# SERIES FOR $1/\pi$ OF SIGNATURE 20

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Abstract. Properties of theta functions and Eisenstein series dating to Jacobi and Ramanujan are used to deduce differential equations associated with McKay Thompson series of level 20. These equations induce expansions for modular forms of level 20 in terms of modular functions. The theory of singular values is applied to derive expansions for $1/\pi$ of signature 20 analogous to those formulated by Ramanujan.

## 1. Introduction

Let $|q| < 1$ and define the three null theta functions by

(1.1)

$$\theta_3(q) = \sum_{n=-\infty}^{\infty} q^{n^2}, \qquad \theta_4(q) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2}, \qquad \theta_2(q) = \sum_{n=-\infty}^{\infty} q^{(n+1/2)^2}.$$

In the Nineteenth Century, Jacobi proved that each $\vartheta = \theta_i(q)$ satisfies a third order differential equation [8] and that the triple of null theta functions satisfies a quartic relation

$$\theta_3^4(q) = \theta_4^4(q) + \theta_2^4(q). \tag{1.2}$$

Jacobi referred to (1.2) as an "*aequatio identica satis abstrusa*" [7, p. 147]. Ramanujan proved an equivalent coupled system of differential equations for Eisenstein series on the full modular group. This system and Ramanujan's parameterizations for Eisenstein series in terms of theta functions will be used to formulate a coupled system for level 2 modular forms

$$q\frac{d}{dq}\theta_3^4 = \frac{1}{3}\left(\theta_2^8 - \theta_4^8 + \theta_3^4 P_2\right), \quad q\frac{d}{dq}\theta_4^4 = \frac{1}{3}\left(\theta_2^8 - \theta_3^8 + \theta_4^4 P_2\right), \tag{1.3}$$

$$q\frac{d}{dq}\theta_2^4 = \frac{1}{3}\left(\theta_3^8 - \theta_4^8 + \theta_2^4 P_2\right), \quad q\frac{d}{dq}P_2 = \frac{2P_2^2 - \theta_2^8 - \theta_3^8 - \theta_4^8}{12}, \tag{1.4}$$

where $P_n = P(q^n)$ is defined in terms of the weight two Eisenstein series

$$P(q) = 1 - 24\sum_{j=1}^{\infty} \frac{jq^j}{1-q^j}.$$

We apply (1.3)–(1.4) to obtain a third order differential equation for $\theta_3^4$ with rational coefficients in the level 4 modular function $\theta_2^4\theta_4^4/\theta_3^8$. From these identities and (1.2), we derive a third order differential equation for modular forms with coefficients in the field of rational functions generated by McKay

Thompson series of level 20. The theory of singular values is then used to construct new Ramanujan-Sato expansions for $1/\pi$.

The McKay-Thompson series for the subgroups $\Gamma$ that will be relevant here are given in terms of the Dedekind eta-function, defined for $q = e^{2\pi i\tau}$ and $\tau \in \mathbb{H}$ by $\eta(\tau) = q^{1/24}\prod_{j=1}^{\infty}(1-q^j)$ with $\eta_\ell = \eta(\ell\tau)$. These are [4]

$$u = \left(\frac{\eta_1\eta_{20}}{\eta_4\eta_5}\right)^2, \quad v = \left(\frac{\eta_2\eta_5\eta_{20}}{\eta_1\eta_4\eta_{10}}\right)^2 \tag{1.5}$$

$$k = \left(\frac{\eta_4\eta_{20}}{\eta_2\eta_{10}}\right)^2, \quad w = \left(\frac{\eta_2\eta_{20}}{\eta_4\eta_{10}}\right)^3. \tag{1.6}$$

In the following table, we list the appropriate group and Hauptmodul using the notation of [4] for $\Gamma_0(20)$ extended by the indicated Atkin-Lehner involutions. The function $t_\Gamma$ represents the normalized Hauptmodul for $\Gamma$.

