Decidability (Continued):

Theorem: Every CFL (Context Free Grammar) is decidable.

Infinity:

Definition: A set \( A \) is countable if either it is finite or it has the same size as the natural numbers \( \mathbb{N} \) (an infinite set).

Definition: Countably Infinite – A set has the same size as \( \mathbb{N} \), i.e. there is a bijection (one-to-one and onto).

Ex.) The number of positive even integers is the same as the number of positive integers.

\[
f(x) = 2x
\]
Maps

$1 - 1, 2 - 2, 3 - \frac{1}{2}, 4 - 3, 5 - 1/3$, and so on.

$\mathbb{N} \rightarrow \mathbb{Z}$

- You wouldn't be able to map the natural numbers with the integers if you started with only positive integers (You would never get to map 0 or the negative integers). By alternating and starting the integers at 0, you can map all values of $\mathbb{N}$ and $\mathbb{Z}$.

Theorem: The set $\mathbb{R}$ of real numbers is uncountable (No way to set up bijection as decimals are infinite).

Proof: $A = \{x \in \mathbb{R} : 0 \leq x \leq 1\}$ (Only looking for decimal digits between 0 and 1.)

$A \subseteq \mathbb{R}$

Assume that $A$ is countable.

$\Rightarrow \exists$ a bijection $f: \mathbb{N} \rightarrow A$

(implies there exists a bijection $f: \mathbb{N} \rightarrow A$)

So $\forall n \in \mathbb{N}$

$A = \{f(1), f(2), \ldots\}$

Since decimal digits

$f(1) = 0.d_{11}d_{12}d_{13}$

$d_{ij} = \{0, 1, 2, \ldots, 9\}$

$f(n) = 0.d_{n1}d_{n2}d_{n3}$

Construct a decimal number as follows,

$x = 0.d_{11}d_{22}d_{33}\ldots$

where $d_n = \{4$ if $d_{nn} \neq 4, 5$ if $d_{nn} = 4\}$

Clearly, $x \in A$, thus $\exists n \in \mathbb{N}$ s.t. $f(n) = x$

The $n^{th}$ digit of $f(n)$ is $d_{nn}$

The $n^{th}$ digit of $x$ is $d_n$

Since $f(n) = x$, their $n^{th}$ digits must be equal, i.e. $d_{nn} = d_n$

However, by definition of $d_n$, we have $d_n \neq d_{nn}$

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Contradiction, the set A is not countable

How x was defined
For each \( n \geq 1 \), the \( n^{th} \) digit of x is not equal to the \( n^{th} \) digit of f(n).
Thus for each \( n \geq 1 \), \( x \neq f(n) \) and therefore \( x \notin A \).

This technique is called Diagonalization (Georg Cantor).

Theorem: If S is a countably infinite set, the power set of S is uncountable.

\[
| 2^S | = | 2^\mathbb{N} |
\]

Assume that \( 2^\mathbb{N} \) is countably infinite.
The subsets of \( \mathbb{N} \) can be listed \( A_0, A_1, A_2, \ldots \)
So that every subset is \( A_i \) for some i
\[
A = \{ i \mid i \geq 0 \text{ and } i \in A_i \}
\]

So \( A \subseteq \mathbb{N} \Rightarrow A = A_j \) for some \( j \)
1. If \( j \in A \), then \( j \notin A \)
2. If \( j \notin A \), then \( j \in A \)
Must be either in or out of A.

\( 2^\mathbb{N} \) is uncountable.

\[
A_0 = \{6, 1, 3, 0\}
\]

Continuum Hypothesis:

\[
\mathbb{N} = \mathbb{N}_0
\]

Wanted to show,

\[
\mathbb{R} = 2^\mathbb{N}
\]

In general,

\[
\mathbb{N}_i = 2^{\mathbb{N}_{i-1}}
\]

\[
A = (0, 1]
\]

-> Levels of Infinity

Theorem: The set of all infinite binary sequences B is uncountable.
Proof: Assume $B$ is countably infinite. \exists a bijection $f: \mathbb{N} \rightarrow B$

Define $x = d_1d_2\ldots$ s.t.

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$d_n = \{0 \text{ if } d_{nn} = 1, \ 1 \text{ if } d_{nn} = 0\}$

Thus, $\forall \ n \geq 1$ the $n^{th}$ digit of $x$ is not equal to the $n^{th}$ digit of $f(n)$.

Thus, $\forall \ n \geq x \notin f(n)$ but $x \in B$.

\[ \Rightarrow \text{ Contradiction} \]