phys. Lett. **B237** (1990) 411.
[15] E.K. Sklyanin, Prog. Theor. Phys. Suppl. **118** (1995) 35.
[16] R.G. ZHOU, Ph. D. Thesis, Fudan University (1997).
[17] O. Babelon and M. Talon, Nucl. Phys. **B379** (1992) 321.
[18] J. Harnad and P. Winternitz, Commun. Math. Phys. **172** (1995) 263.