| $\Gamma$ | $t_\Gamma$ |
|---|---|
| 20+ | $\frac{1}{u} - 2 + u$ |
| 20\|2+ | $\frac{1}{w} - w$ |
| 20+4 | $\frac{1}{v} + 2$ |
| 20\|2+5 | $\frac{1}{k}$ |
| 20\|2+10 | $\frac{1}{w}$ |
| 20+20 | $\frac{1}{u} - 2$ |

As in the lower level analogues, weight two forms that appear in our analysis are theta functions corresponding to binary quadratic forms. The class number $h(-20) = 2$, and the corresponding inequivalent forms are

$$Z = \left(\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} q^{m^2+5n^2}\right)^2, \quad \mathbf{Z} = \left(\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} q^{2m^2+2mn+3n^2}\right)^2.$$

Although we focus on differential equations satisfied by $Z$, sister equations for $\mathbf{Z}$ may be derived using a similar construction to that appearing here.

We require the notion of an eta-product, a function of the form

$$f(\tau) = \prod_{\delta|\ell} (\eta(\delta\tau))^{r_\delta} \tag{1.7}$$

where $\ell$ is a positive integer, the product is taken over the positive divisors of $\ell$, and the $r_\delta$ are integers. Let $M_k(\Gamma_0(\ell))$ be the space of modular forms of weight $k$ with trivial multiplier system for the modular subgroup $\Gamma_0(\ell)$; see, e.g., [10, Chapter 1] for the definitions. When $k$ is an even integer there is a simple test that can be used to determine if an eta-product is in $M_k(\Gamma_0(\ell))$:

**Lemma 1.1.** *Let $\ell$ be a positive integer and consider the eta-product $f(\tau)$ defined by* (1.7). *Let*

$$k = \frac{1}{2}\sum_{\delta|\ell} r_\delta \quad \textit{and} \quad s = \prod_{\delta|\ell} \delta^{|r_\delta|}.$$

*Suppose that*

(1) $k$ is an even integer;
(2) $s$ is the square of an integer;
(3) $\displaystyle\sum_{\delta|\ell} \delta\, r_\delta \equiv 0 \pmod{24}$;
(4) $\displaystyle\sum_{\delta|\ell} \frac{\ell}{\delta}\, r_\delta \equiv 0 \pmod{24}$;
(5) $\displaystyle\sum_{\delta|\ell} \gcd(d,\delta)^2\, \frac{r_\delta}{\delta} \geq 0$ for all $d|\ell$.

*Then $f \in M_k(\Gamma_0(\ell))$.*

*Proof.* This is immediate from [10, Thms. 1.64, 1.65]. The main ideas of the proof are given in [6, Theorem 1]. □

We will need the following result about Eisenstein series of weight 2.

**Lemma 1.2.** *Define $P_\ell = P(q^\ell)$. For any positive integer $\ell \geq 2$,*

$$\ell P_\ell - P_1 \in M_2(\Gamma_0(\ell)).$$

*Proof.* See [12, pp. 177–178]. □

The collection of results in the next theorem will be necessary to develop spanning sets for the spaces of modular forms in the remainder of the paper.

**Theorem 1.3.** *The dimension of the space of modular forms of weight* 2 *for the modular subgroup $\Gamma_0(20)$ is given by*

$$\dim M_2(\Gamma_0(20)) = 6. \tag{1.8}$$

*If $c_1$, $c_2$, $c_4$, $c_5$, $c_{10}$ and $c_{20}$ are any constants that satisfy*

$$20c_1 + 10c_2 + 5c_4 + 4c_5 + 2c_{10} + c_{20} = 0,$$

*then*

$$c_1P_1 + c_2P_2 + c_4P_4 + c_5P_5 + c_{10}P_{10} + c_{20}P_{20} \in M_2(\Gamma_0(20)). \tag{1.9}$$

