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Qualitative Analysis for a New Integrable Two-Component Camassa–Holm System with Peakon and Weak Kink Solutions

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Abstract: This paper is devoted to a new integrable two-component Camassa–Holm system with peaked solitons (peakons) and weak-kink solutions. It is the first integrable system that admits weak kink and kink–peakon interactional solutions. In addition, the new system includes both standard (quadratic) and cubic Camassa–Holm equations as two special cases. In the paper, we first establish the local well-posedness for the Cauchy problem of the system, and then derive a precise blow-up scenario and a new blow-up result for strong solutions to the system with both quadratic and cubic nonlinearity. Furthermore, its peakon and weak kink solutions are discussed as well.

1. Introduction

In the past two decades, a large amount of literature was devoted to the celebrated Camassa–Holm (CH) equation [3]

$$m_t + um_x + 2u_x m + bu_x = 0, \quad m = u - u_{xx},$$

where b is an arbitrary constant, which models the unidirectional propagation of shallow water waves over a flat bottom. Here $u(t, x)$ stands for the fluid velocity at time t in the spatial x direction [3,22,41]. The CH equation is also recognized as a model for the propagation of axially symmetric waves in hyperelastic rods [18]. It has a bi-Hamiltonian structure and is completely integrable with algebro-geometric solutions on a symplectic submanifold [3,29,48]. Its solitary waves vanishing at both infinities are peaked solitons (peakons) [4] when $b = 0$, and they are orbitally stable [17]. It is also worth pointing out that the peakons replicate a feature that is characteristic for the waves of great height—waves of the largest amplitude that are exact traveling wave solutions of the governing equations for irrotational water waves, cf. [11,53].

The Cauchy problem and initial boundary value problem for the CH equation have been studied extensively [8,9,19,27]. It has been shown that this equation is locally well-posed [8,9,19] for initial data $u_0 \in H^s(\mathbb{R})$, $s > \frac{3}{2}$. Moreover, it has both globally

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strong solutions [7–9] and blow-up solutions at a finite time [7–10]. On the other hand, it also has globally weak solutions in $H^1(\mathbb{R})$ [2,16,57]. In comparison with the KdV equation, the advantage of the CH equation lies in the fact that the CH equation has peakons and models wave breaking [4,10] (namely, the wave remains bounded while its slope becomes unbounded in finite time [54]).

Another important integrable equation admitting peakons is the well-known Degasperis–Procesi (DP) equation [21]

$$m_t + um_x + 3u_x m = 0, \quad m = u - u_{xx},$$

which is regarded as another model for nonlinear shallow water dynamics [13,15]. It was proved in [20] that the DP equation has a bi-Hamiltonian structure and an infinite number of conservation laws, and admits peakon solutions which are analogous to the CH peakons. The DP equation was already extended to a completely integrable hierarchy in a 3×3 matrix Lax pair, which possesses involutive representation of solutions under a Neumann constraint on a symplectic submanifold [51], and furthermore it was proven to have algebro-geometric solutions for such a 3×3 integrable system [40].

The Cauchy problem and initial boundary value problem for the DP equation have been studied extensively in [6,25–27,43,62,63]. Although the DP equation is very similar to the CH equation in the aspects of integrability, particularly in the form of equation, there are some significant differences between these two equations. One of the remarkable features of the DP equation is that it has not only (periodic) peakon solutions [20,63], but also (periodic) shock peakons [26,44]. Besides, the CH equation is a re-expression of geodesic flow on the diffeomorphism group [14], while the DP equation is regarded as a non-metric Euler equation [23].

The nonlinear terms in both CH and DP equations are quadratic with slightly different constant coefficients. However, there do exist integrable peakon systems with cubic nonlinearity, which include the cubic CH equation (also called the FORQ equation):

$$m_t + ((u^2 - u_x^2)m)_x + bu_x = 0, \quad m = u - u_{xx} \tag{1.1}$$

with a constant b, and the Novikov equation:

$$m_t + u^2 m_x + 3uu_x m = 0, \quad m = u - u_{xx}.$$

Equation (1.1) was proposed independently in [29,46,49]. It was derived from the two-dimensional Euler equations, and its Lax pair, peakon and cusped soliton (cuspon) solutions have been studied in [49]. Recently, the formation of singularities, wave-breaking mechanism, and the peakon stability of Eq. (1.1) with $b = 0$ were investigated in [36].

The Novikov equation was proposed in [45] and its Lax pair, bi-Hamiltonian structure, peakon solutions, well-posedness, blow-up phenomena and global weak solutions have been studied extensively in [37,39,45,55]. Very recently, the following integrable equation with both quadratic and cubic nonlinearity

$$m_t + \frac{1}{2}k_1((u^2 - u_x^2)m)_x + \frac{1}{2}k_2(um_x + 2u_x m) + bu_x = 0, \tag{1.2}$$

was investigated for its explicit weak solutions [50], where $m = u - u_{xx}$, b, k_1, and k_2 are three arbitrary constants. Eq. (1.2) was first implied in the work of Fokas [28]. Its Lax pair, bi-Hamiltonian structure, peakons, weak kinks, kink-peakon interaction, and classical soliton solutions were studied recently in [50].

A natural idea is to extend such a study to the multi-component systems. One of the popular systems is the following integrable two-component Camassa–Holm shallow water system (2CH) [5,12]:

$$\begin{cases} m_t + um_x + 2u_x m + \sigma\rho\rho_x = 0, \\ \rho_t + (u\rho)_x = 0, \end{cases}$$

where $m = u - u_{xx}$ and $\sigma = \pm 1$, which becomes CH equation when $\rho \equiv 0$. The Cauchy problems of the 2CH system with $\sigma = -1$ and with $\sigma = 1$ have been studied in [24] and [12,31,33,35], respectively. Local well-posedness for the 2CH system with the initial data in Sobolev spaces and in Besov spaces have been established in [12,25,35]. The blow-up phenomena and global existence of strong solutions to the 2CH system in Sobolev spaces have been investigated in [25,31,35]. The analyticity of solutions to the Cauchy problem and the initial boundary value problem for the 2CH system have been studied in [58] and [61], respectively. Recently, the existence of global weak solutions for the 2CH system with $\sigma = 1$ has been proved in [33]. Two other notable systems are the modified two-component Camassa–Holm system (M2CH) [38]:

$$\begin{cases} m_t + um_x + 2u_x m + \sigma\rho\bar{\rho}_x = 0, \\ \rho_t + (u\rho)_x = 0, \end{cases}$$

with $m = u - u_{xx}$, $\rho = (1 - \partial_x^2)(\bar{\rho} - \bar{\rho}_0)$, $\sigma = \pm 1$, and the two-component Degasperis–Procesi system (2DP) [47]:

$$\begin{cases} m_t + um_x + 3u_x m + b\rho\rho_x = 0, \\ \rho_t + u\rho_x + 2u_x\rho = 0, \end{cases}$$

with $m = u - u_{xx}$ and a real constant b. When $\rho \equiv 0$, both M2CH and 2DP equations are reduced to CH and DP equations, respectively. The Cauchy problems and initial boundary value problems for the M2CH and 2DP equations have been studied in many works, for example [32,34,52,60,61] and [59,61]. It is worth pointing out that the nonlinear terms in above three two-component systems are all quadratic.

In this paper, we consider the following integrable two-component Camassa–Holm system with both quadratic and cubic nonlinearity proposed in [50,56]:

$$\begin{cases} m_t + \frac{1}{2}[(uv - u_x v_x)m]_x - \frac{1}{2}(uv_x - u_x v)m + bu_x = 0, \\ n_t + \frac{1}{2}[(uv - u_x v_x)n]_x + \frac{1}{2}(uv_x - u_x v)n + bv_x = 0, \end{cases} \tag{1.3}$$

where $m = u - u_{xx}$, $n = v - v_{xx}$, and b takes an arbitrary value. The system (1.3) is the first two-component system admitting weak kink solutions. It can be reduced to the CH equation, the cubic CH equation Eq. (1.1), and the generalized CH equation Eq. (1.2) as $v = 2$, $v = 2u$, and $v = k_1 u + k_2$, respectively. Integrability of this system, its bi-Hamiltonian structure, and infinitely many conservation laws were already presented in [56]. Let us now set up the Cauchy problem for the above system as follows:

$$\begin{cases} m_t + \frac{1}{2}(uv - u_x v_x)m_x = -\frac{1}{2}(u_x n + v_x m)m + \frac{1}{2}(uv_x - u_x v)m - bu_x, \\ n_t + \frac{1}{2}(uv - u_x v_x)n_x = -\frac{1}{2}(u_x n + v_x m)n - \frac{1}{2}(uv_x - u_x v)n - bv_x, \\ m(0,x) = m_0(x), \\ n(0,x) = n_0(x). \end{cases} \tag{1.4}$$

By using an approach similar to the one in [58], the analytic solutions to the system (1.4) can readily be proved in both variables, globally in space and locally in time. However, the goal of this paper is to establish the local well-posedness regime for the Cauchy problem in Besov spaces, present the precise blow-up scenario and a new blow-up result for strong solutions to the system , and provide the peakon and weak kink solutions.

Regarding the locally well-posed problem, we first adopt the classical Kato semigroup theory to obtain the local well-posedness for the system (1.4) with initial data (m_0, n_0) belonging in Sobolev space $H^s(\mathbb{R}) \times H^s(\mathbb{R})$ as $s \geq 1$. Subsequently, we take advantage of the transport equation theory, Littlewood–Paley's decomposition and some fine estimates of Besov spaces to establish the local well-posedness for the system (1.4) in Besov spaces (see Theorem 3.2 below in Sect. 3), which particularly implies the system is locally well-posed with initial data $(m_0, n_0) \in H^s(\mathbb{R}) \times H^s(\mathbb{R})$ for $\frac{1}{2} < s \neq \frac{3}{2}$. This almost improves the corresponding result by using Kato's semigroup approach.

In order to analyze the blow-up phenomena, here we may make good use of the fine structure of the system (1.4). It is not difficult for us to verify that the system (1.4) possesses the following two conservation laws:

$$H_1 = \frac{1}{2}\int_{\mathbb{R}} (uv + u_x v_x)dx = \frac{1}{2}\int_{\mathbb{R}} undx = \frac{1}{2}\int_{\mathbb{R}} vmdx,$$
$$H_2 = \frac{1}{4}\int_{\mathbb{R}} \Big((u^2 v_x + u_x^2 v_x - 2uu_x v)n + 2b(uv_x - u_x v)\Big)\, dx.$$

In fact, as mentioned before, this system is completely integrable and has infinitely many conservation laws [56]. But, unfortunately, it seems that there is not a good way to control the quantities $||u(t,\cdot)||_{L^\infty}$ and $||v(t,\cdot)||_{L^\infty}$ directly, while it is very important to bound them in studying the blow-up phenomena of the system (1.4). This difficulty may be overcome by exploiting the characteristic ODE related to the system (1.4) to construct some invariant properties of the solutions and sufficiently utilizing the structure of the system itself, which we need to deal with for the two cross-terms $\frac{1}{2}(uv_x - u_x v)m$ and $-\frac{1}{2}(uv_x - u_x v)n$ in the system (see Lemma 4.2 below). For the special case of Eq. (1.1) as $b = 0$, with the conserved quantity $\int_{\mathbb{R}} umdx = ||u||^2_{H^1(\mathbb{R})}$ in hand, one can directly control $||u(t,\cdot)||_{L^\infty}$ by using the Sobolev's embedding theorem, that is,

$$||u(t,\cdot)||_{L^\infty(\mathbb{R})} \leq C||u(t,\cdot)||_{H^1(\mathbb{R})} = C||u_0||_{H^1(\mathbb{R})}, \quad \forall t \in [0, T).$$

In that way, the blow-up phenomena of Eq. (1.1) with $b = 0$ has been studied in [36].

On the other hand, in view of the uselessness of the conservation laws of the system (1.4) again (mainly because the regularity is not high enough), we directly investigate the transport equation in terms of $\frac{1}{2}(u_x n + v_x m)$, which is the slope of $\frac{1}{2}(uv - u_x v_x)$ (see Lemma 4.3 below), to derive a new blow-up result with respect to initial data (see Theorem 4.3 below). Overall, we do not use any conservation laws rather than the almost symmetrical structure of the system (1.4) in the whole paper.

The rest of our paper is organized as follows. In Sect. 2, we recall the Littlewood–Paley analysis and the transport equation theory. In Sect. 3, we establish the local well-posedness of the system (1.4). In Sect. 4, we provide the precise blow-up scenario and a new blow-up result of strong solutions to the system (1.4). Section 5 is devoted to discussing peakon and weak kink solutions.

2. Preliminaries

In this section, we recall some facts on the Littlewood–Paley analysis and transport equation theory, which are frequently used in the following sections.

To begin with, we introduce some notations. Let $s \in \mathbb{R}$, $1 \leq p, r \leq \infty$. The nonhomogeneous Besov space $B^s_{p,r}(\mathbb{R}^n)$ ($B^s_{p,r}$ for short) is defined by

$$B^s_{p,r}(\mathbb{R}^n) \triangleq \{f \in \mathcal{S}'(\mathbb{R}^n) : ||f||_{B^s_{p,r}} \triangleq ||(2^{qs}||\Delta_q f||_{L^p(\mathbb{R}^n)})_{q\geq -1}||_{l^r} < \infty\},$$

where Δ_q is the Littlewood–Paley decomposition operator [1] . If $s = \infty$, then $B^\infty_{p,r} \triangleq \bigcap\limits_{s\in\mathbb{R}} B^s_{p,r}$. In addition, define

$$E^s_{p,r}(T) \triangleq C([0,T]; B^s_{p,r}) \cap C^1([0,T]; B^{s-1}_{p,r}), \quad \text{if } r < \infty,$$
$$E^s_{p,\infty}(T) \triangleq L^\infty(0,T; B^s_{p,\infty}) \cap Lip\,(0,T; B^{s-1}_{p,\infty})$$

for some $T > 0$.

Proposition 2.1. [1] *Let $m \in \mathbb{R}$ and f be an S^m-multiplier. That is, $f : \mathbb{R}^n \to \mathbb{R}$ is smooth and satisfies that for any $\alpha \in \mathbb{N}^n$, there is a constant $C_\alpha > 0$ such that*

$$|\partial^\alpha f(\xi)| \leq C_\alpha(1 + |\xi|)^{m-|\alpha|}, \quad \forall\ \xi \in \mathbb{R}^n.$$

Let $f(D) \triangleq \mathcal{F}^{-1} f \mathcal{F} \in Op(S^m)$. Then the operator $f(D)$ is continuous from $B^s_{p,r}$ to $B^{s-m}_{p,r}$.

For some other basic properties of Besov spaces, one may check [1] for more details. Now, we recall the following 1-D Morse type estimate.

Proposition 2.2. [19,35] (i) *For $s_1 \leq \frac{1}{p} < s_2$ ($s_2 \geq \frac{1}{p}$ if $r = 1$) and $s_1 + s_2 > 0$,*

$$||fg||_{B^{s_1}_{p,r}(\mathbb{R})} \leq C||f||_{B^{s_1}_{p,r}(\mathbb{R})}||g||_{B^{s_2}_{p,r}(\mathbb{R})}. \tag{2.1}$$

(ii) *For $s > 0$,*

$$||fg||_{B^s_{p,r}(\mathbb{R})} \leq C(||f||_{B^s_{p,r}(\mathbb{R})}||g||_{L^\infty(\mathbb{R})} + ||g||_{B^s_{p,r}(\mathbb{R})}||f||_{L^\infty(\mathbb{R})}). \tag{2.2}$$

(iii) *In Sobolev spaces $H^s(\mathbb{R}) = B^s_{2,2}(\mathbb{R})$, for $s > 0$,*

$$||f\partial_x g||_{H^s(\mathbb{R})} \leq C(||f||_{H^{s+1}(\mathbb{R})}||g||_{L^\infty(\mathbb{R})} + ||f||_{L^\infty(\mathbb{R})}||\partial_x g||_{H^s(\mathbb{R})}), \tag{2.3}$$

where C is a positive constant independent of f and g.

Finally, let us state some useful results in the transport equation theory, which are crucial to the proofs of our main theorems.

