A Unisonant r-Matrix Structure of Integrable Systems and Its Reductions *

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A new method is presented to generate finite dimensional integrable systems. Our starting point is a generalized Lax matrix instead of usual Lax pair. Then a unisonant r-matrix structure and a set of generalized Hamiltonian functions are constructed. It can be clearly seen that various constrained integrable flows by nonlinearization method, such as the c-AKNS, c-MKdV, c-Toda, etc., are derived from the reduction of this structure. Furthermore, some new integrable flows are produced.

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It is well-known that the nonlinearization technique[1] is a powerful tool to produce finite dimensional integrable systems. With the help of this method, many new completely integrable systems were found.[2,4] Each integrable system is generated through making nonlinearized procedure for a concrete spectral problem or Lax pair, and it has its own individuality. Then a natural question is whether or not there are a unified structure such that it can contain those individual integrable systems generated by nonlinearization method. In the present letter, we give an affirmable answer. We propose a new procedure to generate finite dimensional integrable systems from a generalized Lax matrix instead of usual Lax pair. To do so, we construct a unisonant r-matrix structure and a set of generalized integrable Hamiltonian functions through studying the fundamental Poisson bracket.

It can be clearly seen that various constrained integrable flows by nonlinearization method, such as the c-AKNS, c-MKdV, c-Toda, etc., are derived from the reduction of the structure. Moreover, some new integrable flows are produced from this structure. Let us first give some necessary notation in this letter: $dp\wedge dq$ stands for the standard symplectic structure in Euclidean space $R^{2N} = \{(p,q)|p=(p_1,\dots,p_N), q=(q_1,\dots,q_N)\}$, $\langle\cdot,\cdot\rangle$ the standard inner product in R^N; in $(R^{2N}, dp\wedge dq)$ the Poisson bracket of two Hamiltonian functions F, G is defined by

$$\{F,G\} = \sum_{i=1}^{N}\left(\frac{\partial F}{\partial q_i}\frac{\partial G}{\partial p_i} - \frac{\partial F}{\partial p_i}\frac{\partial G}{\partial q_i}\right) = \langle\frac{\partial F}{\partial q},\frac{\partial G}{\partial p}\rangle - \langle\frac{\partial F}{\partial p},\frac{\partial G}{\partial q}\rangle. \quad (1)$$

And $\lambda_1,\cdots,\lambda_N$ are N arbitrarily given distinct constants; λ and μ are the two different spectral parameters; $\Lambda = \mathrm{diag}(\lambda_1,\cdots,\lambda_N)$, $I_0 = \langle q,q\rangle$, $J_0 = \langle p,q\rangle$, $K_0 = \langle p,p\rangle$, $I_1 = \langle\Lambda p,p\rangle\langle\Lambda q,q\rangle$, $J_1 = \langle\Lambda p,q\rangle$; $a_0, a_1 = \text{const}$.

Denote all infinitely times differentiable functions on real field R by $C^\infty(R)$.

Consider the following matrix (called Lax matrix)

$$L(\lambda) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & -A(\lambda)\end{pmatrix}, \quad (2)$$

where

$$A(\lambda) = a_{-2}(I_1,J_1)\lambda^{-2} + a_{-1}(J_0)\lambda^{-1} + a_0 + a_1\lambda + \sum_{j=1}^{N}\frac{p_jq_j}{\lambda-\lambda_j}, \quad (3)$$

$$B(\lambda) = b_{-1}(I_0,J_0)\lambda^{-1} + b_0(J_0) - \sum_{j=1}^{N}\frac{q_j^2}{\lambda-\lambda_j}, \quad (4)$$

$$C(\lambda) = c_{-1}(J_0,K_0)\lambda^{-1} + c_0(J_0) + \sum_{j=1}^{N}\frac{p_j^2}{\lambda-\lambda_j}. \quad (5)$$

Now, we make an Assumption (P): $\{A(\lambda),A(\mu)\}$, $\{A(\lambda),B(\mu)\}$, $\{A(\lambda),C(\mu)\}$, $\{B(\lambda),B(\mu)\}$, $\{B(\lambda),C(\mu)\}$, $\{C(\lambda),C(\mu)\}$ are all expressed as some linear combinations of $A(\lambda),A(\mu),B(\lambda),B(\mu),C(\lambda),C(\mu)$, then we have:

Proposition 1: If the Assumption (P) holds, then $L(\lambda)$ only contains the following cases:

1. As $a_{-2}\neq \text{const}$, $a_0=b_0=c_0=a_1=0$, $a_{-1}=-J_0$, $b_{-1}=I_0$, $c_{-1}=-K_0$, a_{-2} satisfies the relation $I_1=(J_1+a_{-2})^2+f(a_{-2})$, $\forall f(a_{-2})\in C^\infty(R)$; as $a_{-2}=\text{const}\neq 0$, $a_0=b_0=c_0=a_1=0$; $a_{-1}=\text{const}$, $b_{-1}=I_0+f(J_0)$, $c_{-1}=-K_0+g(J_0)$, and $f(J_0)$, $g(J_0)\in C^\infty(R)$ satisfy the relation $f(J_0)g(J_0)=-J_0^2-2a_{-1}J_0+\text{const}$, or $a_{-1}=-J_0$, $b_{-1}=I_0$, $c_{-1}=-K_0$.

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2. $a_{-2} = a_{-1} = b_{-1} = c_{-1} = b_0 = c_0 = a_1 = 0$, $a_0 = \text{const}$.

3. $a_{-2} = b_{-1} = a_0 = b_0 = c_0 = a_1 = 0$, $c_{-1} = -K_0, a_{-1} = a_{-1}(J_0), \forall a_{-1}(J_0) \in C^\infty(R)$, but $\mathrm{d}a_{-1}/\mathrm{d}J_0 \neq 0$.

4. $a_{-2} = a_{-1} = b_{-1} = c_{-1} = b_0 = a_1 = 0, a_0 = \text{const}, c_0 \neq 0, \forall c_0 = c_0(J_0) \in C^\infty(R)$.

5. $a_{-2} = c_{-1} = b_0 = a_1 = 0$, $a_{-1} = \text{const}$, $a_0 = \text{const}, b_{-1} = I_0 + f(J_0), c_0 = c_0(J_0)$ satisfies $(\mathrm{d}/\mathrm{d}J_0)(c_0(J_0)\cdot f(J_0)) = -2a_0, \forall f(J_0) \in C^\infty(R)$, as $c_0(J_0)\cdot f(J_0) = \text{const}$, choose $a_0 = 0$.

6. $a_{-2} = c_{-1} = a_0 = b_0 = c_0 = a_1 = 0, a_{-1} = J_0 + \text{const}, b_{-1} = I_0$.

7. As $a_{-2} = a_0 = b_0 = c_0 = a_1 = 0$, there are the following five subcases:

(7.1) $a_{-1} = \text{const}, c_{-1} = -K_0 + f(J_0), b_{-1} = I_0 + g(J_0), \forall f(J_0), g(J_0) \in C^\infty(R)$;

(7.2) $a_{-1} = -J_0, b_{-1} = I_0, c_{-1} = K_0$;

(7.3) $a_{-1} = -J_0 + \text{const}, (\mathrm{d}/\mathrm{d}J_0)(b_{-1}c_{-1})' = 2a_{-1}, \forall b_{-1} = b_{-1}(J_0), c_{-1} = c_{-1}(J_0) \in C^\infty(R)$;

(7.4) $a_{-1} = -J_0 + \text{const}, b_{-1} = I_0, \forall c_{-1} = c_{-1}(J_0) \in C^\infty(R)$;

(7.5) $a_{-1} = -J_0 + \text{const.}, c_{-1} = -K_0, \forall b_{-1} = b_{-1}(J_0) \in C^\infty(R)$.

8. $a_{-2} = a_{-1} = b_{-1} = c_{-1} = 0$, $a_0 = \text{const}, a_1 = \text{const}, b_0 \neq 0, c_0 \neq 0$, which satisfy the relation $(\mathrm{d}/\mathrm{d}J_0)(b_0c_0) = -2a_1, \forall b_0 = b_0(J_0), c_0 = c_0(J_0) \in C^\infty(R)$.

9. $a_{-2} = a_{-1} = b_{-1} = c_{-1} = 0$, c_0, a_1, $a_0 = \text{const}$, $b_0 \neq 0$, $\forall b_0 = b_0(J_0) \in C^\infty(R)$.

10. $a_{-2} = b_{-1} = c_0 = a_1 = 0$, $a_{-1} = \text{const}, a_0 = \text{const}, c_{-1} = -K_0 + f(J_0)$, $b_0 = b_0(J_0)$ satisfies the relation $(\mathrm{d}/\mathrm{d}J_0)(b_0(J_0)\cdot f(J_0)) = -2a_0, \forall f(J_0) \in C^\infty(R)$, as $b_0(J_0)\cdot f(J_0) = \text{const}$, choose $a_0 = 0$.