*Furthermore,*

$$z,\ zu,\ zu^{-1},\ zv,\ zv^{-1} \in M_2(\Gamma_0(20)). \tag{1.10}$$

*Proof.* The dimension formula (1.8) follows from [13, Prop. 6.1]. The result (1.9) follows from Lemma 1.2 and the trivial property that

$$M_k(\Gamma_0(\ell)) \subseteq M_k(\Gamma_0(m)) \quad \text{if} \quad \ell|m.$$

The results in (1.10) are immediate consequences of Lemma 1.1. □

## 2. Spanning Sets

In this section, we develop bases for weight two and four forms for $\Gamma_0(20)$. It is well-known, e.g., [13, p. 83], that

$$M_k(\Gamma_0(\ell)) = E_k(\Gamma_0(\ell)) \oplus S_k(\Gamma_0(\ell)) \tag{2.1}$$

where $E_k(\Gamma_0(\ell))$ and $S_k(\Gamma_0(\ell))$ are the subspaces of Eisenstein series and cusp forms, respectively, of weight $k$ for $\Gamma_0(\ell)$. From the dimension formulas in [13, p. 93] we find $\dim E_2(\Gamma_0(20)) = 5$ and $\dim S_2(\Gamma_0(20)) = 1$. In fact,

$$\begin{aligned} E_2(\Gamma_0(20)) &= \operatorname{span}_{\mathbb{C}} \left\{ \begin{matrix} 2P(q^2)-P(q),\, 4P(q^4)-P(q),\, 5P(q^5)-P(q), \\ 10P(q^{10})-P(q),\, 20P(q^{20})-P(q) \end{matrix} \right\} \\ &= \left\{ \begin{matrix} c_1P_1 + c_2P_2 + c_4P_4 + c_5P_5 \\ +c_{10}P_{10} + c_{20}P_{20} \end{matrix} \;\middle|\; \begin{matrix} 20c_1 + 10c_2 + 5c_4 + 4c_5 \\ +2c_{10} + c_{20} = 0 \end{matrix} \right\} \end{aligned} \tag{2.2}$$

and [5]

$$S_2(\Gamma_0(20)) = \mathbb{C}z, \qquad \text{where } z = \eta_2^2\eta_{10}^2. \tag{2.3}$$

Therefore,

$$M_2(\Gamma_0(20)) = \operatorname{span}_{\mathbb{C}} \left\{ \begin{matrix} 2P(q^2)-P(q),\, 4P(q^4)-P(q),\, 5P(q^5)-P(q), \\ 10P(q^{10})-P(q),\, 20P(q^{20})-P(q),\, z \end{matrix} \right\}. \tag{2.4}$$

Similarly, we compute an explicit basis for $M_4(\Gamma_0(20))$:

$$M_4(\Gamma_0(20)) = \operatorname{span}_{\mathbb{C}} \left\{ \begin{matrix} Q(q),\, Q(q^2),\, Q(q^4),\, Q(q^5),\, Q(q^{10}),\, Q(q^{20}),\, z^2, \\ \frac{z^2}{u},\, z^2v,\, z^2k^2,\, z^2kw \end{matrix} \right\}. \tag{2.5}$$

The above construction can be applied to give a representation for each of $zu$, $zu^{-1}$, $zv$ and $zv^{-1}$ as the sum of Eisenstein series and a cusp form.

**Theorem 2.1.** *Let $z$ be defined by* (2.3). *Then*

$$zu = \frac{1}{72}\left(P_1 - 6P_2 + 20P_4 - 25P_5 + 30P_{10} - 20P_{20}\right) + \frac{1}{3}z, \tag{2.6}$$

$$\frac{z}{u} = \frac{1}{72}\left(-5P_1 + 6P_2 - 4P_4 + 5P_5 - 30P_{10} + 100P_{20}\right) + \frac{1}{3}z, \tag{2.7}$$

$$zv = \frac{1}{72}\left(-P_1 + 4P_4 + P_5 - 4P_{20}\right) - \frac{1}{3}z, \tag{2.8}$$

$$\frac{z}{v} = \frac{1}{72}\left(P_1 - 4P_4 - 25P_5 + 100P_{20}\right) - \frac{5}{3}z. \tag{2.9}$$

*Proof.* These are immediate from (1.10) and (2.4). □

These relations induce identities between the level 20 haptmoduln.

