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Chapter 14

A NOTE ON NONLINEAR INTEGRABLE HAMILTONIAN SYSTEMS

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Abstract

This paper presents an approach to determine what kind of canonical Hamiltonian system is integrable and what is nonintegrable. Examples are analyzed to show integrability and nonintegrability. In particular, some new integrable Hamiltonian systems are found and the remarkable peakon dynamical system is a reduction.

1 Introduction

The Liouville-Arnold theory has been playing a very important role in the investigation of finite-dimensional integrable system [1]. A motivation for studying integrable systems has been brought by the discovery of soliton equations [5]. New techniques, such as the Lax pair [6] and the spectral curve method [4] etc, were successively involved in soliton theory. For a given Hamiltonian it is usually very difficult to verify whether it is integrable or even to check whether it has additional integrals of motion apart from those simply related to the geometrical symmetries of the potential. The Lax representation is quite an effective way to show the integrability. But, how do we find a Lax pair for a given finite dimensional Hamiltonian system? Calogero [2] proposed a general scheme equation with three different functions to be determined for many-body problem on the line. More details of three different functions $g(x), \alpha(x), \gamma(x)$ are in Eq. (2.8) or on page 137 of Calogero's book [2].

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In this paper we simplify the Calogero equation (2.8) and reduce it to one single equation (see Eq. (3.18)) only containing a function $\alpha(x)$ to be determined, and meanwhile the other two $g(x), \gamma(x)$ are able to be expressed in terms of the function $\alpha(x)$. In the paper we provide a procedure to determine what kind of function $g(x)$ is appropriate and what is inappropriate. Several examples are analyzed to show integrability and nonintegrability. In particular, some new integrable Hamiltonian systems are found and the remarkable peakon dynamical system is a reduction.

2 Preliminaries

Let us start from the following two $N \times N$ matrices:

$$L = \sum_{i,j=1}^{N} L_{ij} E_{ij}, \tag{2.1}$$

$$M = \sum_{i,j=1}^{N} M_{ij} E_{ij}, \tag{2.2}$$

where $\{E_{ij}\}$ is the matrix basis, i.e. $(E_{ij})_{kl} = \delta_{ik}\delta_{jl},\ i, j, k, l = 1, ..., N$, and

$$L_{ij} = \sqrt{p_i p_j}\,\alpha(q_i - q_j), \tag{2.3}$$

$$M_{ij} = \sqrt{p_i p_j}\,\gamma(q_i - q_j), \tag{2.4}$$

where α and γ are two functions to be determined.

Calogero [2] proved that the Lax equation

$$\dot{L} = [M, L] \tag{2.5}$$

is equivalent to the following canonical Hamiltonian equation

$$(H): \begin{cases} \dot{q}_i = \frac{\partial H}{\partial p_i} = 2\sum_{j=1}^{N} p_j g(q_i - q_j), \\ \dot{p}_i = -\frac{\partial H}{\partial q_i} = -2p_i \sum_{j=1}^{N} p_j g'(q_i - q_j), \end{cases} \tag{2.6}$$

$$\text{with} \quad H = \sum_{i,j=1}^{N} p_i p_j g(q_i - q_j), \tag{2.7}$$

if and only if the even function $g(x)$, together with other two functions $\alpha(x),\ \gamma(x)$, satisfies

$$\begin{aligned} & 2\alpha'(x+y)[g(x) - g(y)] - \alpha(x+y)[g'(x) - g'(y)] \\ = \ & \alpha(x)\gamma(y) - \alpha(y)\gamma(x),\ \forall x,\ y \in \mathbb{R}, \end{aligned} \tag{2.8}$$

where the superscripts denote the corresponding function's derivative with respect to the argument. Eq. (2.8) actually comes from the Ref. [2] (see page 137, equation (***)).

Definition 1 *The function $g(x)$ is said to be* **appropriate** *for the finite-dimensional integrable system (2.6) if there exist two functions $\alpha(x),\ \gamma(x)$ such that Eq. (2.8) holds (i.e. the Lax equation (2.5) is equivalent to the Hamiltonian equation (2.6)). Otherwise, the function $g(x)$ is said to be* **inappropriate**.