Lemma 2.1. [1,19] (A priori estimates in Besov spaces) *Let $1 \leq p, r \leq \infty$ and $s > -\min(\frac{1}{p}, 1 - \frac{1}{p})$. Assume that $f_0 \in B^s_{p,r}$, $F \in L^1(0,T; B^s_{p,r})$, and $\partial_x v$ belongs to $L^1(0,T; B^{s-1}_{p,r})$ if $s > 1 + \frac{1}{p}$ or to $L^1(0,T; B^{\frac{1}{p}}_{p,r} \cap L^\infty)$ otherwise. If $f \in L^\infty(0,T; B^s_{p,r}) \bigcap C([0,T]; \mathcal{S}')$ solves the following 1-D transport equation:*

$$(T) : \begin{cases} \partial_t f + v\,\partial_x f = F, \\ f|_{t=0} = f_0, \end{cases}$$

then there exists a constant C depending only on s, p and r, and such that the following statements hold:

(1) *If* $r = 1$ *or* $s \neq 1 + \frac{1}{p}$*, then*

$$||f(t)||_{B^s_{p,r}} \leq ||f_0||_{B^s_{p,r}} + \int_0^t ||F(\tau)||_{B^s_{p,r}} d\tau + C \int_0^t V'(\tau)||f(\tau)||_{B^s_{p,r}} d\tau$$

with $V(t) = \int_0^t ||\partial_x v(\tau)||_{B^{\frac{1}{p}}_{p,r} \cap L^\infty} d\tau$ *if* $s < 1 + \frac{1}{p}$*, and* $V(t) = \int_0^t ||\partial_x v(\tau)||_{B^{s-1}_{p,r}} d\tau$ *else.*

(2) *If* $r < \infty$*, then* $f \in C([0, T]; B^s_{p,r})$*; and if* $r = \infty$*, then* $f \in C([0, T]; B^{s'}_{p,1})$ *for all* $s' < s$*.*

Lemma 2.2. [35] (A priori estimate in Sobolev spaces) *Let* $0 < \sigma < 1$*. Assume that* $f_0 \in H^\sigma$*,* $F \in L^1(0, T; H^\sigma)$*, and* $v, \partial_x v \in L^1(0, T; L^\infty)$*. If* $f \in L^\infty(0, T; H^\sigma) \bigcap C([0, T]; \mathcal{S}')$ *solves (T), then* $f \in C([0, T]; H^\sigma)$*, and there exists a constant* C *depending only on* σ *such that*

$$||f(t)||_{H^\sigma} \leq ||f_0||_{H^\sigma} + C \int_0^t ||F(\tau)||_{H^\sigma} d\tau + C \int_0^t V'(\tau)||f(\tau)||_{H^\sigma} d\tau$$

with $V(t) = \int_0^t (||v(\tau)||_{L^\infty} + ||\partial_x v(\tau)||_{L^\infty}) d\tau$*.*

Lemma 2.3. [1] (Existence and uniqueness) *Let* p, r, s, f_0 *and* F *be as in the statement of Lemma 2.1. Assume that* $v \in L^\rho(0, T; B^{-M}_{\infty,\infty})$ *for some* $\rho > 1$ *and* $M > 0$*, and* $\partial_x v \in L^1(0, T; B^{s-1}_{p,r})$ *if* $s > 1 + \frac{1}{p}$ *or* $s = 1 + \frac{1}{p}$ *and* $r = 1$*, and* $\partial_x v \in L^1(0, T; B^{\frac{1}{p}}_{p,\infty} \cap L^\infty)$ *if* $s < 1 + \frac{1}{p}$*. Then (T) has a unique solution* $f \in L^\infty(0, T; B^s_{p,r}) \bigcap \big(\bigcap_{s' < s} C([0, T]; B^{s'}_{p,1}) \big)$ *and the inequalities of Lemma 2.1 hold true. Moreover, if* $r < \infty$*, then* $f \in C([0, T]; B^s_{p,r})$*.*

3. Local Well-Posedness

In this section, we study the local well-posedness for the system (1.4). To do so, we apply the classical Kato's semigroup theory [42] to set up the local well-posedness of the system (1.4) in Sobolev spaces. More precisely, we have

Theorem 3.1. *Suppose that* $(m_0, n_0) \in H^s(\mathbb{R}) \times H^s(\mathbb{R})$ *with* $s \geq 1$*. There exists a maximal existence time* $T = T(||m_0||_{H^s(\mathbb{R})}, ||n_0||_{H^s(\mathbb{R})}) > 0$*, and a unique solution* (m, n) *to the system (1.4) such that*

$$(m, n) \in C([0, T); H^s(\mathbb{R}) \times H^s(\mathbb{R})) \cap C^1([0, T); H^{s-1}(\mathbb{R}) \times H^{s-1}(\mathbb{R})).$$

Moreover, the solution depends continuously on the initial data, that is, the mapping $(m_0, n_0) \mapsto (m, n)$ *:*

$$H^s(\mathbb{R}) \times H^s(\mathbb{R}) \to C([0, T); H^s(\mathbb{R}) \times H^s(\mathbb{R})) \cap C^1([0, T); H^{s-1}(\mathbb{R}) \times H^{s-1}(\mathbb{R}))$$

is continuous.

Proof. By going along the similar line of the proof in [24], one can readily prove the theorem. For the sake of simplicity, we omit the details here. □

Let us now focus on the case in the nonhomogeneous Besov spaces. Uniqueness and continuity with respect to the initial data in some sense can be obtained by the following priori estimates.

Lemma 3.1. *Let* $1 \le p, r \le \infty$ *and* $s > \max(1-\frac{1}{p}, \frac{1}{p}, \frac{1}{2})$. *Suppose that we are given* $(m^{(i)}, n^{(i)}) \in L^\infty(0,T; B^s_{p,r} \times B^s_{p,r}) \cap C([0,T]; \mathcal{S}' \times \mathcal{S}')$ $(i = 1,2)$ *two solutions of the system* (1.4) *with the initial data* $(m_0^{(i)}, n_0^{(i)}) \in B^s_{p,r} \times B^s_{p,r}$ $(i = 1,2)$ *and let* $u^{(12)} \triangleq u^{(2)} - u^{(1)}$, $v^{(12)} \triangleq v^{(2)} - v^{(1)}$, $m^{(12)} \triangleq m^{(2)} - m^{(1)}$, *and* $n^{(12)} \triangleq n^{(2)} - n^{(1)}$. *Then for all* $t \in [0,T]$, *we have*

(1) *if* $s > \max(1-\frac{1}{p}, \frac{1}{p}, \frac{1}{2})$, *but* $s \neq 2 + \frac{1}{p}$, *then*

$$\begin{aligned}
&||m^{(12)}(t)||_{B^{s-1}_{p,r}} + ||n^{(12)}(t)||_{B^{s-1}_{p,r}} \\
&\le (||m_0^{(12)}||_{B^{s-1}_{p,r}} + ||n_0^{(12)}||_{B^{s-1}_{p,r}}) \\
&\quad \times e^{C\int_0^t (||m^{(1)}(\tau)||_{B^s_{p,r}} + ||m^{(2)}(\tau)||_{B^s_{p,r}} + ||n^{(1)}(\tau)||_{B^s_{p,r}} + ||n^{(2)}(\tau)||_{B^s_{p,r}} + 1)^2 d\tau} \\
&\triangleq L(s-1;t);
\end{aligned} \tag{3.1}$$

(2) *if* $s = 2 + \frac{1}{p}$, *then*

$$\begin{aligned}
&||m^{(12)}(t)||_{B^{s-1}_{p,r}} + ||n^{(12)}(t)||_{B^{s-1}_{p,r}} \\
&\le CL^\theta(s-1;t)((||m^{(1)}(t)||_{B^s_{p,r}} + ||m^{(2)}(t)||_{B^s_{p,r}})^{1-\theta} \\
&\quad +(||n^{(1)}(t)||_{B^s_{p,r}} + ||n^{(2)}(t)||_{B^s_{p,r}})^{1-\theta}),
\end{aligned}$$

where $\theta \in (0,1)$.

Proof. Apparently, $(m^{(12)}, n^{(12)}) \in L^\infty(0,T; B^s_{p,r} \times B^s_{p,r}) \cap C([0,T]; \mathcal{S}' \times \mathcal{S}')$ solves the following Cauchy problem of the transport equations:

$$\begin{cases}
\partial_t m^{(12)} + \frac{1}{2}(u^{(1)}v^{(1)} - u_x^{(1)}v_x^{(1)})\partial_x m^{(12)} = F(t,x), \\
\partial_t n^{(12)} + \frac{1}{2}(u^{(1)}v^{(1)} - u_x^{(1)}v_x^{(1)})\partial_x n^{(12)} = G(t,x), \\
m^{(12)}|_{t=0} = m_0^{(12)} \triangleq m_0^{(2)} - m_0^{(1)}, \\
n^{(12)}|_{t=0} = n_0^{(12)} \triangleq n_0^{(2)} - n_0^{(1)},
\end{cases} \tag{3.2}$$

where $F(t,x) \triangleq \frac{1}{2}\big((u^{(1)} - m^{(1)})v_x^{(1)} - (v^{(1)} + n^{(1)})u_x^{(1)} - v_x^{(2)}m^{(2)}\big)m^{(12)} - \frac{1}{2}u_x^{(2)}m^{(2)}n^{(12)} + \frac{1}{2}\big(v_x^{(1)}m_x^{(2)} - (v^{(1)} + n^{(1)})m^{(2)} - 2b\big)u_x^{(12)} + \frac{1}{2}\big(u_x^{(2)}m_x^{(2)} - (m^{(1)} - u^{(2)})m^{(2)}\big)v_x^{(12)} + \frac{1}{2}\big(v_x^{(1)}m^{(2)} - v^{(1)}m_x^{(2)}\big)u^{(12)} - \frac{1}{2}(u^{(2)}m^{(2)})_x v^{(12)}$, and
$G(t,x) \triangleq \frac{1}{2}\big((v^{(1)} - n^{(1)})u_x^{(1)} - (u^{(1)} + m^{(1)})v_x^{(1)} - u_x^{(2)}n^{(2)}\big)n^{(12)} - \frac{1}{2}v_x^{(2)}n^{(2)}m^{(12)} + \frac{1}{2}\big(u_x^{(1)}n_x^{(2)} - (u^{(1)} + m^{(1)})n^{(2)} - 2b\big)v_x^{(12)} + \frac{1}{2}\big(v_x^{(2)}n_x^{(2)} - (n^{(1)} - v^{(2)})n^{(2)}\big)u_x^{(12)} + \frac{1}{2}\big(u_x^{(1)}n^{(2)} - u^{(1)}n_x^{(2)}\big)v^{(12)} - \frac{1}{2}(v^{(2)}n^{(2)})_x u^{(12)}$.

Here, we make a Claim: For all $s > \max(\frac{1}{p}, \frac{1}{2})$ and $t \in [0,T]$, we have

$$\begin{aligned}
&||F(t)||_{B^{s-1}_{p,r}},\ \ ||G(t)||_{B^{s-1}_{p,r}} \\
&\le C(||m^{(12)}(t)||_{B^{s-1}_{p,r}} + ||n^{(12)}(t)||_{B^{s-1}_{p,r}}) \\
&\quad \times(||m^{(1)}(t)||_{B^s_{p,r}} + ||m^{(2)}(t)||_{B^s_{p,r}} + ||n^{(1)}(t)||_{B^s_{p,r}} + ||n^{(2)}(t)||_{B^s_{p,r}} + 1)^2,
\end{aligned} \tag{3.3}$$

where $C = C(s,p,r,b)$ is a positive constant.

Indeed, for $s > 1 + \frac{1}{p}$, $B^{s-1}_{p,r}$ is an algebra, we then have

$$||(u^{(1)} - m^{(1)})v^{(1)}_x m^{(12)}||_{B^{s-1}_{p,r}} \leq (||u^{(1)}||_{B^{s-1}_{p,r}} + ||m^{(1)}||_{B^{s-1}_{p,r}})||v^{(1)}_x||_{B^{s-1}_{p,r}}||m^{(12)}||_{B^{s-1}_{p,r}}.$$

According to Proposition 2.1 and noticing $(1-\partial_x^2)^{-1} \in Op(S^{-2})$, we obtain

$$||u^{(i)}||_{B^{s+2}_{p,r}} \approx ||m^{(i)}||_{B^{s}_{p,r}} \quad \text{and} \quad ||v^{(i)}||_{B^{s+2}_{p,r}} \approx ||n^{(i)}||_{B^{s}_{p,r}}, \quad i = 1, 2, 12, \ \forall s \in \mathbb{R}. \tag{3.4}$$

Therefore

$$||(u^{(1)} - m^{(1)})v^{(1)}_x m^{(12)}||_{B^{s-1}_{p,r}} \leq C||m^{(1)}||_{B^{s}_{p,r}}||n^{(1)}||_{B^{s}_{p,r}}||m^{(12)}||_{B^{s-1}_{p,r}}.$$

Similarly, we are able to get the following estimates:

$$\begin{aligned}
&||(v^{(1)} + n^{(1)})u^{(1)}_x m^{(12)}||_{B^{s-1}_{p,r}} + ||v^{(2)}_x m^{(2)} m^{(12)}||_{B^{s-1}_{p,r}} \\
&\quad \leq C(||m^{(1)}||_{B^{s}_{p,r}}||n^{(1)}||_{B^{s}_{p,r}} + ||m^{(2)}||_{B^{s}_{p,r}}||n^{(2)}||_{B^{s}_{p,r}})||m^{(12)}||_{B^{s-1}_{p,r}}, \\
&||u^{(2)}_x m^{(2)} n^{(12)}||_{B^{s-1}_{p,r}} + ||(u^{(2)}m^{(2)})_x v^{(12)}||_{B^{s-1}_{p,r}} \leq C||m^{(2)}||^2_{B^{s}_{p,r}}||n^{(12)}||_{B^{s-1}_{p,r}}, \\
&||\big(v^{(1)}_x m^{(2)}_x - (v^{(1)} + n^{(1)})m^{(2)} - 2b\big)u^{(12)}_x||_{B^{s-1}_{p,r}} + ||\big(v^{(1)}_x m^{(2)} - v^{(1)} m^{(2)}_x\big)u^{(12)}||_{B^{s-1}_{p,r}} \\
&\quad \leq C(||m^{(2)}||_{B^{s}_{p,r}}||n^{(1)}||_{B^{s}_{p,r}} + 1)||m^{(12)}||_{B^{s-1}_{p,r}},
\end{aligned}$$

and

$$\begin{aligned}
&||\big(u^{(2)}_x m^{(2)}_x - (m^{(1)} - u^{(2)})m^{(2)}\big)v^{(12)}_x||_{B^{s-1}_{p,r}} \\
&\quad \leq C(||m^{(1)}||_{B^{s}_{p,r}} + ||m^{(2)}||_{B^{s}_{p,r}})||m^{(2)}||_{B^{s}_{p,r}}||n^{(12)}||_{B^{s-1}_{p,r}}.
\end{aligned}$$

So, if $s > 1 + \frac{1}{p}$, we have

$$\begin{aligned}
||F(t)||_{B^{s-1}_{p,r}} &\leq C(||m^{(12)}(t)||_{B^{s-1}_{p,r}} + ||n^{(12)}(t)||_{B^{s-1}_{p,r}}) \\
&\quad \times (||m^{(1)}(t)||_{B^{s}_{p,r}} + ||m^{(2)}(t)||_{B^{s}_{p,r}} + ||n^{(1)}(t)||_{B^{s}_{p,r}} + ||n^{(2)}(t)||_{B^{s}_{p,r}} + 1)^2.
\end{aligned}$$

$||G(t)||_{B^{s-1}_{p,r}}$ $(s > 1 + \frac{1}{p})$ can be dealt with likewise.