**Theorem 2.2.** *Let $u, v$ be defined by* (1.5). *Then the following identity holds.*

$$\frac{1}{u} + u = \frac{1}{v} + 4 + 5v. \tag{2.10}$$

*Proof.* By (1.8) and (1.10) we have that $z$, $zu$, $zu^{-1}$, $zv$, $zv^{-1}$ are linearly dependent; in fact, from Theorem 2.1 we have

$$4z + 5zv + \frac{z}{v} - zu - \frac{z}{u} = 0.$$

The claimed identity may be obtained by dividing both sides by $z$. □

The preceding proof is a prototype for the derivation of a plethora of relations between the other Hauptmoduln of level 20. For our purposes, we next derive theta quotient representations for certain linear functions in $v$.

**Theorem 2.3.**

$$1 + v = \frac{\eta_{10}^8}{\eta_1\eta_4\eta_5^3\eta_{20}^3}, \qquad 1 + 5v = \frac{\eta_2^{10}\eta_5\eta_{20}}{\eta_1^5\eta_4^5\eta_{10}^2}. \tag{2.11}$$

*Proof.* By the definition (2.3) of $z$ and Lemma 1.1, we can check that

$$z\frac{\eta_{10}^8}{\eta_1\eta_4\eta_5^3\eta_{20}^3} = \frac{\eta_2^2\eta_{10}^{10}}{\eta_{20}^3\eta_5^3\eta_4\eta_1} \in M_2(\Gamma_0(20)).$$

By (2.4), it can be shown that

$$z\frac{\eta_{10}^8}{\eta_1\eta_4\eta_5^3\eta_{20}^3} = -\frac{1}{72}P(q) + \frac{1}{18}P(q^4) + \frac{1}{72}P(q^5) - \frac{1}{18}P(q^{20}) + \frac{2}{3}z.$$

In addition, by Theorem 2.1, we find that

$$z + zv = -\frac{1}{72}P(q) + \frac{1}{18}P(q^4) + \frac{1}{72}P(q^5) - \frac{1}{18}P(q^{20}) + \frac{2}{3}z.$$

Therefore,

$$z + zv = z\frac{\eta_{10}^8}{\eta_1\eta_4\eta_5^3\eta_{20}^3},$$

and dividing both sides by $z$ gives the first identity of (2.11). The second identity can be deduced in a similar way. We omit the details. □

## 3. Differential Equations

In this section, we prove the differential system (1.3)–(1.4) and deduce third order differential equations from the computations in the prior section.

**Theorem 3.1.** *The differential sytem* (1.3)–(1.4) *holds.*

*Proof.* Ramanujan's parameterization [1, Equation 5.4.5]

$$P(q^2) = (1-2x)\theta_3^4 + 6x(1-x)\frac{d\theta_3^2}{dx}, \qquad x = \frac{\theta_2^4}{\theta_3^4} \tag{3.1}$$

coupled with Jacobi's relation (1.2) and [1]

$$q\frac{dx}{dq} = \theta_3^4 x(1-x) \tag{3.2}$$

together imply the claimed expression for $qd\theta_3^4/dq$. The differential equation for $\theta_4^4$ follows from that of $\theta_3^4$ since the map $q \mapsto -q$ fixes $\theta_2, P_2$ and interchanges $\theta_3$ and $\theta_4$. We derive the differential equation for $\theta_2^4$ directly from those for $\theta_3^4, \theta_4^4$ and Jacobi's relation (1.2). The expression for $qdP_2/dq$ is equivalent to Ramanujan's differential equation for the Eisenstein series of weight two on the full modular group.