Based on the functional equation (2.8), we shall concretely discuss what kind of the function $g(x)$ is appropriate or inappropriate. We will construct a governing equation of $\alpha(x)$, which is available to express $g(x)$ in terms of $\alpha(x)$ and as well as easy to make $g(x)$ appropriate or inappropriate.

3 A General Formula for $\alpha(x)$

Since Eq. (2.8) holds for any $x,\ y \in \mathbb{R}$, let us first choose $y = -x$. Then Eq. (2.8) reads

$$2\alpha_0 g'(x) = \alpha(-x)\gamma(x) - \alpha(x)\gamma(-x), \tag{3.9}$$

where $\alpha_0 = \alpha(0)$. However, setting $y = 0$ yields

$$\begin{aligned} 2\alpha'(x)[g(x) - g_0] - \alpha(x)[g'(x) + \gamma_0] &= -\alpha_0\gamma(x), &(3.10)\\ 2\alpha'(-x)[g(x) - g_0] + \alpha(-x)[g'(x) - \gamma_0] &= -\alpha_0\gamma(-x), &(3.11) \end{aligned}$$

which implies

$$\begin{aligned} &2[\alpha'(x) + \alpha'(-x)][g(x) - g_0] + [\alpha(-x) - \alpha(x)]g'(x) - \gamma_0[\alpha(x) + \alpha(-x)]\\ = \; &-\alpha_0[\gamma(x) + \gamma(-x)], \end{aligned} \tag{3.12}$$

and

$$\alpha_0^2 g'(x) = \Big(g(x) - g_0\Big)\Big(\alpha(x)\alpha'(-x) - \alpha(-x)\alpha'(x)\Big) + \alpha(x)\alpha(-x)g'(x), \tag{3.13}$$

where $\gamma_0 = \gamma(0),\ g_0 = g(0)$. Solving Eq. (3.13) yields the relation between $g(x)$ and $\alpha(x)$

$$g(x) = c\alpha(x)\alpha(-x) - c\alpha_0^2 + g_0, \tag{3.14}$$

where c is a non-zero constant and thereafter all c's in the examples are non-zero constants.

To get $\gamma(x)$ in terms of $\alpha(x)$, let us take $y = x + \epsilon\Delta x$ and the derivative $\frac{d}{d\epsilon}\big|_{\epsilon=0}$ in both sides of Eq. (2.8). Then we have

$$-2\alpha'(2x)g'(x) + \alpha(2x)g''(x) = \alpha(x)\gamma'(x) - \alpha'(x)\gamma(x),$$

i.e.

$$\frac{d}{dx}\frac{g'(x)}{\alpha(2x)} = \frac{\alpha^2(x)}{\alpha^2(2x)} \cdot \frac{d}{dx}\frac{\gamma(x)}{\alpha(x)}.$$

The integration of this equality directly leads to a relation between $\gamma(x)$ and $\alpha(x)$

$$\begin{aligned} \gamma(x) &= c\alpha(x)\Gamma(\alpha(x), \alpha(2x)), &(3.15)\\ \Gamma(\alpha(x), \alpha(2x)) &= \int \frac{\alpha^2(2x)}{\alpha^2(x)}\Big(\frac{\alpha'(x)\alpha(-x) - \alpha(x)\alpha'(-x)}{\alpha(2x)}\Big)' dx, &(3.16) \end{aligned}$$

where $'$ is derivatives with respect to x. Substituting Eqs. (3.14) and (3.15) into Eq. (2.8) gives the following equation:

$$\frac{1}{2}(A(x, y) - A(y, x)) = \frac{1}{2c}\Big(\frac{\gamma(y)}{\alpha(y)} - \frac{\gamma(x)}{\alpha(x)}\Big). \tag{3.17}$$

So, we obtain an equation only interfering $\alpha(x)$:

$$A(x,y) - A(y,x) = \Gamma(\alpha(y), \alpha(2y)) - \Gamma(\alpha(x), \alpha(2x)), \quad \forall x, y, \tag{3.18}$$

where

$$A(x,y) = \frac{2\alpha'(x+y)\alpha(-x)\alpha(x) - \alpha(x+y)(\alpha'(x)\alpha(-x) - \alpha'(-x)\alpha(x))}{\alpha(x)\alpha(y)}. \tag{3.19}$$

Therefore, we have the following theorem.