Let $L_1(\lambda) = L(\lambda)\otimes I$, $L_2(\mu) = I \otimes L(\mu)$, where I is the 2×2 unit matrix, $\otimes$ is the tensor product of matrix. In the following, we search for a general 4×4 r-matrix structure $r_{12}(\lambda,\mu)$ such that the fundamental Poisson bracket:[3]

$$\{L(\lambda) \overset{\otimes}{,} L(\mu)\} = [r_{12}(\lambda,\mu), L_1(\lambda)] - [r_{21}(\mu,\lambda), L_2(\mu)] \tag{6}$$

holds, where $r_{21}(\lambda,\mu) = Pr_{12}(\lambda,\mu)P$, $P = (1/2)\sum_{i=0}^{3}\sigma_i\otimes\sigma_i$, σ_i is the standard Pauli matrices.

Theorem 1: Under the Assumption (P),

$$r_{12}(\lambda,\mu) = \frac{2}{\mu-\lambda}P + S \tag{7}$$

is an r-matrix structure satisfying Eq. (6), where

$$S = \begin{pmatrix} \frac{2\lambda}{\mu^2}\frac{\partial a_{-2}}{\partial J_1} + \frac{2}{\mu}\frac{\mathrm{d}a_{-1}}{\mathrm{d}J_0} & \frac{2}{\mu}\frac{\partial b_{-1}}{\partial J_0} & \frac{2\lambda}{\mu^2}\langle\Lambda q, q\rangle\frac{\partial a_{-2}}{\partial I_1} & 0 \\ 2\frac{\mathrm{d}c_0}{\mathrm{d}J_0} & 0 & \frac{2}{\mu}\frac{\partial c_{-1}}{\partial K_0} & -\frac{2\lambda}{\mu^2}\langle\Lambda q, q\rangle\frac{\partial a_{-2}}{\partial I_1} \\ -\frac{2\lambda}{\mu^2}\langle\Lambda p, p\rangle\frac{\partial a_{-2}}{\partial I_1} & -\frac{2}{\mu}\frac{\partial b_{-1}}{\partial I_0} & 0 & -2\frac{\mathrm{d}b_0}{\mathrm{d}J_0} \\ 0 & \frac{2\lambda}{\mu^2}\langle\Lambda p, p\rangle\frac{\partial a_{-2}}{\partial I_1} & -\frac{2}{\mu}\frac{\partial c_{-1}}{\partial J_0} & \frac{2\lambda}{\mu^2}\frac{\partial a_{-2}}{\partial J_1} + \frac{2}{\mu}\frac{\mathrm{d}a_{-1}}{\mathrm{d}J_0} \end{pmatrix}.$$

The proof of Proposition 1 and Theorem 1 can be seen in Ref. 5.

Through considering the determinant of $L(\lambda)$ and combining it with Eq. (6), we can easily obtain the following theorem.

Theorem 2: Under the assumption (P), the following equalities

$$\{E_i, E_j\} = 0,\ \{H_l, E_j\} = 0,\ \{F_m, E_j\} = 0,$$
$$i,j = 1,2,\cdots,N,\ l = -4,\cdots,2,\ m = 0,1,2,\cdots, \tag{8}$$

hold. Hence, the Hamiltonian systems (H_l) and (F_m)

$$(H_l)\colon q_x = \frac{\partial H_l}{\partial p},\ p_x = -\frac{\partial H_l}{\partial q},\ l = -4,\cdots,2, \tag{9}$$

$$(F_m)\colon q_{t_m} = \frac{\partial F_m}{\partial p},\ p_{t_m} = -\frac{\partial F_m}{\partial q},\quad m = 0,1,2,\cdots, \tag{10}$$

are completely integrable in Liouville's sense. Here, the expressions E_j $(j = 1,2,\cdots,N)$, H_l $(l = -4,\cdots,2)$, and F_m $(m = 0,1,2,\cdots)$ are the same as ones in Ref. 5.

Because Lax matrix (2) includes various cases displayed in Proposition 1, we call matrix (2) as a generalized Lax matrix. In the following it can be seen that various constrained flows are reduced from the unisonant r-matrix (7).

The following numbers of title coincide with the ones in Proposition 1, i.e. the corresponding conditions are coincidental. For simplicity, here we only give a new reduction and several well-known examples from the unisonant r-matrix (7). Other cases are similar. In what follows we take the prime "$'$" for $\mathrm{d}/\mathrm{d}J_0$.