$$q\frac{dP}{dq} = \frac{P^2 - Q}{12}, \qquad Q(q^2) = 1 + 240\sum_{n=1}^{\infty} \frac{n^3 q^{2n}}{1 - q^{2n}} = \frac{\theta_2^8 + \theta_3^8 + \theta_4^8}{2}. \tag{3.3}$$

Ramanujan derived the parameterization for $Q(q^2)$ [1, Equation 5.4.8]. $\square$

**Theorem 3.2.** *Let $X$ be defined by*

$$X = \frac{z}{Z}.$$

*Then*

$$X = \frac{u}{(1+u)^2} = \frac{v}{(1+v)(1+5v)}. \tag{3.4}$$

*Proof.* Since $Z \in M_2(\Gamma_0(20))$, then by (2.4), it is easy to show that

$$Z = -\frac{1}{18}P(q) + \frac{2}{9}P(q^4) - \frac{5}{18}P(q^5) + \frac{10}{9}P(q^{20}) + \frac{8}{3}z.$$

By Theorem 2.1, it follows that

$$zu + \frac{z}{u} + 2z = \frac{\eta_1^{24}\eta_4^{24}}{\eta_2^{48}}.$$

Therefore

$$Z = zu + \frac{z}{u} + 2z = z\frac{(1+u)^2}{u},$$

which implies the first equality of (3.4). The second equality follows from Theorem 2.2. $\square$

**Theorem 3.3.**

$$q\frac{d}{dq}\log u = \frac{z}{u}\sqrt{1 - 8u - 2u^2 - 8u^3 + u^4}, \tag{3.5}$$

$$q\frac{d}{dq}\log X = Z\sqrt{(1-4X)(1-12X+16X^2)}. \tag{3.6}$$

*Proof.* It is easy to verify that

$$q\frac{d}{dq}\log u = \frac{1}{12}P(q) - \frac{1}{3}P(q^4) - \frac{5}{12}P(q^5) + \frac{5}{3}P(q^{20}).$$

By Theorem 2.1, we have

$$q\frac{d}{dq}\log u = -5zv + \frac{z}{v}.$$

Then together with Theorem 2.2, the above identity implies (3.5). By Theorem 3.2, we have

$$X = \frac{u}{(1+u)^2} \quad \text{and} \quad Z = z\frac{(1+u)^2}{u}.$$

The differentiation formula (3.6) follows from (3.5) via changes of variables. □

**Theorem 3.4.**

$$\begin{aligned} &X^2(1-4X)(1-12X+16X^2)\frac{d^3Z}{dX^3} + 3X(1-24X+128X^2-160X^3)\frac{d^2Z}{dX^2} \\ (3.7) \qquad &+(1-56X+464X^2-784X^3)\frac{dZ}{dX} - 4(1-20X+54X^2)Z = 0. \end{aligned}$$

*Proof.* Let $F$ and $T$ be defined by

$$F = \theta_3^4 \quad \text{and} \quad T = \frac{t}{(1+16t)^2} = \frac{\theta_2^4\theta_4^4}{16\theta_3^8}.$$

From the differential system (1.3)–(1.4) and Jacobi's quartic identity,

$$q\frac{dT}{dq} = \frac{\theta_3^4}{16}\frac{\theta_2^4\theta_4^4}{\theta_3^8}\left(1 - 2\frac{\theta_2^4}{\theta_3^4}\right) = \frac{\theta_2^4\theta_4^4}{16\theta_3^4}\left(1 - 2\frac{\theta_2^4}{\theta_3^4}\right). \tag{3.8}$$

By the chain rule,

$$\frac{dF}{dT} = \frac{dF}{dq}\Big/\frac{dT}{dq} = \frac{16\theta_3^8\left(\theta_2^8 - \theta_4^8 - \theta_3^4P_2\right)}{3\theta_2^4\theta_4^4(\theta_3^4 - 2\theta_2^4)}. \tag{3.9}$$