Theorem 1 *If the Lax equation (2.5) has the Hamiltonian canonical form (2.6) with an even function $g(-x) = g(x)$, then $\alpha(x)$ satisfies Eq. (3.18). In addition, $g(x)$ and $\gamma(x)$ are given in terms of $\alpha(x)$ by Eqs. (3.14) and (3.15), respectively.*

This theorem is telling us that the key problem in solving Calogero's equation (2.8) is to find the solution $\alpha(x)$ of equation (3.18). Let us give some examples as follows.

Examples

1. Choosing $\alpha(x) = A \sin \frac{a}{2} x$ (A, a are two constants, and apparently $\alpha(x)$ is an odd function) yields

$$g(x) = \lambda \cos ax + \mu, \ \lambda = -\frac{cA^2}{2}, \ \mu = g_0 - \lambda. \tag{3.20}$$

By Eqs. (3.16) and (3.15), we take $\gamma(x) \equiv 0$. Because this $\alpha(x)$ satisfies

$$A(x,y) = A(y,x), \ \forall x, y,$$

the function $g(x) = \lambda \cos ax + \mu$ is appropriate for the Lax equation (2.5).

2. Choosing $\alpha(x) = A \ \mathrm{sgn}(x) \sin \frac{a}{2} x$ (A, a are two constants, and $\alpha(-x) = \alpha(x)$) yields

$$g(x) = \lambda \cos ax + \mu, \ \lambda = -\frac{cA^2}{2}, \ \mu = g_0 - \lambda. \tag{3.21}$$

By Eqs. (3.16) and (3.15), we have

$$\begin{aligned} \Gamma(\alpha(x), \alpha(2x)) &= d, \ \ d = constant, \\ \gamma(x) &= B\alpha(x), \ \ B = cd. \end{aligned}$$

Additionally, $\alpha(x)$ satisfies

$$A(x,y) = A(y,x), \ \forall x, y.$$

Thus, in this example $g(x) = \lambda \cos ax + \mu$ is again proved to be appropriate.

The above two examples have same $g(x)$, but different $\alpha(x)$ and $\gamma(x)$. This illustrates that the Lax representation is not **unique**. Such $g(x)$ is a main example of Calogero's book [2].

3. Choosing $\alpha(x) = x^n,\ n \in \mathbb{R}$, we get

$$\begin{aligned}\Gamma(\alpha(x),\alpha(2x)) &= n(-1)^n 2^{n+1} x^{n-1},\\ A(x,y) &= 2n(-1)^{n-1}\frac{x^{n-1}(x+y)^{n-1}}{y^{n-1}},\end{aligned}$$

which imply that Eq. (3.18) holds iff

$$n = 1.$$

Thus, $\alpha(x) = x$, $\gamma(x) = -4cx$, and

$$g(x) = -cx^2 + g_0.$$

Therefore, the canonical Hamiltonian system (2.6) with $g(x) = -cx^2 + g_0$ ($c \neq 0, g_0$ are any constants) is a new integrable system. In particular, we take

$$g(x) = 1 - x^2 \tag{3.22}$$

as an appropriate function for the Lax equation (2.5). However, the following function

$$g(x) = \begin{cases} 1 - x^2, & |x| < 1,\\ 0, & |x| \geq 1,\end{cases} \tag{3.23}$$

is not appropriate, because at $x = \pm 1$ $g(x)$ does not satisfy the equation (2.8).