On the other hand, if $\max(\frac{1}{p}, \frac{1}{2}) < s \leq 1 + \frac{1}{p}$, $B^{s}_{p,r}$ is an algebra. In light of (2.1) and (3.4), one may infer the following results:

$$\begin{aligned}
&||\big((u^{(1)} - m^{(1)})v^{(1)}_x - (v^{(1)} + n^{(1)})u^{(1)}_x - v^{(2)}_x m^{(2)}\big)m^{(12)}||_{B^{s-1}_{p,r}} \\
&\quad \leq C(||(u^{(1)} - m^{(1)})v^{(1)}_x||_{B^{s}_{p,r}} + ||(v^{(1)} + n^{(1)})u^{(1)}_x||_{B^{s}_{p,r}} + ||v^{(2)}_x m^{(2)}||_{B^{s}_{p,r}})||m^{(12)}||_{B^{s-1}_{p,r}} \\
&\quad \leq C(||m^{(1)}||_{B^{s}_{p,r}}||n^{(1)}||_{B^{s}_{p,r}} + ||m^{(2)}||_{B^{s}_{p,r}}||n^{(2)}||_{B^{s}_{p,r}})||m^{(12)}||_{B^{s-1}_{p,r}}, \\
&\quad ||u^{(2)}_x m^{(2)} n^{(12)}||_{B^{s-1}_{p,r}} \leq C||u^{(2)}_x m^{(2)}||_{B^{s}_{p,r}}||n^{(12)}||_{B^{s-1}_{p,r}} \leq C||m^{(2)}||^2_{B^{s}_{p,r}}||n^{(12)}||_{B^{s-1}_{p,r}}, \\
&\quad ||\big(v^{(1)}_x m^{(2)}_x - (v^{(1)} + n^{(1)})m^{(2)} - 2b\big)u^{(12)}_x||_{B^{s-1}_{p,r}} \\
&\qquad \leq C||v^{(1)}_x u^{(12)}_x||_{B^{s}_{p,r}}||m^{(2)}_x||_{B^{s-1}_{p,r}} + C(||(v^{(1)} + n^{(1)})m^{(2)}||_{B^{s}_{p,r}} + |b|)||u^{(12)}_x||_{B^{s-1}_{p,r}}
\end{aligned}$$

$$
\begin{aligned}
&\le C(||m^{(2)}||_{B^s_{p,r}}||n^{(1)}||_{B^s_{p,r}}+1)||u^{(12)}||_{B^{s+1}_{p,r}}\\
&\le C(||m^{(2)}||_{B^s_{p,r}}||n^{(1)}||_{B^s_{p,r}}+1)||m^{(12)}||_{B^{s-1}_{p,r}},\\
&||\big(u^{(2)}_x m^{(2)}_x-(m^{(1)}-u^{(2)})m^{(2)}\big)v^{(12)}_x||_{B^{s-1}_{p,r}}\\
&\le C(||u^{(2)}_x v^{(12)}_x||_{B^s_{p,r}}||m^{(2)}_x||_{B^{s-1}_{p,r}}+||(m^{(1)}-u^{(2)})m^{(2)}||_{B^s_{p,r}}||v^{(12)}_x||_{B^{s-1}_{p,r}})\\
&\le C(||m^{(1)}||_{B^s_{p,r}}+||m^{(2)}||_{B^s_{p,r}})||m^{(2)}||_{B^s_{p,r}}||v^{(12)}||_{B^{s+1}_{p,r}}\\
&\le C(||m^{(1)}||_{B^s_{p,r}}+||m^{(2)}||_{B^s_{p,r}})||m^{(2)}||_{B^s_{p,r}}||n^{(12)}||_{B^{s-1}_{p,r}},\\
&||\big(v^{(1)}_x m^{(2)}-v^{(1)}m^{(2)}_x\big)u^{(12)}||_{B^{s-1}_{p,r}}\\
&\le C(||v^{(1)}_x m^{(2)}||_{B^s_{p,r}}||u^{(12)}||_{B^{s-1}_{p,r}}+||v^{(1)}u^{(12)}||_{B^s_{p,r}}||m^{(2)}_x||_{B^{s-1}_{p,r}})\\
&\le C||m^{(2)}||_{B^s_{p,r}}||n^{(1)}||_{B^s_{p,r}}||m^{(12)}||_{B^{s-1}_{p,r}},
\end{aligned}
$$

and

$$
\begin{aligned}
&||(u^{(2)}m^{(2)})_x v^{(12)}||_{B^{s-1}_{p,r}}\\
&\quad\le C(||u^{(2)}_x m^{(2)}||_{B^s_{p,r}}||v^{(12)}||_{B^{s-1}_{p,r}}+||u^{(2)}v^{(12)}||_{B^s_{p,r}}||m^{(2)}_x||_{B^{s-1}_{p,r}})\\
&\quad\le C||m^{(2)}||^2_{B^s_{p,r}}||n^{(12)}||_{B^{s-1}_{p,r}}.
\end{aligned}
$$

Thus, we obtain

$$
\begin{aligned}
||F(t)||_{B^{s-1}_{p,r}}\le{}& C(||m^{(12)}(t)||_{B^{s-1}_{p,r}}+||n^{(12)}(t)||_{B^{s-1}_{p,r}})\\
&\times(||m^{(1)}(t)||_{B^s_{p,r}}+||m^{(2)}(t)||_{B^s_{p,r}}+||n^{(1)}(t)||_{B^s_{p,r}}+||n^{(2)}(t)||_{B^s_{p,r}}+1)^2
\end{aligned}
$$

provided that $\max(\frac{1}{p},\frac{1}{2})<s\le 1+\frac{1}{p}$. We can also treat $||G(t)||_{B^{s-1}_{p,r}}$ for $\max(\frac{1}{p},\frac{1}{2})<s\le 1+\frac{1}{p}$ in a similar way. Therefore, our Claim (3.3) is guaranteed.

By Lemma 2.1 (1) and the following fact

$$
\begin{aligned}
V(t)\triangleq{}& ||\partial_x(u^{(1)}v^{(1)}-u^{(1)}_x v^{(1)}_x)||_{B^{\frac{1}{p}}_{p,r}\cap L^\infty}+||\partial_x(u^{(1)}v^{(1)}-u^{(1)}_x v^{(1)}_x)||_{B^{s-2}_{p,r}}\\
\le{}& C||\partial_x(u^{(1)}v^{(1)}-u^{(1)}_x v^{(1)}_x)||_{B^s_{p,r}}\\
\le{}& C||u^{(1)}||_{B^{s+2}_{p,r}}||v^{(1)}||_{B^{s+2}_{p,r}}\\
\le{}& C||m^{(1)}||_{B^s_{p,r}}||n^{(1)}||_{B^s_{p,r}}
\end{aligned}
$$

for $s>\max(1-\frac{1}{p},\frac{1}{p},\frac{1}{2})$, but $s\neq 2+\frac{1}{p}$, we have

$$
||m^{(12)}(t)||_{B^{s-1}_{p,r}}\le ||m^{(12)}_0||_{B^{s-1}_{p,r}}+\int_0^t ||F(\tau)||_{B^{s-1}_{p,r}}d\tau+C\int_0^t V(\tau)||m^{(12)}(\tau)||_{B^{s-1}_{p,r}}d\tau
$$

and

$$
||n^{(12)}(t)||_{B^{s-1}_{p,r}}\le ||n^{(12)}_0||_{B^{s-1}_{p,r}}+\int_0^t ||G(\tau)||_{B^{s-1}_{p,r}}d\tau+C\int_0^t V(\tau)||n^{(12)}(\tau)||_{B^{s-1}_{p,r}}d\tau
$$

which together with (3.3) lead to

$$
\begin{aligned}
&||m^{(12)}(t)||_{B^{s-1}_{p,r}} + ||n^{(12)}(t)||_{B^{s-1}_{p,r}} \\
&\le ||m_0^{(12)}||_{B^{s-1}_{p,r}} + ||n_0^{(12)}||_{B^{s-1}_{p,r}} + C\int_0^t (||m^{(12)}(\tau)||_{B^{s-1}_{p,r}} + ||n^{(12)}(\tau)||_{{}^{s-1}_{p,r}}) \\
&\quad \times(||m^{(1)}(\tau)||_{B^{s}_{p,r}} + ||m^{(2)}(\tau)||_{B^{s}_{p,r}} + ||n^{(1)}(\tau)||_{B^{s}_{p,r}} + ||n^{(2)}(\tau)||_{B^{s}_{p,r}} + 1)^2 d\tau.
\end{aligned}
$$

Taking advantage of Gronwall's inequality gives rise to (3.1).

For the critical case (2) $s = 2 + \frac{1}{p}$, here we use the interpolation method to cope with it. Let us choose $s_1 \in (\max(1 - \frac{1}{p}, \frac{1}{p}, \frac{1}{2}) - 1, s - 1)$, $s_2 \in (s - 1, s)$ and $\theta = \frac{s_2-(s-1)}{s_2-s_1} \in (0, 1)$, then $s - 1 = \theta s_1 + (1 - \theta)s_2$. As per the interpolation inequality and the consequence of case (1), we have

$$
\begin{aligned}
&||m^{(12)}(t)||_{B^{s-1}_{p,r}} + ||n^{(12)}(t)||_{B^{s-1}_{p,r}} \\
&\le ||m^{(12)}(t)||^{\theta}_{B^{s_1}_{p,r}} ||m^{(12)}(t)||^{1-\theta}_{B^{s_2}_{p,r}} + ||n^{(12)}(t)(t)||^{\theta}_{B^{s_1}_{p,r}} ||n^{(12)}(t)(t)||^{1-\theta}_{B^{s_2}_{p,r}} \\
&\le CL^{\theta}(s_1; t)((||m^{(1)}(t)||_{B^{s_2}_{p,r}} + ||m^{(2)}(t)||_{B^{s_2}_{p,r}})^{1-\theta} \\
&\quad +(||n^{(1)}(t)||_{B^{s_2}_{p,r}} + ||n^{(2)}(t)||_{B^{s_2}_{p,r}})^{1-\theta}) \\
&\le CL^{\theta}(s-1; t)((||m^{(1)}(t)||_{B^{s}_{p,r}} + ||m^{(2)}(t)||_{B^{s}_{p,r}})^{1-\theta} \\
&\quad +(||n^{(1)}(t)||_{B^{s}_{p,r}} + ||n^{(2)}(t)||_{B^{s}_{p,r}})^{1-\theta})
\end{aligned}
$$

which completes the proof of Lemma 3.1. □

Next we construct the smooth approximation of solutions to the system (1.4).

Lemma 3.2. *Let p and r be as in the statement of Lemma 3.1. Assume that* $s > \max(1 - \frac{1}{p}, \frac{1}{p}, \frac{1}{2})$ *and* $s \neq 1 + \frac{1}{p}$, $(m_0, n_0) \in B^s_{p,r} \times B^s_{p,r}$ *and* $(m^0, n^0) = (0, 0)$. *Then* (1) *there exists a sequence of smooth functions* $(m^k, n^k)_{k\in\mathbb{N}}$ *belonging to* $C(\mathbb{R}^+; B^{\infty}_{p,r} \times B^{\infty}_{p,r})$ *and solving the following linear transport equations*

$$
(T_k) : \begin{cases}
\partial_t m^{k+1} + \frac{1}{2}(u^k v^k - u^k_x v^k_x)\partial_x m^{k+1} = R^k_1(t, x), \\
\partial_t n^{k+1} + \frac{1}{2}(u^k v^k - u^k_x v^k_x)\partial_x n^{k+1} = R^k_2(t, x), \\
m^{k+1}|_{t=0} \triangleq m^{k+1}_0(x) = S_{k+1} m_0, \\
n^{k+1}|_{t=0} \triangleq n^{k+1}_0(x) = S_{k+1} n_0,
\end{cases}
$$

where $R^k_1(t, x) \triangleq -\frac{1}{2}\big((u^k v^k - u^k_x v^k_x)_x - (u^k v^k_x - u^k_x v^k)\big)m^k - bu^k_x$, $R^k_2(t, x) \triangleq -\frac{1}{2}\big((u^k v^k - u^k_x v^k_x)_x + (u^k v^k_x - u^k_x v^k)\big)n^k - bv^k_x$, *and* $S_{k+1} \triangleq \sum_{p=-1}^{k} \Delta_p$ *is the low frequency cut-off operator.*

(2) *there exists* $T > 0$ *such that the solution* $(m^k, n^k)_{k\in\mathbb{N}}$ *is uniformly bounded in* $E^s_{p,r}(T) \times E^s_{p,r}(T)$ *and a Cauchy sequence in* $C([0, T]; B^{s-1}_{p,r} \times B^{s-1}_{p,r})$ *so that it converges to some limit* $(m, n) \in C([0, T]; B^{s-1}_{p,r} \times B^{s-1}_{p,r})$.

Proof. Since all the data $S_{k+1}m_0,\ S_{k+1}n_0 \in B^{\infty}_{p,r}$, it follows from Lemma 2.3 and by induction with respect to the index k that (1) holds.

To prove (2), analogizing the proof of Lemma 3.1 (1), for $s > \max(1-\frac{1}{p}, \frac{1}{p}, \frac{1}{2})$ and $s \neq 1+\frac{1}{p}$, we obtain

$$\begin{aligned} & a_{k+1}(t) \\ & \leq Ce^{CU^k(t)}\left(A + \int_0^t e^{-CU^k(\tau)}(||R_1^k(\tau)||_{B^s_{p,r}} + ||R_2^k(\tau)||_{B^s_{p,r}})d\tau\right), \end{aligned} \tag{3.5}$$

where $a_k(t) \triangleq ||m^k(t)||_{B^s_{p,r}} + ||n^k(t)||_{B^s_{p,r}}$, $A \triangleq ||m_0||_{B^s_{p,r}} + ||n_0||_{B^s_{p,r}}$ and $U^k(t) \triangleq \int_0^t ||m^k(\tau)||_{B^s_{p,r}} ||n^k(\tau)||_{B^s_{p,r}} d\tau$. Since $B^s_{p,r}$ is an algebra, by (3.4), one can have

$$\begin{aligned} & ||R_1^k(t)||_{B^s_{p,r}} + ||R_2^k(t)||_{B^s_{p,r}} \\ & \leq C(||m^k(t)||_{B^s_{p,r}} + ||n^k(t)||_{B^s_{p,r}})(1 + ||m^k(t)||_{B^s_{p,r}} ||n^k(t)||_{B^s_{p,r}}) \\ & \leq C(a_k(t) + a_k^3(t)). \end{aligned}$$

If $a_k(t) < 1$, then by (3.5), $a_{k+1}(t) \leq C(A+t)$, which implies that $(m^k, n^k)_{k\in\mathbb{N}}$ is uniformly bounded in $C([0,T]; B^s_{p,r} \times B^s_{p,r})$. If $a_k(t) \geq 1$, from (3.5) we have

$$a_{k+1}(t) \leq Ce^{CU^k(t)}\left(A + \int_0^t e^{-CU^k(\tau)} a_k^3(\tau) d\tau\right). \tag{3.6}$$

Choose T satisfying $0 < T < \frac{1}{4C^3A^2}$ and suppose

$$a_k(t) \leq \frac{CA}{\sqrt{1-4C^3A^2t}}, \quad \forall t \in [0,T]. \tag{3.7}$$

Due to $e^{C(U^k(t)-U^k(\tau))} \leq \sqrt[4]{\frac{1-4C^3A^2\tau}{1-4C^3A^2t}}$, substituting (3.7) into (3.6) yields

$$\begin{aligned} a_{k+1}(t) & \leq \frac{CA}{\sqrt[4]{1-4C^3A^2t}} + \frac{C}{\sqrt[4]{1-4C^3A^2t}} \int_0^t \frac{C^3A^3}{(1-4C^3A^2\tau)^{\frac{5}{4}}} d\tau \\ & = \frac{CA}{\sqrt[4]{1-4C^3A^2t}} + \frac{C}{\sqrt[4]{1-4C^3A^2t}}\left(\frac{A}{\sqrt[4]{1-4C^3A^2t}} - A\right) \\ & \leq \frac{CA}{\sqrt{1-4C^3A^2t}}, \end{aligned}$$

which implies that

$$(m^k, n^k)_{k\in\mathbb{N}} \quad \text{is uniformly bounded in} \quad C([0,T]; B^s_{p,r} \times B^s_{p,r}).$$

Using the equations (T_k) and the similar argument in the proof of Lemma 3.1 (1), one can easily prove that

$$(\partial_t m^{k+1}, \partial_t m^{k+1})_{k\in\mathbb{N}} \quad \text{is uniformly bounded in} \quad C([0,T]; B^{s-1}_{p,r} \times B^{s-1}_{p,r}).$$