Likewise, $d^2F/dT^2, d^3F/dT^3$ may be expressed by way of the chain rule as rational functions of $\theta_2^4, \theta_3^4, \theta_4^4, P_2$. By applying Jacobi's quartic identity (1.2), we derive

$$T^2(1-64T)\frac{d^3F}{dT^3} + 3T(1-96T)\frac{d^2F}{dT^2} + (1-208T)\frac{dF}{dT} = 8F. \tag{3.10}$$

Theorem 2.1 and the Jacobi triple product identity [1] for the theta functions imply

$$F = z\frac{(1+5v)^2}{v} \quad \text{and} \quad T = \frac{v}{(1+v)(1+5v)^5}.$$

Making use of these changes of variables and chain rule, the differential equation (3.10) implies that $z$ satisfies a third order differential equation with respect to $v$. This implies (3.7) by the change of variables from Theorem 3.2

$$Z = z\frac{(1+v)(1+5v)}{v} \quad \text{and} \quad X = \frac{v}{(1+v)(1+5v)}.$$

□

Writing $Z_X = X\frac{d}{dX}$, the last theorem takes a more compact form.

**Theorem 3.5.**

$$4x\left(54x^2-20x+1\right)Z+16x\left(27x^2-13x+1\right)Z_X+24x(2x-1)(6x-1)Z_{XX}$$
$$+(4x-1)\left(16x^2-12x+1\right)Z_{XXX}=0.$$

Theorem 3.5 induces a power series expansion for $Z$ in terms of $X$.

**Corollary 3.6.** *For* $|X|<\frac{1}{8}\left(3-\sqrt{5}\right)$,

$$Z=\sum_{n=0}^{\infty}a_nX^n,$$

*where* $a_0=1, a_1=4, a_2=20$ *and*

$$a_{n+1}=4\frac{(2n+1)\left(2n^2+2n+1\right)}{(n+1)^3}a_n-16\frac{n\left(4n^2+1\right)}{(n+1)^3}a_{n-1}+8\frac{(2n-1)^3}{(n+1)^3}a_{n-2}.$$

## 4. Singular Values and Ramanujan-Sato Series of Level 20

In this section, we given an abbreviated proof of the expansions for $1/\pi$ resulting from the the differential equations for $Z$ and singular values for the modular function $X(\tau)$. Well known results from the theory of singular values and properties of the $j$-invariant are used. Most of the results needed may be found in [3, 4, 9]

For any natural number $N$ and $e||N$, consider the set of Atkin-Lehner involutions

$$W_e=\left\{\left(\begin{array}{cc} ea & b\\ Nc & ed\end{array}\right)\,\middle|\, \begin{array}{c}(a,b,c,d)\in\mathbb{Z}^4\\ ead-\frac{N}{e}bc=1\end{array}\right\}.$$

Each $W_e$ is a coset of $\Gamma_0(N)$ with the multiplication rule

$$W_eW_f\equiv W_{ef/\gcd(e,f)^2} \bmod \Gamma_0(N).$$

For any set of indices $e$ closed under this rule, the group $\Gamma=\cup_e W_e$ is a subgroup of the normalizer in $PSL(2,\mathbb{R})$ of $\Gamma_0(N)$. Such a group is denoted as $\Gamma_0(N)+w_{e_1},w_{e_2}\cdots$, or more succinctly as $N+e_1,e_2,\cdots$. This is shortened to $N+$ when all of the indices are present. For each such $\Gamma$ of genus 0, let $t_\Gamma(\tau)$ be its normalized Hauptmodul

$$t_\Gamma(\tau)=\frac{1}{q}+\sum_{n=1}^{\infty}a_nq^n.$$

The modular function $X(\tau)$ defined in Theorem 3.2 is the normalized Hauptmodul for $\Gamma_0(20)+$ and is therefore invariant under action by any element of $\Gamma_0(20)+$. To find explicit evaluations for $X(\tau)$ in the upper half plane $\mathbb{H}$, we construct modular equations of level 20 and degree $n\geq 2$.