4. Choosing $\alpha(x) = |x|^n,\ n \in \mathbb{R}$, we have

$$\begin{aligned}\Gamma(\alpha(x),\alpha(2x)) &= \begin{cases} 2^{n+1} n|x|^{n-1}\mathrm{sgn}(x), & n \neq 1,\\ constant, & n = 1,\end{cases}\\ A(x,y) &= 2n|x|^{n-1}|x+y|^{n-1}\frac{|x|\mathrm{sgn}(x+y) - |x+y|\mathrm{sgn}(x)}{|y|^n},\end{aligned}$$

which imply that Eq. (3.18) holds only for $n = 1$ because $A(x,y) = A(y,x)$ and $\Gamma(\alpha(x),\alpha(2x)) = \Gamma(\alpha(y),\alpha(2y))$ only when $n = 1$. Therefore the function $g(x) = g_0 + cx^2$ (g_0 and $c \neq 0$ are constants) corresponding $n = 1$ is appropriate (also see example 3). But when we take $n = \frac{1}{2}$, $c = -1$, and $g_0 = 1$, the function

$$g(x) = 1 - |x| \tag{3.24}$$

is not appropriate. Therefore,

$$g(x) = \begin{cases} 1 - |x|, & |x| < 1,\\ 0, & |x| \geq 1,\end{cases} \tag{3.25}$$

is not an appropriate function for the Lax equation (2.5).

5. Choosing $\alpha(x) = e^{-\frac{a}{2}|x|}$ ($a \in \mathbb{R}$ is a constant), then we have

$$\begin{aligned} \Gamma(\alpha(x), \alpha(2x)) &= -a(1 + \mathrm{sgn}(x)), \\ A(x, y) &= a e^{-\frac{a}{2}(|x+y|+|x|-|y|)} \Big(\mathrm{sgn}(x) - \mathrm{sgn}(x + y)\Big), \end{aligned}$$

which imply

$$A(x, y) - A(y, x) = \Gamma(\alpha(y), \alpha(2y)) - \Gamma(\alpha(x), \alpha(2x)),\ \forall x, y.$$

Therefore, $\alpha(x) = e^{-\frac{a}{2}|x|}$ is a solution of Eq. (3.18). In this case

$$\begin{aligned} g(x) &= c e^{-a|x|} + g_0 - c, \\ \gamma(x) &= -ac\, e^{-\frac{a}{2}|x|}\mathrm{sgn}(x), \end{aligned}$$

where $c \neq 0,\ a,\ g_0$ are constants. So, $g(x) = ce^{-a|x|} + g_0 - c$ is appropriate, and the canonical Hamiltonian system (2.6) with $g(x) = ce^{-a|x|} + g_0 - c$ (a, g_0 are any constants) is a new integrable system. In particular, this system includes the integrable peakon dynamics [3] as a special reduction with $c = a = g_0 = 1$ (i.e. $g(x) = e^{-|x|}$).

Two natural questions arise here:

1. If $g_1(x)$ and $g_2(x)$ are appropriate, then is their sum function $g(x) = g_1(x) + g_2(x)$ appropriate? If so, what are the conditions for $g_1(x)$ and $g_2(x)$?

2. If $\alpha_1(x)$ and $\alpha_2(x)$ along with their corresponding functions $\gamma_1(x)$ and $\gamma_2(x)$ satisfy Eq. (3.17), do their sum functions $\alpha(x) = \alpha_1(x) + \alpha_2(x)$ and $\gamma(x) = \gamma_1(x) + \gamma_2(x)$ still satisfy the equation (3.17)?

Due to the length limit of the paper, we shall discuss the above two problems elsewhere.

4 Conclusion

Here we presented a fairly general construction of finite dimensional completely integrable Hamiltonian systems associated with the family of metric function $g(x)$. Based on the above discussions, we conclude

1. All canonical Hamiltonian systems (2.6) corresponding to those appropriate functions $g(x)$ are integrable.
2. All canonical Hamiltonian systems (2.6) corresponding to those inappropriate functions $g(x)$ are nonintegrable.

Basically, the choice of the metric function $g(x)$ depends on the $\alpha(x)'$s. The latter satisfies the nonlinear integro-differential equation (3.18) which is only respect to $\alpha(x)$. An open problem is how to solve equation (3.18) in general. This is really hard. In this paper, we provided some special solutions of $\alpha(x)$ and gave some theorems and propositions to judge whether the metric function $g(x)$ is appropriate or not. We also presented some examples to show whether $g(x)$ is appropriate. However, the more general solution of equation (3.18) is still unknown, which we defer to another time.

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