Hence,

$$(m^k, n^k)_{k\in\mathbb{N}} \quad \text{is uniformly bounded in} \quad E^s_{p,r}(T)\times E^s_{p,r}(T).$$

Now it suffices to show that $(m^k, n^k)_{k\in\mathbb{N}}$ is a Cauchy sequence in $C([0,T]; B^{s-1}_{p,r})\times C([0,T]; B^{s-1}_{p,r})$. Actually, for all $k, l\in\mathbb{N}$, from (T_k) we have

$$\begin{aligned}
&\partial_t(m^{k+l+1}-m^{k+1})+\frac{1}{2}(u^{k+l}v^{k+l}-u^{k+l}_x v^{k+l}_x)\partial_x(m^{k+l+1}-m^{k+1})\\
&=-\frac{1}{2}\big((u^{k+l}v^{k+l}-u^{k+l}_x v^{k+l}_x)_x-(u^{k+l}v^{k+l}_x-u^{k+l}_x v^{k+l})\big)(m^{k+l}-m^k)\\
&\quad-\frac{1}{2}\big((u^{k+l}v^{k+l}-u^kv^k)_x-(u^{k+l}_x v^{k+l}_x-u^k_xv^k_x)_x-(u^{k+l}v^{k+l}_x-u^kv^k_x)\\
&\quad+(u^{k+l}_xv^{k+l}-u^k_xv^k)\big)m^k-\frac{1}{2}\big((u^{k+l}v^{k+l}-u^kv^k)-(u^{k+l}_xv^{k+l}_x-u^k_xv^k_x)\big)m^{k+1}_x\\
&\quad-b(u^{k+l}_x-u^k_x)
\end{aligned}$$

and

$$\begin{aligned}
&\partial_t(n^{k+l+1}-n^{k+1})+\frac{1}{2}(u^{k+l}v^{k+l}-u^{k+l}_x v^{k+l}_x)\partial_x(n^{k+l+1}-n^{k+1})\\
&=-\frac{1}{2}\big((u^{k+l}v^{k+l}-u^{k+l}_x v^{k+l}_x)_x+(u^{k+l}v^{k+l}_x-u^{k+l}_x v^{k+l})\big)(n^{k+l}-n^k)\\
&\quad-\frac{1}{2}\big((u^{k+l}v^{k+l}-u^kv^k)_x-(u^{k+l}_x v^{k+l}_x-u^k_xv^k_x)_x-(u^{k+l}_xv^{k+l}-u^k_xv^k)\\
&\quad+(u^{k+l}v^{k+l}_x-u^kv^k_x)\big)n^k-\frac{1}{2}\big((u^{k+l}v^{k+l}-u^kv^k)-(u^{k+l}_xv^{k+l}_x-u^k_xv^k_x)\big)n^{k+1}_x\\
&\quad-b(v^{k+l}_x-v^k_x).
\end{aligned}$$

Similar to the proof of Lemma 3.1 (1), for $s>\max(1-\frac{1}{p},\frac{1}{p},\frac{1}{2})$ and $s\neq 1+\frac{1}{p}, 2+\frac{1}{p}$, we have

$$b^l_{k+1}(t)\le Ce^{CU^{k+l}(t)}\left(b^l_{k+1}(0)+\int_0^t e^{-CU^{k+l}(\tau)}d^l_k(\tau)b^l_k(\tau)d\tau\right),$$

where $b^l_k(t)\triangleq ||(m^{k+l}-m^k)(t)||_{B^{s-1}_{p,r}}+||(n^{k+l}-n^k)(t)||_{B^{s-1}_{p,r}}$, $U^{k+l}(t)\triangleq\int_0^t ||m^{k+l}(\tau)||_{B^s_{p,r}}$ $||m^{k+l}(\tau)||_{B^s_{p,r}}d\tau$, and $d^l_k(t)\triangleq(||m^k(t)||_{B^s_{p,r}}+||m^{k+1}(t)||_{B^s_{p,r}}+||m^{k+l}(t)||_{B^s_{p,r}}$ $+||n^k(t)||_{B^s_{p,r}}+||n^{k+1}(t)||_{B^s_{p,r}}+||n^{k+l}(t)||_{B^s_{p,r}})^2+1$.

Note that

$$\begin{aligned}
||\sum_{q=k+1}^{k+l}\Delta_q m_0||_{B^{s-1}_{p,r}}&=\left(\sum_{j\ge -1}2^{j(s-1)r}||\Delta_j(\sum_{q=k+1}^{k+l}\Delta_q m_0)||^r_{L^p}\right)^{\frac{1}{r}}\\
&\le C\left(\sum_{j=k}^{k+l+1}2^{-jr}2^{jsr}||\Delta_j m_0||^r_{L^p}\right)^{\frac{1}{r}}\\
&\le C2^{-k}||m_0||_{B^s_{p,r}}.
\end{aligned}$$

Likewise,

$$||\sum_{q=k+1}^{k+l}\Delta_q n_0||_{B^{s-1}_{p,r}} \le C2^{-k}||n_0||_{B^s_{p,r}}.$$

Therefore, we have

$$b^l_{k+1}(0) \le C2^{-k}(||m_0||_{B^s_{p,r}} + ||n_0||_{B^s_{p,r}}).$$

According to the fact that $(m^k, n^k)_{k\in\mathbb{N}}$ is uniformly bounded in $E^s_{p,r}(T)\times E^s_{p,r}(T)$, there is a positive constant C_T independent of k, l such that

$$b^l_{k+1}(t) \le C_T\left(2^{-k} + \int_0^t b^l_k(\tau)d\tau\right), \quad \forall t\in[0,T].$$

Finally by induction with respect to the index k, we arrive at

$$\begin{aligned} b^l_{k+1}(t) &\le C_T\left(2^{-k}\sum_{j=0}^{k}\frac{(2TC_T)^j}{j!} + C_T^{k+1}\int_0^t\frac{(t-\tau)^k}{k!}d\tau\right)\\ &\le \left(C_T\sum_{j=0}^{k}\frac{(2TC_T)^j}{j!}\right)2^{-k} + C_T\frac{(TC_T)^{k+1}}{(k+1)!}, \end{aligned}$$

which implies the desired result as $k\to+\infty$.

On the other hand, for the critical point $2+\frac{1}{p}$, we can apply the interpolation method which has been used in the proof of Lemma 3.1 to show that $(m^k, n^k)_{k\in\mathbb{N}}$ is also a Cauchy sequence in $C([0,T]; B^{s-1}_{p,r}\times B^{s-1}_{p,r})$ for this critical case. Therefore, we have completed the proof of Lemma 3.2. □

Now, it is our turn to prove the main theorem of this section.

Theorem 3.2. *Assume that* $1\le p, r\le\infty$ *and* $s>\max(1-\frac{1}{p},\frac{1}{p},\frac{1}{2})$ *but* $s\ne 1+\frac{1}{p}$. *Let* $(m_0, n_0)\in B^s_{p,r}\times B^s_{p,r}$ *and* (m,n) *be the limit of the existing Cauchy sequence in Lemma 3.2 (2). Then there exists a time* $T>0$ *such that* $(m,n)\in E^s_{p,r}(T)\times E^s_{p,r}(T)$ *is the unique solution to the system (1.4), and the mapping* $(m_0,n_0)\mapsto(m,n)$ *is continuous from* $B^s_{p,r}\times B^s_{p,r}$ *into*

$$C([0,T]; B^{s'}_{p,r}\times B^{s'}_{p,r})\cap C^1([0,T]; B^{s'-1}_{p,r}\times B^{s'-1}_{p,r})$$

for all $s'<s$ *if* $r=\infty$, *and* $s'=s$ *if* $1\le r<\infty$.

Proof. We first claim that $(m,n)\in E^s_{p,r}(T)\times E^s_{p,r}(T)$ solves the system (1.4).

In fact, according to Lemma 3.2 (2) and the Fatou lemma, one can have

$$(m,n)\in L^\infty([0,T]; B^s_{p,r}\times B^s_{p,r}).$$

For all $s'<s$, utilizing Lemma 3.2 (2) together with an interpolation argument yields

$$(m^k, n^k)\to(m,n), \text{ as } n\to\infty, \text{ in } C([0,T]; B^{s'}_{p,r}\times B^{s'}_{p,r}).$$

Taking limit in (T_k), we can see that (m, n) solves the system (1.4) in the sense of $C([0, T]; B^{s'-1}_{p,r} \times B^{s'-1}_{p,r})$ for all $s' < s$. By the system (1.4), adopting a similar proof to (3.4) together with Lemma 2.1 (2) and Lemma 2.3 leads to $(m, n) \in E^s_{p,r}(T) \times E^s_{p,r}(T)$.

On the other hand, the continuity with respect to the initial data in

$$C([0, T]; B^{s'}_{p,r} \times B^{s'}_{p,r}) \cap C^1([0, T]; B^{s'-1}_{p,r} \times B^{s'-1}_{p,r}) \quad (\forall s' < s)$$

can be shown by Lemma 3.1 and a simple interpolation argument. The continuity in $C([0, T]; B^s_{p,r} \times B^s_{p,r}) \cap C^1([0, T]; B^{s-1}_{p,r} \times B^{s-1}_{p,r})$ for $r < \infty$ can be proved through using a sequence of viscosity approximation of solutions $(m_\varepsilon, n_\varepsilon)_{\varepsilon>0}$ to the system (1.4). The sequence uniformly converges in $C([0, T]; B^s_{p,r} \times B^s_{p,r}) \cap C^1([0, T]; B^{s-1}_{p,r} \times B^{s-1}_{p,r})$. This completes the proof of Theorem 3.2. □

Remark 3.1. Apparently, for every $s \in \mathbb{R}$, $B^s_{2,2} = H^s$. Theorem 3.2 holds true in the corresponding Sobolev spaces with $\frac{1}{2} < s \neq \frac{3}{2}$, which almost improves the result of Theorem 3.1 demonstrated by Kato's theory with $s \geq 1$ required. Therefore, Theorem 3.2 implies that the conclusion of Theorem 3.1 holds true for initial data $(m_0, n_0) \in H^s(\mathbb{R}) \times H^s(\mathbb{R})$ with $s > \frac{1}{2}$ or for all initial data $(u_0, v_0) \in H^s(\mathbb{R}) \times H^s(\mathbb{R})$ with $s > \frac{5}{2}$.

4. Blow-Up

In this section, we derive the precise blow-up scenario of strong solutions to the system (1.4) and then provide a new blow-up result with respect to initial data.

Theorem 4.1. *Let $(m_0, n_0) \in H^s(\mathbb{R}) \times H^s(\mathbb{R})$ with $s > \frac{1}{2}$ and T be the maximal existence time of the solution (m, n) to the system (1.4), which is guaranteed by Remark 3.1. If $T < \infty$, then*

$$\int_0^T ||m(\tau)||_{L^\infty} ||n(\tau)||_{L^\infty} d\tau = \infty.$$

Proof. Let us prove the theorem using the mathematical induction with respect to the regular index s $(s > \frac{1}{2})$.

Step 1. For $s \in (\frac{1}{2}, 1)$, by Lemma 2.2 and The system (1.4), we have

$$\begin{aligned} ||m(t)||_{H^s} \leq ||m_0||_{H^s} + C \int_0^t ||m(\tau)||_{H^s} (||uv - u_x v_x||_{L^\infty} + ||u_x n + v_x m||_{L^\infty}) d\tau \\ + C \int_0^t (|| \big((u_x n + v_x m) - (u v_x - u_x v) \big) m(\tau)||_{H^s} + ||u_x(\tau)||_{H^s}) d\tau \end{aligned}$$

and

$$\begin{aligned} ||n(t)||_{H^s} \leq ||n_0||_{H^s} + C \int_0^t ||n(\tau)||_{H^s} (||uv - u_x v_x||_{L^\infty} + ||u_x n + v_x m||_{L^\infty}) d\tau \\ + C \int_0^t \left(|| \big((u_x n + v_x m) + (u v_x - u_x v) \big) n(\tau)||_{H^s} + ||v_x(\tau)||_{H^s} \right) d\tau. \end{aligned}$$

Noticing that $u = (1 - \partial_x^2)^{-1} m = p * m$ with $p(x) \triangleq \frac{1}{2} e^{-|x|}$ $(x \in \mathbb{R})$, $u_x = \partial_x p * m$, $u_{xx} = u - m$ and $||p||_{L^1} = ||\partial_x p||_{L^1} = 1$, together with the Young inequality, for all $s \in \mathbb{R}$ we have

$$||u||_{L^\infty},\ ||u_x||_{L^\infty},\ ||u_{xx}||_{L^\infty} \leq C ||m||_{L^\infty} \tag{4.1}$$

and

$$||u||_{H^s},\ ||u_x||_{H^s},\ ||u_{xx}||_{H^s} \leq C||m||_{H^s}. \tag{4.2}$$

Similarly, the identity $v = p * n$ ensures

$$||v||_{L^\infty},\ ||v_x||_{L^\infty},\ ||v_{xx}||_{L^\infty} \leq C||n||_{L^\infty} \tag{4.3}$$

and

$$||v||_{H^s},\ ||v_x||_{H^s},\ ||v_{xx}||_{H^s} \leq C||n||_{H^s}. \tag{4.4}$$

Therefore, we get the following two estimates:

$$\begin{aligned} &||\big((u_x n + v_x m) - (uv_x - u_x v)\big)m||_{H^s} + ||u_x||_{H^s} \\ &\leq C(||m||_{L^\infty}||n||_{L^\infty} + 1)||m||_{H^s} \end{aligned} \tag{4.5}$$

and

$$||uv - u_x v_x||_{L^\infty} + ||u_x n + v_x m||_{L^\infty} \leq C||m||_{L^\infty}||n||_{L^\infty}. \tag{4.6}$$

Hence, we obtain

$$||m(t)||_{H^s} \leq ||m_0||_{H^s} + C\int_0^t (||m(\tau)||_{L^\infty}||n(\tau)||_{L^\infty} + 1)||m(\tau)||_{H^s} d\tau.$$

Likewise for $n(t)$, we have

$$||n(t)||_{H^s} \leq ||n_0||_{H^s} + C\int_0^t (||m(\tau)||_{L^\infty}||n(\tau)||_{L^\infty} + 1)||n(\tau)||_{H^s} d\tau.$$

So, we arrive at

$$\begin{aligned} &||m(t)||_{H^s} + ||n(t)||_{H^s} \\ &\quad \leq ||m_0||_{H^s} + ||n_0||_{H^s} + C\int_0^t (||m||_{L^\infty}||n||_{L^\infty} + 1)(||m||_{H^s} + ||n||_{H^s}) d\tau. \end{aligned} \tag{4.7}$$

Taking advantage of the Gronwall's inequality gives

$$\begin{aligned} ||m(t)||_{H^s} + ||n(t)||_{H^s} \leq &(||m_0||_{H^s} + ||n_0||_{H^s}) \\ &\times e^{C\int_0^t (||m||_{L^\infty}||n||_{L^\infty}+1)d\tau}. \end{aligned} \tag{4.8}$$

Therefore, if $T < \infty$ satisfies $\int_0^T ||m(\tau)||_{L^\infty}||n(\tau)||_{L^\infty} d\tau < \infty$, then we deduce from (4.8) that

$$\limsup_{t \to T}(||m(t)||_{H^s} + ||n(t)||_{H^s}) < \infty, \tag{4.9}$$

which contradicts with the assumption that $T < \infty$ is the maximal existence time. This completes the proof of the theorem for $s \in (\frac{1}{2}, 1)$.