2.

$$r_{12}(\lambda,\mu) = \frac{2}{\mu-\lambda}P. \tag{11}$$

This is nothing but the r-matrix of the well-known constrained AKNS (c-AKNS) system.[1]

6.

$$r_{12}(\lambda,\mu)=\frac{2}{\mu-\lambda}P+\frac{2}{\mu}S,$$

$$S=\begin{pmatrix}1&0&0&0\\0&0&0&0\\0&1&0&0\\0&0&0&1\end{pmatrix}. \qquad (12)$$

This is a new r-matrix.

9.

$$r_{12}(\lambda,\mu)=\frac{2}{\mu-\lambda}P+S,$$

$$S=b_0'(J_0)\begin{pmatrix}0&1&0&0\\0&0&0&0\\0&0&0&-1\\0&0&0&0\end{pmatrix}. \qquad (13)$$

As $b_0=-J_0$, Eq. (13) is reduced to the common r-matrix of the constrained Toda (c-Toda) system and the constrained CKdV (c-CKdV) system.[8]

10.

$$r_{12}(\lambda,\mu)=\frac{2}{\mu-\lambda}P+S,$$

$$S=\begin{pmatrix}0&b_0'(J_0)&0&0\\ \frac{1}{\mu}f'(J_0)&0&-\frac{2}{\mu}&0\\0&0&0&-b_0'(J_0)\\0&0&-\frac{1}{\mu}f'(J_0)&0\end{pmatrix}. \qquad (14)$$

As $f(J_0)=\text{const}$, $b_0(J_0)=0$, Eq. (14) reads the r-matrix[6] of the constrained MKdV (c-MKdV) system.

The reduction procedure of the above four cases and the r-matrices related to other cases in Theorem 1 are detailedly presented in Ref. 5.

Here, we simply give the integrable system arising from the new r-matrix (12). The corresponding involutive systems are

$$\begin{aligned}E_j^1=&2(\langle p,q\rangle+c)\lambda_j^{-1}p_jq_j\\&+\langle q,q\rangle\lambda_j^{-1}p_j^2-\Gamma_j,\ j=1,\cdots,N,\end{aligned} \qquad (15)$$

$$\Gamma_j=\sum_{k=1,k\neq j}^{N}\frac{(p_jq_k-p_kq_j)^2}{\lambda_j-\lambda_k},$$

where c is an arbitrarily given constant. Thus, the finite dimesional Hamiltonian systems (F_m^1) defined by $F_m^1=\sum_{j=1}^N\lambda_j^mE_j^1$, $m=0,\cdots$, i.e.

$$\begin{aligned}F_m^1=&2(\langle p,q\rangle+c)\langle\lambda_j^{m-1}p,q\rangle\\&+\langle q,q\rangle\langle\lambda_j^{m-1}p,p\rangle\\&-\sum_{i+j=m-1}(\langle\Lambda^iq,q\rangle\langle\Lambda^jp,p\rangle\\&-\langle\Lambda^iq,p\rangle\langle\Lambda^jp,q\rangle)\end{aligned} \qquad (16)$$

are completely integrable. Particularly, as $m=2$ the Hamiltonian system (F_2^1):

$$\begin{cases}q_x=\dfrac{\partial F_2^1}{\partial p}=2c\Lambda q-2\langle\Lambda q,q\rangle p\\ \qquad+4\langle\Lambda p,q\rangle q+4\langle p,q\rangle\Lambda q,\\ p_x=-\dfrac{\partial F_2^1}{\partial q}=-2c\Lambda p+2\langle p,p\rangle\Lambda q\\ \qquad-4\langle\Lambda p,q\rangle p-4\langle p,q\rangle\Lambda p,\end{cases} \qquad (17)$$

is a new finite dimensional integrable system, which can be changed as the following spectral problem

$$\phi_x=\begin{pmatrix}(2c+4v)\lambda+4u&-2w\\2s\lambda&-(2c+4v)\lambda-4u\end{pmatrix}\phi \qquad (18)$$

with the constraint condition $u=\langle\Lambda p,q\rangle$, $v=\langle p,q\rangle$, $w=\langle\Lambda q,q\rangle$, $s=\langle p,p\rangle$, and $\lambda=\lambda_j$, $\phi=(q_j,p_j)^{\mathrm T}$, $j=1,\cdots,N$. Apparently, spectral problem (18) is a new one and has never been studied before.

As we see as above, r-matrix indeed plays a very important role in guaranteeing integrability as well as in unifying each concrete integrable system by nonlinearization method. Recently, we found two different (even a continuous and a discrete) constrained flows share a common r-matrix, Lax matrix, and even involutive systems.[7,8] This, more or less, will be helpful to the classification of finite dimensional integrable systems. In addition, the above new integrable system also implies an interesting procedure how to connect r-matrix with the spectral problem.

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