Step 2. For $s \in [1, \frac{3}{2})$, applying Lemma 2.1 (1) to the first equation of the system (1.4), we have

$$||m(t)||_{H^s} \leq ||m_0||_{H^s} + C\int_0^t ||m(\tau)||_{H^s}||u_x n + v_x m||_{H^{\frac{1}{2}}\cap L^\infty} d\tau \\ + C\int_0^t (||\big((u_x n + v_x m) - (uv_x - u_x v)\big)m(\tau)||_{H^s} + ||u_x(\tau)||_{H^s})d\tau.$$

Noticing that

$$||u_x n + v_x m||_{H^{\frac{1}{2}}\cap L^\infty} \leq C||u_x n + v_x m||_{H^{\frac{1}{2}+\varepsilon_0}} \leq C||m||_{H^{\frac{1}{2}+\varepsilon_0}}||n||_{H^{\frac{1}{2}+\varepsilon_0}},$$

where $\varepsilon_0 \in (0, \frac{1}{2})$. Using (4.5) and the fact that $H^{\frac{1}{2}+\varepsilon_0}(\mathbb{R}) \hookrightarrow H^{\frac{1}{2}}(\mathbb{R}) \cap L^\infty(\mathbb{R})$, leads to

$$||m(t)||_{H^s} \leq ||m_0||_{H^s} + C\int_0^t (||m||_{H^{\frac{1}{2}+\varepsilon_0}}||n||_{H^{\frac{1}{2}+\varepsilon_0}} + 1)||m(\tau)||_{H^s} d\tau.$$

For the second equation of the system (1.4), we can deal with it in a similar way and obtain that

$$||n(t)||_{H^s} \leq ||n_0||_{H^s} + C\int_0^t (||m||_{H^{\frac{1}{2}+\varepsilon_0}}||n||_{H^{\frac{1}{2}+\varepsilon_0}} + 1)||n(\tau)||_{H^s} d\tau.$$

Hence, we have

$$||m(t)||_{H^s} + ||n(t)||_{H^s} \\ \leq ||m_0||_{H^s} + ||n_0||_{H^s} + C\int_0^t (||m||_{H^{\frac{1}{2}+\varepsilon_0}}||n||_{H^{\frac{1}{2}+\varepsilon_0}} + 1)(||m||_{H^s} + ||n||_{H^s})d\tau,$$

which implies the following results by the Gronwall's inequality

$$||m(t)||_{H^s} + ||n(t)||_{H^s} \leq (||m_0||_{H^s} + ||n_0||_{H^s}) \\ \times e^{C\int_0^t (||m||_{H^{\frac{1}{2}+\varepsilon_0}}||n||_{H^{\frac{1}{2}+\varepsilon_0}} + 1)d\tau}. \quad (4.10)$$

Therefore, if $T < \infty$ satisfies $\int_0^T ||m(\tau)||_{L^\infty}||n(\tau)||_{L^\infty} d\tau < \infty$, then we deduce from the uniqueness of the solution to the system (1.4) and (4.9) with $\frac{1}{2} + \varepsilon_0 \in (\frac{1}{2}, 1)$ that

$$||m(t)||_{H^{\frac{1}{2}+\varepsilon_0}}||n(t)||_{H^{\frac{1}{2}+\varepsilon_0}} \quad \text{is uniformly bounded in} \quad t \in (0, T).$$

This along with (4.10) implies that

$$\limsup_{t\to T}(||m(t)||_{H^s} + ||n(t)||_{H^s}) < \infty, \quad (4.11)$$

which contradicts with the assumption that $T < \infty$ is the maximal existence time. Thus, the theorem is also correct for $s \in [1, \frac{3}{2})$.

Step 3. For $s \in (1, 2)$, differentiating the system (1.4) with respect to x, we have

$$\begin{aligned}
&\partial_t m_x + \frac{1}{2}(uv - u_x v_x)\partial_x m_x \\
&\quad = \left(\frac{1}{2}(uv_x - u_x v) - (u_x n + v_x m)\right) m_x - \frac{1}{2}((u_x n + v_x m) \\
&\qquad -(uv_x - u_x v))_x m - bu_{xx} \\
&\quad \triangleq R_1(t, x)
\end{aligned}$$

and

$$\begin{aligned}
&\partial_t n_x + \frac{1}{2}(uv - u_x v_x)\partial_x n_x \\
&\quad = \left(-\frac{1}{2}(uv_x - u_x v) - (u_x n + v_x m)\right) n_x - \frac{1}{2}((u_x n + v_x m) \\
&\qquad +(uv_x - u_x v))_x n - bv_{xx} \\
&\quad \triangleq R_2(t, x).
\end{aligned}$$

By Lemma 2.2 with $s - 1 \in (0, 1)$, we get

$$\begin{aligned}
||m_x(t)||_{H^{s-1}} \le\ & ||\partial_x m_0||_{H^{s-1}} + C\int_0^t ||R_1(\tau)||_{H^{s-1}} d\tau \\
& + C\int_0^t ||m_x(\tau)||_{H^{s-1}}(||uv - u_x v_x||_{L^\infty} + ||u_x n + v_x m||_{L^\infty}) d\tau
\end{aligned}$$

and

$$\begin{aligned}
||n_x(t)||_{H^{s-1}} \le\ & ||\partial_x n_0||_{H^{s-1}} + C\int_0^t ||R_2(\tau)||_{H^{s-1}} d\tau \\
& + C\int_0^t ||n_x(\tau)||_{H^{s-1}}(||uv - u_x v_x||_{L^\infty} + ||u_x n + v_x m||_{L^\infty}) d\tau.
\end{aligned}$$

Due to (2.3) and (4.1)–(4.4), we have

$$\begin{aligned}
&||[\frac{1}{2}(uv_x - u_x v) - (u_x n + v_x m)]m_x||_{H^{s-1}} \\
&\quad \le C(||\frac{1}{2}(uv_x - u_x v) - (u_x n + v_x m)||_{H^s}||m||_{L^\infty} \\
&\qquad + ||\frac{1}{2}(uv_x - u_x v) - (u_x n + v_x m)||_{L^\infty}||m_x||_{H^{s-1}}) \\
&\quad \le C||m||_{L^\infty}||n||_{L^\infty}||m||_{H^s},
\end{aligned}$$

and

$$\begin{aligned}
&|| - \frac{1}{2}[(u_x n + v_x m) - (uv_x - u_x v)]_x m + bu_{xx}||_{H^{s-1}} \\
&\quad \le C(||m||_{H^s}||(u_x n + v_x m) - (uv_x - u_x v)||_{L^\infty} \\
&\qquad + ||m||_{L^\infty}||(u_x n + v_x m) - (uv_x - u_x v)||_{H^s}) + |b|||m||_{H^{s-1}} \\
&\quad \le C(||m||_{L^\infty}||n||_{L^\infty} + 1)||m||_{H^s},
\end{aligned}$$

which together with (4.6) imply

$$||m_x(t)||_{H^{s-1}} \leq ||m_0||_{H^s} + C\int_0^t (||m(\tau)||_{L^\infty}||n(\tau)||_{L^\infty}+1)||m(\tau)||_{H^s}d\tau,$$

and

$$||n_x(t)||_{H^{s-1}} \leq ||n_0||_{H^s} + C\int_0^t (||m(\tau)||_{L^\infty}||n(\tau)||_{L^\infty}+1)||n(\tau)||_{H^s}d\tau.$$

Thus, we have

$$\begin{aligned}&||m_x(t)||_{H^{s-1}}+||n_x(t)||_{H^{s-1}}\\ &\quad\leq ||m_0||_{H^s}+||n_0||_{H^s}+C\int_0^t (||m||_{L^\infty}||n||_{L^\infty}+1)(||m||_{H^s}+||n||_{H^s})d\tau.\end{aligned}$$

Casting (4.7) with $s-1$ and using the above inequality lead to

$$\begin{aligned}&||m(t)||_{H^{s}}+||n(t)||_{H^{s}}\\ &\quad\leq ||m_0||_{H^s}+||n_0||_{H^s}+C\int_0^t (||m||_{L^\infty}||n||_{L^\infty}+1)(||m||_{H^s}+||n||_{H^s})d\tau.\end{aligned}$$

Adopting the procedure similar to Step 1 guarantees the theorem is valid for $s\in(1,2)$.
Step 4. For $s=k\in\mathbb{N}$ and $k\geq 2$, differentiating The system (1.4) $k-1$ times with respect to x gives

$$\begin{aligned}\left(\partial_t+\frac{1}{2}(uv-u_xv_x)\partial_x\right)\partial_x^{k-1}m &= -\frac{1}{2}\sum_{l=0}^{k-2}C_{k-1}^l\partial_x^{k-l-1}(uv-u_xv_x)\partial_x^{l+1}m - b\partial_x^{k-1}u_x\\ &\quad -\frac{1}{2}\partial_x^{k-1}[((u_xn+v_xm)-(uv_x-u_xv))\,m]\\ &\triangleq F_1(t,x)\end{aligned}$$

and

$$\begin{aligned}\left(\partial_t+\frac{1}{2}(uv-u_xv_x)\partial_x\right)\partial_x^{k-1}n &= -\frac{1}{2}\sum_{l=0}^{k-2}C_{k-1}^l\partial_x^{k-l-1}(uv-u_xv_x)\partial_x^{l+1}n - b\partial_x^{k-1}v_x\\ &\quad -\frac{1}{2}\partial_x^{k-1}[((u_xn+v_xm)+(uv_x-u_xv))\,n]\\ &\triangleq F_2(t,x),\end{aligned}$$

which together with Lemma 2.1 (1) imply

$$||\partial_x^{k-1}m(t)||_{H^1}\leq ||m_0||_{H^k}+\int_0^t ||F_1(\tau)||_{H^1}d\tau + C\int_0^t ||u_xn+v_xm||_{H^{\frac{1}{2}}\cap L^\infty}||m||_{H^k}d\tau$$

and

$$||\partial_x^{k-1}n(t)||_{H^1}\leq ||n_0||_{H^k}+\int_0^t ||F_2(\tau)||_{H^1}d\tau + C\int_0^t ||u_xn+v_xm||_{H^{\frac{1}{2}}\cap L^\infty}||n||_{H^k}d\tau.$$

Because of inequalities (4.1)–(4.4), we have

$$\begin{aligned}
&||-\frac{1}{2}\sum_{l=0}^{k-2}C_{k-1}^{l}\partial_x^{k-l-1}(uv-u_xv_x)\partial_x^{l+1}m||_{H^1}\\
&\le C(k)\sum_{l=0}^{k-2}||\partial_x^{k-l-1}(uv-u_xv_x)||_{L^\infty}||m||_{H^{l+2}}\\
&\le C(k)\sum_{l=0}^{k-2}||uv-u_xv_x||_{H^{k-l-\frac{1}{2}+\varepsilon_0}}||m||_{H^{l+2}}\\
&\le C(k)||uv-u_xv_x||_{H^{k-\frac{1}{2}+\varepsilon_0}}||m||_{H^k}\\
&\le C(k)||m||_{H^{k-\frac{1}{2}+\varepsilon_0}}||n||_{H^{k-\frac{1}{2}+\varepsilon_0}}||m||_{H^k}, \qquad (4.12)\\
&||-\frac{1}{2}\partial_x^{k-1}[((u_xn+v_xm)-(uv_x-u_xv))m]-b\partial_x^{k-1}u_x||_{H^1}\\
&\le C||[(u_xn+v_xm)-(uv_x-u_xv)]m||_{H^k}+|b|||u_x||_{H^k}\\
&\le C(||m||_{L^\infty}||m||_{L^\infty}+1)||m||_{H^k}\\
&\le C(||m||_{H^{k-\frac{1}{2}+\varepsilon_0}}||n||_{H^{k-\frac{1}{2}+\varepsilon_0}}+1)||m||_{H^k}, \qquad (4.13)
\end{aligned}$$

and

$$\begin{aligned}
||u_xn+v_xm||_{H^{\frac{1}{2}}\cap L^\infty} &\le C||u_xn+v_xm||_{H^{k-\frac{1}{2}+\varepsilon_0}}\\
&\le C||m||_{H^{k-\frac{1}{2}+\varepsilon_0}}||n||_{H^{k-\frac{1}{2}+\varepsilon_0}},
\end{aligned}$$

where $\varepsilon_0\in(0,\frac{1}{2})$ and

$$H^{k-\frac{1}{2}+\varepsilon_0}(\mathbb{R})\hookrightarrow H^{\frac{1}{2}+\varepsilon_0}(\mathbb{R})\hookrightarrow H^{\frac{1}{2}}(\mathbb{R})\cap L^\infty(\mathbb{R})\quad\text{with}\quad k\ge 2, \qquad (4.14)$$

is used in the above derivation. So, we obtain

$$||\partial_x^{k-1}m(t)||_{H^1}\le||m_0||_{H^k}+C\int_0^t(||m||_{H^{k-\frac{1}{2}+\varepsilon_0}}||n||_{H^{k-\frac{1}{2}+\varepsilon_0}}+1)||m||_{H^k}d\tau,$$

and

$$||\partial_x^{k-1}n(t)||_{H^1}\le||n_0||_{H^k}+C\int_0^t(||m||_{H^{k-\frac{1}{2}+\varepsilon_0}}||n||_{H^{k-\frac{1}{2}+\varepsilon_0}}+1)||n||_{H^k}d\tau,$$

which lead to

$$\begin{aligned}
&||\partial_x^{k-1}m(t)||_{H^1}+||\partial_x^{k-1}n(t)||_{H^1}\\
&\le||m_0||_{H^k}+||n_0||_{H^k}+C\int_0^t(||m||_{H^{k-\frac{1}{2}+\varepsilon_0}}||n||_{H^{k-\frac{1}{2}+\varepsilon_0}}+1)(||m||_{H^k}+||n||_{H^k})d\tau.
\end{aligned}$$

Therefore by the Gronwall's inequality and (4.10) with $s=1$, we have

$$\begin{aligned}
||m(t)||_{H^k}+||n(t)||_{H^k}\le&(||m_0||_{H^k}+||n_0||_{H^k})\\
&\times e^{C\int_0^t(||m||_{H^{k-\frac{1}{2}+\varepsilon_0}}||n||_{H^{k-\frac{1}{2}+\varepsilon_0}}+1)d\tau}. \qquad (4.15)
\end{aligned}$$

If $T < \infty$ satisfies $\int_0^T ||m(\tau)||_{L^\infty}||n(\tau)||_{L^\infty}d\tau < \infty$, applying Step 3 with $\frac{3}{2}+\varepsilon_0 \in (1,2)$ and by induction with respect to $k \geq 2$, we see that $||m(t)||_{H^{k-\frac{1}{2}+\varepsilon_0}}||n(t)||_{H^{k-\frac{1}{2}+\varepsilon_0}}$ is uniformly bounded in $t \in (0,T)$. By (4.15), we have

$$\limsup_{t\to T}(||m(t)||_{H^k}+||n(t)||_{H^k}) < \infty, \tag{4.16}$$

which contradicts with the assumption that $T < \infty$ is the maximal existence time. This completes the proof of the theorem for $s = k \in \mathbb{N}$ and $k \geq 2$.

Step 5. For $s \in (k, k+1)$, $k \in \mathbb{N}$ and $k \geq 2$, differentiating the system (1.4) k times with respect to x yields

$$\begin{aligned}\left(\partial_t+\frac{1}{2}(uv-u_xv_x)\partial_x\right)\partial_x^k m &= -\frac{1}{2}\sum_{l=0}^{k-1}C_k^l\partial_x^{k-l}(uv-u_xv_x)\partial_x^{l+1}m - b\partial_x^k u_x\\ &\quad -\frac{1}{2}\partial_x^k[((u_xn+v_xm)-(uv_x-u_xv))m]\\ &\triangleq G_1(t,x)\end{aligned}$$

and

$$\begin{aligned}\left(\partial_t+\frac{1}{2}(uv-u_xv_x)\partial_x\right)\partial_x^k n &= -\frac{1}{2}\sum_{l=0}^{k-1}C_k^l\partial_x^{k-l}(uv-u_xv_x)\partial_x^{l+1}n - b\partial_x^k v_x\\ &\quad -\frac{1}{2}\partial_x^k[((u_xn+v_xm)+(uv_x-u_xv))n]\\ &\triangleq G_2(t,x),\end{aligned}$$

which together with Lemma 2.2 as $s-k \in (0,1)$ imply

$$\begin{aligned}||\partial_x^k m(t)||_{H^{s-k}} &\leq ||\partial_x^k m_0||_{H^{s-k}} + C\int_0^t ||G_1(\tau)||_{H^{s-k}}d\tau\\ &\quad +C\int_0^t (||uv-u_xv_x||_{L^\infty}+||u_xn+v_xm||_{L^\infty})||\partial_x^k m(\tau)||_{H^{s-k}}d\tau\end{aligned}$$

and

$$\begin{aligned}||\partial_x^k n(t)||_{H^{s-k}} &\leq ||\partial_x^k n_0||_{H^{s-k}} + C\int_0^t ||G_2(\tau)||_{H^{s-k}}d\tau\\ &\quad +C\int_0^t (||uv-u_xv_x||_{L^\infty}+||u_xn+v_xm||_{L^\infty})||\partial_x^k n(\tau)||_{H^{s-k}}d\tau.\end{aligned}$$

By (4.14) and using the procedure similar to (4.12)–(4.13), we obtain

$$\begin{aligned}&||-\frac{1}{2}\partial_x^k[((u_xn+v_xm)-(uv_x-u_xv))m]-b\partial_x^k u_x||_{H^{s-k}}\\ &\leq C(||m||_{H^{k-\frac{1}{2}+\varepsilon_0}}||n||_{H^{k-\frac{1}{2}+\varepsilon_0}}+1)||m||_{H^s},\end{aligned}$$

and

$$||-\frac{1}{2}\sum_{l=1}^{k-1}C_k^l\partial_x^{k-l}(uv-u_xv_x)\partial_x^{l+1}m||_{H^{s-k}} \leq C(k)||m||_{H^{k-\frac{1}{2}+\varepsilon_0}}||n||_{H^{k-\frac{1}{2}+\varepsilon_0}}||m||_{H^s}.$$

Applying (2.3), (4.1)–(4.4), and (4.14) leads to

$$\begin{aligned}
&|| -\frac{1}{2}C_k^0\partial_x^k(uv-u_xv_x)m_x||_{H^{s-k}}\\
&\le C(||m_x||_{H^{s-k+1}}||\partial_x^{k-1}(uv-u_xv_x)||_{L^\infty}+||m_x||_{L^\infty}||\partial_x^k(uv-u_xv_x)||_{H^{s-k}})\\
&\le C(||m||_{H^{s-k+2}}||uv-u_xv_x||_{H^{k-\frac{1}{2}+\varepsilon_0}}+||m||_{H^{k-\frac{1}{2}+\varepsilon_0}}||uv-u_xv_x||_{H^s})\\
&\le C||m||_{H^{k-\frac{1}{2}+\varepsilon_0}}||n||_{H^{k-\frac{1}{2}+\varepsilon_0}}||m||_{H^s}.
\end{aligned}$$

Thus, we have

$$||\partial_x^k m(t)||_{H^{s-k}}\le ||m_0||_{H^s}+C\int_0^t(||m||_{H^{k-\frac{1}{2}+\varepsilon_0}}||n||_{H^{k-\frac{1}{2}+\varepsilon_0}}+1)||m||_{H^s}d\tau,$$

and

$$||\partial_x^k n(t)||_{H^{s-k}}\le ||n_0||_{H^s}+C\int_0^t(||m||_{H^{k-\frac{1}{2}+\varepsilon_0}}||n||_{H^{k-\frac{1}{2}+\varepsilon_0}}+1)||n||_{H^s}d\tau,$$

which imply

$$\begin{aligned}
&||\partial_x^k m(t)||_{H^{s-k}}+||\partial_x^k n(t)||_{H^{s-k}}\\
&\le ||m_0||_{H^s}+||n_0||_{H^s}+C\int_0^t(||m||_{H^{k-\frac{1}{2}+\varepsilon_0}}||n||_{H^{k-\frac{1}{2}+\varepsilon_0}}+1)(||m||_{H^s}+||n||_{H^s})d\tau.
\end{aligned}$$

Casting with (4.7) with $s-k\in(0,1)$ and using the above inequality lead to

$$\begin{aligned}
&||m(t)||_{H^s}+||n(t)||_{H^s}\\
&\le ||m_0||_{H^s}+||n_0||_{H^s}+C\int_0^t(||m||_{H^{k-\frac{1}{2}+\varepsilon_0}}||n||_{H^{k-\frac{1}{2}+\varepsilon_0}}+1)(||m||_{H^s}+||n||_{H^s})d\tau.
\end{aligned}$$

By adopting Gronwall's inequality, Step 3 with $\frac{3}{2}+\varepsilon_0\in(1,2)$, and the similar argument as shown in Step 4, we can arrive at the desired result.

In summary, the above 5 steps complete the proof of the theorem. □

Remark 4.1. The maximal existence time T in Theorem 4.1 can be chosen independent of the regularity index s. Let $(m_0,n_0)\in H^s\times H^s$ with $s>\frac{1}{2}$ and some $s'\in(\frac{1}{2},s)$. Then, Remark 3.1 ensures that there exists a unique $H^s\times H^s$ (resp., $H^{s'}\times H^{s'}$) solution (m_s,n_s) (resp., $(m_{s'},n_{s'})$) to the system (1.4) with the maximal existence time T_s (resp., $T_{s'}$). Since $H^s\hookrightarrow H^{s'}$, it follows from the uniqueness that $T_s\le T_{s'}$ and $(m_s,n_s)\equiv(m_{s'},n_{s'})$ on $[0,T_s)$. On the other hand, suppose that $T_s<T_{s'}$, then $(m_{s'},n_{s'})\in C([0,T_s];H^{s'}\times H^{s'})$. Hence, $(m_{s'},n_{s'})\in L^2(0,T_s;L^\infty\times L^\infty)$, which together with the Holder inequality leads to a contraction to Theorem 4.1. Therefore, $T_s=T_{s'}$.

Utilizing the Sobolev's embedding theorem and Theorem 4.1, we have the following blow-up criterion.

Corollary 4.1. *Let $(m_0,n_0)\in H^s(\mathbb{R})\times H^s(\mathbb{R})$ $(s>\frac{1}{2})$ and $T>0$ be the maximal existence time of the corresponding solution (m,n) to the system (1.4).*

Then, the solution (m,n) blows up in finite time if and only if

$$\limsup_{t\to T}||m(t,\cdot)||_{L^\infty}=\infty\quad or\quad \limsup_{t\to T}||n(t,\cdot)||_{L^\infty}=\infty.$$

Let us now turn our attention to the precise blow-up scenario for sufficiently regular solutions to the system (1.4) with $b = 0$, which is required for discussion in the remaining parts of this section. To do so, let us first consider the following initial value problem:

$$\begin{cases} \partial_t q(t,x) = \frac{1}{2}(uv - u_x v_x)(t, q(t,x)), & (t,x) \in (0,T) \times \mathbb{R}, \\ q(0,x) = x, & x \in \mathbb{R}, \end{cases} \tag{4.17}$$

for the flow q generated by $\frac{1}{2}(uv - u_x v_x)$.

The following lemmas are very crucial to the blow-up phenomena of strong solutions to the system (1.4) with $b = 0$.

Lemma 4.1. *Let* $(m_0, n_0) \in H^s(\mathbb{R}) \times H^s(\mathbb{R})$ *(*$s > \frac{1}{2}$*) and* $T > 0$ *be the maximal existence time of the corresponding solution* (m, n) *to the system* (1.4) *with* $b = 0$*. Then Eq.* (4.17) *has a unique solution* $q \in C^1([0,T) \times \mathbb{R}; \mathbb{R})$*. Moreover, the mapping* $q(t, \cdot)$ *is an increasing diffeomorphism of* $\mathbb{R}$ *with*

$$q_x(t,x) = \exp\left(\frac{1}{2}\int_0^t (u_x n + v_x m)(s, q(s,x))ds\right) > 0, \tag{4.18}$$

for all $(t,x) \in [0,T) \times \mathbb{R}$.

Proof. Since $(u, v) \in C([0,T); H^s(\mathbb{R}) \times H^s(\mathbb{R})) \cap C^1([0,T); H^{s-1}(\mathbb{R}) \times H^{s-1}(\mathbb{R}))$ as $s > \frac{5}{2}$, it follows from the fact $H^{s-1}(\mathbb{R}) \hookrightarrow Lip(\mathbb{R})$ ($s > \frac{5}{2}$) that $\frac{1}{2}(uv - u_x v_x)$ is bounded and Lipschitz continuous in the space variable x and of class C^1 in time variable t. Then the classical ODE theory ensures that Eq. (4.17) has a unique solution $q \in C^1([0,T) \times \mathbb{R}; \mathbb{R})$. Differentiating Eq. (4.17) with respect to x gives

$$\begin{cases} \partial_t q_x(t,x) = \frac{1}{2}(u_x n + v_x m)(t, q(t,x)) q_x(t,x), & (t,x) \in (0,T) \times \mathbb{R}, \\ q_x(0,x) = 1, & x \in \mathbb{R}, \end{cases}$$

which leads to (4.18).

On the other hand, for all $t < T$, by Sobolev's embedding theorem, we have

$$\sup_{(s,x) \in [0,T) \times \mathbb{R}} \left| \frac{1}{2}(u_x n + v_x m)(s,x) \right| < \infty.$$

This along with (4.18) implies that there exists a constant $C > 0$ such that

$$q_x(t,x) \geq e^{-Ct}, \quad \forall (t,x) \in [0,T) \times \mathbb{R}.$$

So, the mapping $q(t, \cdot)$ is an increasing diffeomorphism of $\mathbb{R}$ before its blow-up. $\square$

Lemma 4.2. *Let* $(m_0, n_0) \in H^s(\mathbb{R}) \times H^s(\mathbb{R})$ *(*$s > \frac{1}{2}$*) and* $T > 0$ *be the maximal existence time of the solution* (m, n) *corresponding to the system* (1.4) *with* $b = 0$*. Then, we have*

$$m(t, q(t,x)) q_x(t,x) = m_0(x) \exp\left(\frac{1}{2}\int_0^t (u v_x - u_x v)(s, q(s,x))ds\right), \tag{4.19}$$

and

$$n(t, q(t,x)) q_x(t,x) = n_0(x) \exp\left(-\frac{1}{2}\int_0^t (u v_x - u_x v)(s, q(s,x))ds\right). \tag{4.20}$$

for all $(t,x)\in[0,T)\times\mathbb{R}$. *Moreover, if there exists a* $C>0$ *such that* $(u_xn+v_xm)(t,x)\geq -C$ *and* $||(uv_x-u_xv)(t,\cdot)||_{L^\infty}\leq C$ *for all* $(t,x)\in[0,T)\times\mathbb{R}$, *then*

$$||m(t,\cdot)||_{L^\infty}\leq Ce^{Ct}||m_0||_{H^s} \quad and \quad ||n(t,\cdot)||_{L^\infty}\leq Ce^{Ct}||n_0||_{H^s}, \tag{4.21}$$

for all $t\in[0,T)$.

Proof. Differentiating the left-hand side of (4.19)–(4.20) with respect to t and making use of (4.17)–(4.18) and the system (1.4), we have

$$\begin{aligned}
&\frac{d}{dt}(m(t,q(t,x))q_x(t,x))\\
&\quad=(m_t(t,q)+m_x(t,q)q_t(t,x))q_x(t,x)+m(t,q)q_{xt}(t,x)\\
&\quad=\big(m_t+\frac{1}{2}(uv-u_xv_x)m_x+\frac{1}{2}(u_xn+v_xm)m\big)(t,q(t,x))q_x(t,x)\\
&\quad=\frac{1}{2}(uv_x-u_xv)(t,q(t,x))m(t,q(t,x))q_x(t,x)
\end{aligned}$$

and

$$\begin{aligned}
&\frac{d}{dt}(n(t,q(t,x))q_x(t,x))\\
&\quad=(n_t(t,q)+n_x(t,q)q_t(t,x))q_x(t,x)+n(t,q)q_{xt}(t,x)\\
&\quad=\big(n_t+\frac{1}{2}(uv-u_xv_x)n_x+\frac{1}{2}(u_xn+v_xm)n\big)(t,q(t,x))q_x(t,x)\\
&\quad=-\frac{1}{2}(uv_x-u_xv)(t,q(t,x))n(t,q(t,x))q_x(t,x),
\end{aligned}$$

which guarantee (4.19) and (4.20). By Lemma 4.1, in light of (4.18)–(4.20) and the assumption, for all $t\in[0,T)$ we obtain

$$\begin{aligned}
||m(t,\cdot)||_{L^\infty}&=||m(t,q(t,\cdot))||_{L^\infty}\\
&=||e^{\frac{1}{2}\int_0^t(uv_x-u_xv)(s,\cdot)ds}q_x^{-1}(t,\cdot)m_0(\cdot)||_{L^\infty}\\
&\leq Ce^{Ct}||m_0||_{H^s}
\end{aligned}$$

and

$$\begin{aligned}
||n(t,\cdot)||_{L^\infty}&=||n(t,q(t,\cdot))||_{L^\infty}\\
&=||e^{-\frac{1}{2}\int_0^t(uv_x-u_xv)(s,\cdot)ds}q_x^{-1}(t,\cdot)n_0(\cdot)||_{L^\infty}\\
&\leq Ce^{Ct}||n_0||_{H^s},
\end{aligned}$$

which complete the proof of the lemma. □

The following theorem shows the precise blow-up scenario for sufficiently regular solutions to the system (1.4) with $b=0$.

Theorem 4.2. *Let* $(m_0,n_0)\in H^s(\mathbb{R})\times H^s(\mathbb{R})$ $(s>\frac{1}{2})$ *and* $T>0$ *be the maximal existence time of the solution* (m,n) *corresponding to the system* (1.4) *with* $b=0$. *Then the solution* (m,n) *blows up in finite time if and only if*

$$\liminf_{t\to T}\inf_{x\in\mathbb{R}}\{(u_xn+v_xm)(t,x)\}=-\infty \quad or \quad \limsup_{t\to T}(||(uv_x-u_xv)(t,\cdot)||_{L^\infty})=\infty.$$

Proof. Assume that the solution (m, n) blows up in finite time ($T < \infty$) and there exists a constant $C > 0$ such that

$$(u_x n + v_x m)(t, x) \geq -C \quad \text{and} \quad ||(uv_x - u_x v)(t, \cdot)||_{L^\infty} \leq C, \quad \forall\,(t, x) \in [0, T) \times \mathbb{R}.$$

By (4.21), we have

$$\int_0^T ||m(t)||_{L^\infty} ||n(t)||_{L^\infty} dt \leq C^2 T e^{2CT} ||m_0||_{H^s} ||n_0||_{H^s} < \infty,$$

which contradicts to Theorem 4.1.

On the other hand, by Sobolev's embedding theorem, we can see that if

$$\liminf_{t\to T} \inf_{x\in\mathbb{R}} \{(u_x n + v_x m)(t, x)\} = -\infty \quad \text{or} \quad \limsup_{t\to T}(||(uv_x - u_x v)(t, \cdot)||_{L^\infty}) = \infty,$$

then the solution (m, n) will blow up in finite time. Now, the proof of the theorem is completed. □

Remark 4.2. If $v = 2u$, then Theorem 4.2 recovers the corresponding result in [36].

In order to have a new blow-up criterion with respect to the initial data of strong solutions to the system (1.4) with $b = 0$, we first investigate the transport equation in terms of $\frac{1}{2}(u_x n + v_x m)$, which is actually the slope of $\frac{1}{2}(uv - u_x v_x)$.

Lemma 4.3. *Let $(m_0, n_0) \in H^s(\mathbb{R}) \times H^s(\mathbb{R})$ with $s > \frac{1}{2}$ and $T > 0$ be the maximal existence time of the solution (m, n) corresponding to the system (1.4) with $b = 0$. Set $M = M(t, x) \triangleq (u_x n + v_x m)(t, x)$. Then for all $(t, x) \in [0, T) \times \mathbb{R}$,*

$$\begin{aligned} &M_t + \frac{1}{2}(uv - u_x v_x) M_x \\ &= -\frac{1}{2}M^2 - \frac{1}{2}n(1 - \partial_x^2)^{-1}(u_x M) - \frac{1}{2}m(1 - \partial_x^2)^{-1}(v_x M) - \frac{1}{2}n\partial_x(1 - \partial_x^2)^{-1}(uM) \\ &\quad -\frac{1}{2}m\partial_x(1 - \partial_x^2)^{-1}(vM) - \frac{1}{2}(uv_x - u_x v)(u_x n - v_x m) \\ &\quad +\frac{1}{2}n\partial_x(1 - \partial_x^2)^{-1}\left((uv_x - u_x v)m\right) - \frac{1}{2}m\partial_x(1 - \partial_x^2)^{-1}\left((uv_x - u_x v)n\right). \end{aligned} \tag{4.22}$$

Moreover, if $m_0(x),\ n_0(x) \geq 0$ for all $x \in \mathbb{R}$, then

$$|u_x(t, x)| \leq u(t, x), \quad |v_x(t, x)| \leq v(t, x),$$

and

$$M_t + \frac{1}{2}(uv - u_x v_x) M_x \leq -\frac{1}{2}M^2 + \frac{7}{2}||u||_{L^\infty}||v||_{L^\infty}^2 m + \frac{7}{2}||u||_{L^\infty}^2 ||v||_{L^\infty} n,$$

for all $(t, x) \in [0, T) \times \mathbb{R}$.

Proof. As per Remark 4.1, here we just prove the lemma for the case of $s \geq 3$. Apparently, a direct calculation leads to

$$\begin{aligned} &M_t + \frac{1}{2}(uv - u_x v_x)M_x \\ &\quad = u_{xt}n + v_{xt}m + u_x n_t + v_x m_t + \frac{1}{2}(uv - u_x v_x)(u_x n_x + v_x m_x + u_{xx}n + v_{xx}m). \end{aligned} \tag{4.23}$$

From the system (1.4), we infer that

$$\begin{aligned} &(1 - \partial_x^2)(u_t + \frac{1}{2}(uv - u_x v_x)u_x) \\ &\quad = m_t + \frac{1}{2}(1 - \partial_x^2)((uv - u_x v_x)u_x) \\ &\quad = -\frac{1}{2}(uv - u_x v_x)m_x - \frac{1}{2}(u_x n + v_x m)m + \frac{1}{2}(uv_x - u_x v)m + \frac{1}{2}(uv - u_x v_x)u_x \\ &\qquad -\frac{1}{2}\partial_x^2((uv - u_x v_x)u_x) \\ &\quad = -\frac{1}{2}(u_x n + v_x m)m + \frac{1}{2}(uv_x - u_x v)m - \frac{1}{2}(u_x n + v_x m)_x u_x - (u_x n + v_x m)u_{xx} \\ &\quad = -\frac{1}{2}(uM - (uv_x - u_x v)m + (u_x M)_x). \end{aligned}$$

Hence,

$$\begin{aligned} &u_t + \frac{1}{2}(uv - u_x v_x)u_x \\ &\quad = -\frac{1}{2}(1 - \partial_x^2)^{-1}(uM - (uv_x - u_x v)m + (u_x M)_x). \end{aligned} \tag{4.24}$$

Likewise,

$$\begin{aligned} &v_t + \frac{1}{2}(uv - u_x v_x)v_x \\ &\quad = -\frac{1}{2}(1 - \partial_x^2)^{-1}(vM + (uv_x - u_x v)n + (v_x M)_x). \end{aligned} \tag{4.25}$$

According to (4.24)–(4.25) and the system (1.4), we have

$$\begin{aligned} &u_{xt}n + v_{xt}m \\ &\quad = -\frac{1}{2}(uv - u_x v_x)(u_{xx}n + v_{xx}m) - \frac{1}{2}n\partial_x(1 - \partial_x^2)^{-1}(uM - (uv_x - u_x v)m) \\ &\qquad -\frac{1}{2}m\partial_x(1 - \partial_x^2)^{-1}(vM + (uv_x - u_x v)n) - \frac{1}{2}n(1 - \partial_x^2)^{-1}(u_x M) \\ &\qquad -\frac{1}{2}m(1 - \partial_x^2)^{-1}(v_x M) \end{aligned} \tag{4.26}$$

and

$$\begin{aligned} u_x n_t + v_x m_t = &-\frac{1}{2}(uv - u_x v_x)(u_x n_x + v_x m_x) \\ &-\frac{1}{2}M^2 - \frac{1}{2}(uv_x - u_x v)(u_x n - v_x m), \end{aligned}$$

which together with (4.23) and (4.26) yield (4.22). Since $m_0(x)$, $n_0(x) \geq 0$ for all $x \in \mathbb{R}$, it follows from (4.18)–(4.20) that

$$m(t,x),\ n(t,x) \geq 0, \quad \forall\,(t,x) \in [0,T) \times \mathbb{R}. \tag{4.27}$$

Noticing

$$u(t,x) = (1-\partial_x^2)^{-1} m(t,x) = (p * m)(t,x) = \frac{1}{2}\int_{\mathbb{R}} e^{-|x-y|} m(t,y)dy,$$

then, we obtain

$$u(t,x) = \frac{e^{-x}}{2}\int_{-\infty}^{x} e^{y} m(t,y)dy + \frac{e^{x}}{2}\int_{x}^{\infty} e^{-y} m(t,y)dy$$

and

$$u_x(t,x) = -\frac{e^{-x}}{2}\int_{-\infty}^{x} e^{y} m(t,y)dy + \frac{e^{x}}{2}\int_{x}^{\infty} e^{-y} m(t,y)dy,$$

which together with (4.27) imply

$$u(t,x) + u_x(t,x) = e^{x}\int_{x}^{\infty} e^{-y} m(t,y)dy \geq 0$$

and

$$u(t,x) - u_x(t,x) = e^{-x}\int_{-\infty}^{x} e^{y} m(t,y)dy \geq 0.$$

Hence, we have

$$|u_x(t,x)| \leq u(t,x), \quad \forall\,(t,x) \in [0,T) \times \mathbb{R}, \tag{4.28}$$

as well as

$$|v_x(t,x)| \leq v(t,x), \quad \forall\,(t,x) \in [0,T) \times \mathbb{R}. \tag{4.29}$$

Noticing

$$\begin{aligned}
|\partial_x(1-\partial_x^2)^{-1} f(x)| &= \left|\frac{1}{2}\int_{\mathbb{R}} sgn(x-y) e^{-|x-y|} f(y)dy\right| \\
&\leq \frac{1}{2}\int_{\mathbb{R}} e^{-|x-y|} |f(y)|dy \\
&= (p * |f|)(x),
\end{aligned}$$

and applying (4.28)–(4.29) and the facts that $u = p * m,\ v = p * n$, we arrive at

$$\begin{aligned}
&-\frac{1}{2}n(1-\partial_x^2)^{-1}(u_x M) - \frac{1}{2}m(1-\partial_x^2)^{-1}(v_x M)\\
&\le -\frac{1}{2}n(p * (u_x v_x m)) - \frac{1}{2}m(p * (u_x v_x n))\\
&\le \frac{1}{2}||uv||_{L^\infty}(un + vm),\\
&\quad -\frac{1}{2}n\partial_x(1-\partial_x^2)^{-1}(uM) - \frac{1}{2}m\partial_x(1-\partial_x^2)^{-1}(vM)\\
&\le \frac{1}{2}n\left(||u^2||_{L^\infty}(p * n) + ||uv||_{L^\infty}(p * m)\right)\\
&\quad +\frac{1}{2}m\left(||uv||_{L^\infty}(p * n) + ||v^2||_{L^\infty}(p * m)\right)\\
&= \frac{1}{2}||u^2||_{L^\infty}vn + \frac{1}{2}||v^2||_{L^\infty}um + \frac{1}{2}||uv||_{L^\infty}(un + vm),\\
&\quad -\frac{1}{2}(uv_x - u_x v)(u_x n - v_x m) \le ||uv||_{L^\infty}(un + vm),
\end{aligned}$$

and

$$\begin{aligned}
&\frac{1}{2}n\partial_x(1-\partial_x^2)^{-1}((uv_x - u_x v)m) - \frac{1}{2}m\partial_x(1-\partial_x^2)^{-1}((uv_x - u_x v)n)\\
&\le ||uv||_{L^\infty}\left(n(p * m) + m(p * n)\right)\\
&= ||uv||_{L^\infty}(un + vm),
\end{aligned}$$

which along with (4.22) complete the proof of the lemma. □

It is worth pointing out that Theorem 4.2, Lemma 4.2 and (4.1)–(4.3) tell us that a sufficient condition for the fine structure of finite time singularities is that there exists a constant $C = C(||m_0||_{H^s}, ||n_0||_{H^s}) > 0$ such that

$$||u(t,\cdot)||_{L^\infty},\ ||v(t,\cdot)||_{L^\infty} \le Ce^{Ct}, \quad \text{for all } t \in [0, T).$$

Also, as mentioned in the Sect. 1, for the special case of Eq. (1.1) with $b = 0$, one can apply the following conservation laws

$$H \triangleq \int_{\mathbb{R}} umdx = \int_{\mathbb{R}} (u^2 + u_x^2)dx$$

and Sobolev's embedding theorem to uniformly bound $||u(t,\cdot)||_{L^\infty(\mathbb{R})}$. However, one cannot utilize any appropriate conservation laws of the system (1.4) to control $||u(t,\cdot)||_{L^\infty}$ and $||v(t,\cdot)||_{L^\infty}$ directly. Anyway, the uniform boundedness for the solution u to Eq. (1.1) with $b = 0$ can be viewed as a special case of the above exponential increase assumed in finite time.

Theorem 4.3. *Suppose that* $(m_0, n_0) \in H^s(\mathbb{R}) \times H^s(\mathbb{R})$ *(*$s > \frac{1}{2}$*) and* $T > 0$ *be the maximal existence time of the solution* (m, n) *corresponding to the system* (1.4) *with* $b = 0$*. Assume that there exists a constant* $C = C(||m_0||_{H^s}, ||n_0||_{H^s}) > 0$ *such that*

$$||u(t,\cdot)||_{L^\infty},\ ||v(t,\cdot)||_{L^\infty} \le Ce^{Ct}, \quad \textit{for all } t \in [0, T).$$

Let $M(t) \triangleq M(t, q(t, y_0)) = \inf_{x\in\mathbb{R}} M(t,x)$ *for some* $y_0 \in \mathbb{R}$*, which is guaranteed by Remark 3.1 and Lemma 4.1, and* $N(t) \triangleq (m+n)(t, q(t, y_0))$*. Let* $m_0(x)$*,* $n_0(x) \geq 0$ *for all* $x \in \mathbb{R}$*, and* $m_0(x_0)$*,* $n_0(x_0) > 0$ *for some* $x_0 = q(0, y_0)$*, which is ensured by Lemma 4.1. If* $M(0) < -2C$ *and*

$$\frac{M(0)}{N(0)} < -e^{(e^{C\eta}-1)}\left(\frac{2}{\eta N(0)}+1\right)+1, \tag{4.30}$$

where η *is the unique positive solution to the following equation w.r.t.* t*:*

$$e^{(e^{Ct}-1)}\left(\frac{C}{N(0)}e^{Ct}+\frac{1}{2}Cte^{Ct}+\frac{1}{2}\right)+\frac{1}{2}\left(\frac{M(0)}{N(0)}-1\right)=0, \quad t\geq 0.$$

Then the solution (m,n) *blows up at a time* $T_0 \in (0,\eta]$*.*

Proof. In view of Remark 4.1, here let us prove the theorem for the case of $s \geq 3$. By (4.17), Lemma 4.3 and the assumption of the theorem, we have

$$\begin{aligned}\frac{d}{dt}M(t) &= \frac{d}{dt}M(t,q(t,y_0))\\ &= (M_t + \frac{1}{2}(uv-u_xv_x)M_x)(t,q(t,y_0))\\ &\leq -\frac{1}{2}M^2(t) + Ce^{Ct}N(t).\end{aligned} \tag{4.31}$$

From the system (1.4), we get

$$\begin{aligned}\frac{d}{dt}N(t) &= \frac{d}{dt}m(t,q(t,y_0)) + \frac{d}{dt}n(t,q(t,y_0))\\ &= -\frac{1}{2}M(t)N(t) + \frac{1}{2}(uv_x - u_xv)(m-n)(t,q(t,y_0)).\end{aligned} \tag{4.32}$$

Apparently, (4.19)–(4.20) and the assumption imply $N(t) > 0$ for all $t \in [0,T)$. By (4.28)–(4.29) and (4.31)–(4.32), a direct computation leads to

$$\begin{aligned}&N(t)\frac{d}{dt}M(t) - M(t)\frac{d}{dt}N(t)\\ &\leq Ce^{Ct}N^2(t) - \frac{1}{2}(uv_x - u_xv)(m-n)(t,q(t,y_0))M(t)\\ &\leq Ce^{Ct}N^2(t) + ||uv||_{L^\infty}(||u||_{L^\infty}n + ||v||_{L^\infty}m)N(t)\\ &\leq Ce^{Ct}N^2(t),\end{aligned}$$

which gives

$$\frac{d}{dt}\left(\frac{M(t)}{N(t)}\right) \leq Ce^{Ct}.$$

Integrating from 0 to t yields

$$\frac{M(t)}{N(t)} \leq \frac{M(0)}{N(0)} + C\int_0^t e^{C\tau}d\tau = \frac{M(0)}{N(0)} + e^{Ct} - 1,$$

which implies

$$M(t) \leq \left(\frac{M(0)}{N(0)} - 1 + e^{Ct}\right) N(t). \tag{4.33}$$

On the other hand, according to (4.32) and Lemma 4.3, we have

$$M(t) \geq \frac{-2}{N(t)} \frac{d}{dt} N(t) - Ce^{Ct},$$

which along with (4.33) gives rise to

$$\frac{d}{dt}\left(\frac{1}{N(t)}\right) \leq Ce^{Ct} \frac{1}{N(t)} + \frac{1}{2}\left(\frac{M(0)}{N(0)} - 1 + e^{Ct}\right). \tag{4.34}$$

Since $M(0) < -2C$, $\frac{M(0)}{N(0)} < 1$, it follows from the Gronwall's inequality for (4.34) and the fact $x \leq e^x - 1$ that

$$\begin{aligned} 0 < \frac{1}{N(t)} &\leq \frac{1}{N(0)} e^{\int_0^t Ce^{C\tau} d\tau} + \frac{1}{2}\int_0^t e^{\int_\tau^t Ce^{Cs} ds}\left(\frac{M(0)}{N(0)} - 1 + e^{C\tau}\right) d\tau \\ &= \frac{1}{N(0)} e^{(e^{Ct}-1)} + \frac{1}{2}\int_0^t e^{(e^{Ct}-e^{C\tau})} e^{C\tau} d\tau \\ &\quad + \frac{1}{2}\int_0^t e^{\int_\tau^t Ce^{Cs} ds}\left(\frac{M(0)}{N(0)} - 1\right) d\tau \\ &\leq \frac{1}{N(0)} e^{(e^{Ct}-1)} + \frac{1}{2}\int_0^t e^{(e^{Ct}-1)} d\tau + \frac{1}{2}\left(\frac{M(0)}{N(0)} - 1\right) t \\ &= e^{(e^{Ct}-1)}\left(\frac{1}{N(0)} + \frac{1}{2} t\right) + \frac{1}{2}\left(\frac{M(0)}{N(0)} - 1\right) t \\ &\triangleq f(t), \end{aligned} \tag{4.35}$$

which generates

$$f'(t) = e^{(e^{Ct}-1)}\left(\frac{C}{N(0)} e^{Ct} + \frac{1}{2} Cte^{Ct} + \frac{1}{2}\right) + \frac{1}{2}\left(\frac{M(0)}{N(0)} - 1\right).$$

Since $M(0) < -2C$ ensures $f'(0) = \frac{C}{N(0)} + \frac{M(0)}{2N(0)} < 0$, it follows from the facts

$$\begin{aligned} f''(t) = e^{(e^{Ct}-1)}\Bigg(&Ce^{Ct}\left(\frac{C}{N(0)} e^{Ct} + \frac{Ct}{2} e^{Ct} + \frac{1}{2}\right) \\ &+ C^2 e^{Ct}\left(\frac{1}{N(0)} + \frac{t}{2}\right) + \frac{C}{2} e^{Ct}\Bigg) > 0 \end{aligned}$$

and $\lim_{t\to+\infty} f'(t) = +\infty$ that there exists an unique $\eta > 0$ such that $f'(\eta) = 0$ and

$$f'(t) \begin{cases} < 0, & \text{if} \quad 0 \leq t < \eta, \\ > 0, & \text{if} \quad t > \eta. \end{cases}$$

By (4.30), we have $f(\eta) < 0$. Noticing $f(0) = \frac{1}{N(0)} > 0$ and $f(t) \in C[0, +\infty)$, we can find a finite number $T_0 \in (0, \eta]$ such that

$$f(t) \to 0^+, \quad \text{as } t \to T_0,$$

which together with (4.35) yields

$$N(t) \to +\infty, \quad \text{as}\, t \to T_0.$$

By using (4.30) again, we get $\frac{M(0)}{N(0)} - 1 + e^{C\eta} < 0$. This along with (4.33) ensures

$$\inf_{x\in\mathbb{R}} (u_x n + v_x m)(t, x) = M(t) \to -\infty, \quad \text{as } t \to T_0.$$

So, according to Theorem 4.2, the solution (m, n) blows up at the time $T_0 \in (0, \eta]$, which completes the proof of the theorem. □

5. Peakon and Weak Kink Solutions

In this section, we provide some explicit solutions to the system (1.3), such as peakon and weak kink solutions. To see this, let us first write the weak form of the system (1.3). Apparently, for all $f \in L^2(\mathbb{R})$, we have $(1-\partial_x^2)^{-1} f = p * f$ where p is Geen's function $p(x) \triangleq \frac{1}{2}e^{-|x|}$, which then yields $u = p * m$, and $v = p * n$. So, we can rewrite the system (1.3) in the following weak form:

$$\begin{cases} u_t + \frac{1}{2}(uv - u_x v_x)u_x - \frac{1}{2}\partial_x p * [(u_x v_x)_x u_x - 2bu] \\ \quad +\frac{1}{2} p * [2uvu_x + (uu_x)_x v_x + (u_x v)_x u_x] = 0, \\ v_t + \frac{1}{2}(uv - u_x v_x)v_x - \frac{1}{2}\partial_x p * [(u_x v_x)_x v_x - 2bv] \\ \quad +\frac{1}{2} p * [2uvv_x + (vv_x)_x u_x + (uv_x)_x v_x] = 0. \end{cases} \tag{5.1}$$

Let us now present the peakon solution to (1.3) with $b = 0$ in the following theorem.

Theorem 5.1.

$$u = c_1 e^{-|x-ct|}, \quad v = c_2 e^{-|x-ct|} \tag{5.2}$$

are the single peakon solutions to the system (5.1) with $b = 0$ in the sense of distribution, where c_1 and c_2 are two arbitrary nonzero constants satisfying $c_1 c_2 = 3c$ and c is the wave speed of u and v.

Proof. From (5.2), in the sense of distribution, one can easily get

$$\begin{aligned} u_t &= c \cdot sgn(x - ct)u, \quad u_x = -sgn(x - ct)u, \\ v_t &= c \cdot sgn(x - ct)v, \quad v_x = -sgn(x - ct)v, \end{aligned} \tag{5.3}$$

which generate

$$\begin{aligned} & u_t + \frac{1}{2}(uv - u_x v_x)u_x \\ &= c \cdot sgn(x - ct)u - \frac{1}{2} sgn(x - ct)u \left(uv - sgn^2(x - ct)uv\right) \\ &= sgn(x - ct)u \left(c - \frac{1}{2}\left(1 - sgn^2(x - ct)\right) uv\right) \\ &= cc_1 sgn(x - ct)e^{-|x-ct|} \qquad (\text{if } x \neq ct), \end{aligned} \tag{5.4}$$

and

$$(u_xv_x)_xu_x = -\left(sgn^2(x-ct)uv\right)_x sgn(x-ct)u$$
$$= -2sgn^2(x-ct)\left((sgn(x-ct))_x - sgn^2(x-ct)\right)u^2v.$$

Then, we have

$$-\frac{1}{2}\partial_x p * ((u_xv_x)_xu_x)$$
$$= \partial_x p * \left(sgn^2(x-ct)\,(sgn(x-ct))_x\, u^2v - sgn^4(x-ct)u^2v\right)$$
$$\triangleq I_1 + I_2,$$

where

$$I_1 = \frac{c_1^2c_2}{6}\partial_x \int_{-\infty}^{\infty} e^{-|x-y|}e^{-3|y-ct|}\left(sgn^3(y-ct)\right)_y dy$$
$$= -\frac{c_1^2c_2}{6}\partial_x \int_{-\infty}^{\infty} e^{-|x-y|}e^{-3|y-ct|}sgn^3(y-ct)\,(sgn(x-y) - 3sgn(y-ct))\,dy,$$

and

$$I_2 = \left(-\frac{1}{2}sgn(x)e^{-|x|}\right) * \left(-sgn^4(x-ct)u^2v\right)$$
$$= \frac{c_1^2c_2}{2}\int_{-\infty}^{\infty} e^{-|x-y|}e^{-3|y-ct|}sgn(x-y)sgn^4(y-ct)dy.$$

In the above calculation, $\partial_x p(x) = -\frac{1}{2}sgn(x)e^{-|x|}$ is used. Thus, we obtain

$$-\frac{1}{2}\partial_x p * ((u_xv_x)_xu_x)$$
$$= -\frac{c_1^2c_2}{6}\partial_x \int_{-\infty}^{\infty} e^{-|x-y|}e^{-3|y-ct|}sgn^3(y-ct)\,(sgn(x-y) - 3sgn(y-ct))\,dy$$
$$+\frac{c_1^2c_2}{2}\int_{-\infty}^{\infty} e^{-|x-y|}e^{-3|y-ct|}sgn(x-y)sgn^4(y-ct)dy, \tag{5.5}$$

and

$$2uvu_x + (uu_x)_xv_x + (u_xv)_xu_x$$
$$= -2sgn(x-ct)u^2v + \left(sgn(x-ct)u^2\right)_x sgn(x-ct)v$$
$$+ (sgn(x-ct)uv)_x\, sgn(x-ct)u$$
$$= 2sgn(x-ct)\left((sgn(x-ct))_x - 2sgn^2(x-ct) - 1\right)u^2v,$$

which lead to

$$\begin{aligned}
&\frac{1}{2}p*(2uvu_x+(uu_x)_xv_x+(u_xv)_xu_x)\\
&\quad=p*\Big(sgn(x-ct)\,(sgn(x-ct))_x\,u^2v\Big)\\
&\qquad-p*\Big(sgn(x-ct)\,\Big(2sgn^2(x-ct)+1\Big)\,u^2v\Big)\\
&\quad\triangleq II_1+II_2,
\end{aligned}$$

where

$$\begin{aligned}
II_1&=\frac{c_1^2c_2}{4}\int_{-\infty}^{\infty}e^{-|x-y|}e^{-3|y-ct|}\Big(sgn^2(y-ct)\Big)_y\,dy\\
&=-\frac{c_1^2c_2}{4}\int_{-\infty}^{\infty}e^{-|x-y|}e^{-3|y-ct|}sgn^2(y-ct)\,(sgn(x-y)-3sgn(y-ct))\,dy,
\end{aligned}$$

and

$$II_2=-\frac{c_1^2c_2}{2}\int_{-\infty}^{\infty}e^{-|x-y|}e^{-3|y-ct|}sgn(y-ct)\Big(2sgn^2(y-ct)+1\Big)\,dy.$$

Thus, we get

$$\begin{aligned}
&\frac{1}{2}p*(2uvu_x+(uu_x)_xv_x+(u_xv)_xu_x)\\
&\quad=-\frac{c_1^2c_2}{4}\int_{-\infty}^{\infty}e^{-|x-y|}e^{-3|y-ct|}sgn(y-ct)\\
&\qquad\times\Big(2+sgn^2(y-ct)+sgn(x-y)sgn(y-ct)\Big)\,dy,
\end{aligned}$$

which along with (5.5) leads to

$$\begin{aligned}
&-\frac{1}{2}\partial_xp*((u_xv_x)_xu_x)+\frac{1}{2}p*(2uvu_x+(uu_x)_xv_x+(u_xv)_xu_x)\\
&=-\frac{c_1^2c_2}{6}\partial_x\int_{-\infty}^{\infty}e^{-|x-y|}e^{-3|y-ct|}sgn^3(y-ct)\,(sgn(x-y)-3sgn(y-ct))\,dy\\
&\quad-\frac{c_1^2c_2}{4}\int_{-\infty}^{\infty}e^{-|x-y|}e^{-3|y-ct|}sgn(y-ct)\{sgn(x-y)sgn(y-ct)\\
&\quad\times\Big(1-2sgn^2(y-ct)\Big)+2+sgn^2(y-ct)\}dy\\
&\triangleq III+IV.
\end{aligned}$$

Next, we calculate $III+IV$ in the following two cases.

Case 1: $x > ct$.

$$\begin{aligned}
&\int_{-\infty}^{\infty} e^{-|x-y|}e^{-3|y-ct|}sgn^3(y-ct)\left(sgn(x-y)-3sgn(y-ct)\right)dy\\
&\quad=\left(\int_{-\infty}^{ct}+\int_{ct}^{x}+\int_{x}^{\infty}\right)e^{-|x-y|}e^{-3|y-ct|}sgn^3(y-ct)(sgn(x-y)\\
&\qquad-3sgn(y-ct))dy\\
&\quad=-4e^{-x-3ct}\int_{-\infty}^{ct}e^{4y}dy-2e^{-x+3ct}\int_{ct}^{x}e^{-2y}dy-4e^{x+3ct}\int_{x}^{\infty}e^{-4y}dy\\
&\quad=-e^{-x+ct}+e^{-3x+3ct}-e^{-x+ct}-e^{-3x+3ct}\\
&\quad=-2e^{-(x-ct)}
\end{aligned}$$

yields

$$III=-\frac{c_1^2c_2}{6}\partial_x(-2e^{-(x-ct)})=-\frac{c_1^2c_2}{3}e^{-(x-ct)}.$$

Similarly, we have

$$\begin{aligned}
IV&=-\frac{c_1^2c_2}{4}\left(-4e^{-x-3ct}\int_{-\infty}^{ct}e^{4y}dy+2e^{-x+3ct}\int_{ct}^{x}e^{-2y}dy+4e^{x+3ct}\int_{x}^{\infty}e^{-4y}dy\right)\\
&=-\frac{c_1^2c_2}{4}\left(-e^{-x+ct}-e^{-3x+3ct}+e^{-x+ct}+e^{-3x+3ct}\right)\\
&=0.
\end{aligned}$$

So,

$$III+IV=-\frac{c_1^2c_2}{3}e^{-(x-ct)},\qquad \text{if } x>ct. \tag{5.6}$$

Case 2: $x \le ct$.

$$\begin{aligned}
&\int_{-\infty}^{\infty} e^{-|x-y|}e^{-3|y-ct|}sgn^3(y-ct)\left(sgn(x-y)-3sgn(y-ct)\right)dy\\
&\quad=\left(\int_{-\infty}^{x}+\int_{x}^{ct}+\int_{ct}^{\infty}\right)e^{-|x-y|}e^{-3|y-ct|}sgn^3(y-ct)(sgn(x-y)\\
&\qquad-3sgn(y-ct))dy\\
&\quad=-4e^{-x-3ct}\int_{-\infty}^{x}e^{4y}dy-2e^{x-3ct}\int_{x}^{ct}e^{2y}dy-4e^{x+3ct}\int_{ct}^{\infty}e^{-4y}dy\\
&\quad=-e^{3x-3ct}+e^{3x-3ct}-e^{x-ct}-e^{x-ct}\\
&\quad=-2e^{x-ct},
\end{aligned}$$

leads to

$$III=-\frac{c_1^2c_2}{6}\partial_x(-2e^{x-ct})=\frac{c_1^2c_2}{3}e^{x-ct}.$$

Likewise, we get

$$
\begin{aligned}
IV &= -\frac{c_1^2 c_2}{4}\left(-4e^{-x-3ct}\int_{-\infty}^{x} e^{4y}dy - 2e^{x-3ct}\int_{x}^{ct} e^{2y}dy + 4e^{x+3ct}\int_{ct}^{\infty} e^{-4y}dy\right)\\
&= -\frac{c_1^2 c_2}{4}\left(-e^{3x-3ct} + e^{3x-3ct} - e^{x-ct} + e^{x-ct}\right)\\
&= 0.
\end{aligned}
$$

So,

$$
III + IV = \frac{c_1^2 c_2}{3} e^{x-ct}, \qquad \text{if } \; x \le ct. \tag{5.7}
$$

Therefore combining (5.6) with (5.7) gives

$$
III + IV = -\frac{c_1^2 c_2}{3} sgn(x-ct) e^{-|x-ct|}. \tag{5.8}
$$

By (5.4) and (5.8) with the assumption $c_1 c_2 = 3c$, one may immediately see that the first equation of the system (5.1) holds in the sense of distribution. So does the second one in the system (5.1) duo to the symmetry of u and v. Therefore, we complete the proof of the theorem. □

Remark 5.1. In particular, if $c_2 = 2c_1$ in Theorem 5.1, we recover the single peakon solution $u = \pm\sqrt{\frac{3c}{2}} e^{-|x-ct|}$ of the cubic CH equation (1.1) with $b = 0$ [36,50].

Next, let us show that the system (5.1) with $b \neq 0$ possesses a weak kink solution. We assume the system (5.1) admits the following wave solutions

$$
u = C_1 sgn(x-ct)\left(e^{-|x-ct|} - 1\right), \quad v = C_2 sgn(x-ct)\left(e^{-|x-ct|} - 1\right), \tag{5.9}
$$

where C_1 and C_2 are two nonzero constants to be determined, and c is the wave speed. the solution form (5.9) is called the weak kink wave, which is recently proposed in [50,56]. In fact, if $C_1 \neq 0$ and $C_2 \neq 0$, then the potentials u and v in (5.9) are kink wave solutions due to

$$
\begin{aligned}
\lim_{x\to+\infty} u &= -\lim_{x\to-\infty} u = -C_1,\\
\lim_{x\to+\infty} v &= -\lim_{x\to-\infty} v = -C_2.
\end{aligned} \tag{5.10}
$$

One may easily check that in the sense of distribution, the first order partial derivatives of (5.9) read

$$
\begin{aligned}
u_t &= cC_1 e^{-|x-ct|}, \quad u_x = -C_1 e^{-|x-ct|},\\
v_t &= cC_2 e^{-|x-ct|}, \quad v_x = -C_2 e^{-|x-ct|}.
\end{aligned} \tag{5.11}
$$

Similar to the proof of Theorem 5.1, we can readily prove the following result.

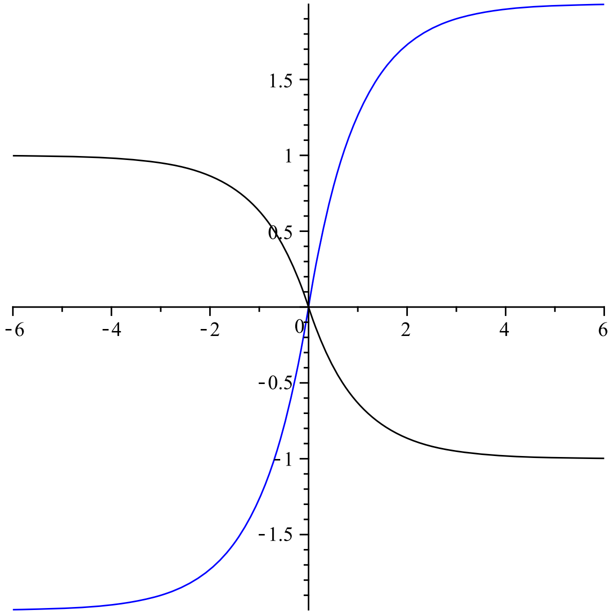

Fig. 1. The weak kink solution at $t = 0$. *Black line* $u(x, 0)$ and *Blue line* $v(x, 0)$ (color figure online)

Theorem 5.2. *Assume that*

$$\begin{cases} C_1 C_2 = -b, \\ c = \frac{1}{2} b. \end{cases} \tag{5.12}$$

Then (5.9) *is the weak kink solution to the system* (5.1) *with* $b \neq 0$ *in the sense of distribution.*

Remark 5.2. (1) The second identity in (5.12) implies that the weak kink wave speed is exactly equal to $\frac{1}{2}b$, that is, the weak kink wave occurs only when its wave speed $c = \frac{1}{2}b$.

(2) In particular, if we take $b = 2$ and $C_1 = 1$, then $c = 1, C_2 = -2$, and the corresponding weak kink solutions are cast into

$$u = sgn(x - t)\left(e^{-|x-t|} - 1\right), \quad v = -2sgn(x - t)\left(e^{-|x-t|} - 1\right).$$

See the following Fig. 1 for the details of the profile for the weak kink wave solution.

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