**Definition 4.1.** *Let $t_\Gamma$ be a normalized Hauptmodul for $\Gamma$ and suppose that* $\gcd(n,N)=1$. *Then the modular equation for $t_\Gamma$ is*

$$\Psi_n^{t_\Gamma}(Y) = \prod_{\substack{\alpha\delta=n\\ 0\le\beta<\delta\\ (\alpha,\beta,\delta)=1}} \left(Y - t_\Gamma\left(\frac{\alpha\tau+\beta}{\delta}\right)\right). \tag{4.1}$$

The modular equations may be expressed as a polynomials with integer coefficients in the parameter $Y$ and $t_\Gamma$ [3, Proposition 2.5].

**Theorem 4.2.** *For each normalized Hauptmodul $t_\Gamma$ and* $\gcd(n,N)=1$,

$$\Psi_n^{t_\Gamma}(Y) \in \mathbb{Z}[t_\Gamma, Y].$$

For each $\tau$ in Table 3, we demonstrate that $X(\tau)$ is a root of a modular equation $\Psi_n^X(X)$ for some $n$ relatively prime to 20.

**Theorem 4.3.** *For $n \ge 2$ with* $\gcd(n,20)=1$, *and for each $\tau\in\mathbb{H}$ defined in Table 3, we have $\Psi_n^X(X)=0$, where $X=X(\tau)$.*

*Proof.* To show each $\tau$ in Table 3 satisfies $\Psi_n^X(X)=0$, where $X=X(\tau)$, we determine an integer $n$ and pairwise relatively prime integers $\alpha,\beta,\delta$ such that $\alpha\delta=n$ and $0\le\beta<\delta$ with $(\alpha\tau+\beta)/\delta$ equal to the image of $\tau$ under an Atkin-Lehner involution from $W_e$ for some $e\ ||\ 20$. Since $X(\tau)$ is invariant under each Atkin-Lehner involution, we see that $X(\tau)-X((\alpha\tau+\beta)/\delta)=0$, so $X=X(\tau)$ satisfies $\Psi_n^X(X)=0$. □

To evaluate $X(\tau)$ for each quadratic irrational $\tau$ listed in Table 3, a sufficiently accurate approximation of $X(\tau)$ may be used to distinguish the corresponding root of $\Psi_n^X(X)$ and to evaluate $X(\tau)$. We illustrate the technique for deriving expansions for $1/\pi$ in the next example. The procedure relies on computing modular equations as in [2].

Let $\tau_0=\tau(20,-40,23)=\frac{1}{10}\left(10+i\sqrt{15}\right)$. Then

$$\frac{\tau_0+2}{3} = \frac{20\tau_0-21}{20\tau_0-20} \tag{4.2}$$

Hence, with $(a,b,c,d)=(1,-21,1,-1)$ and $e=N=20$, we have M:=

$$\begin{pmatrix} ea & b\\ Nc & ed\end{pmatrix} = \begin{pmatrix} 20 & -21\\ 20 & -20\end{pmatrix} \in W_{20}. \tag{4.3}$$

Therefore,

$$X\left(\frac{\tau_0+2}{3}\right) = X(\tau_0), \tag{4.4}$$

and so $X=X(\tau)$ is a root of the modular equation

$$\begin{aligned}\Psi_3(X,Y) &= X^4-256X^3Y^3+192X^3Y^2-30X^3Y+192X^2Y^3-93X^2Y^2\\ &\quad +12X^2Y-30XY^3+12XY^2-XY+Y^4=0.\end{aligned} \tag{4.5}$$

Hence, $X(\tau)$ is a root of

$$\Psi_3(X,X) = -X^2(X-1)(16X-1)\left(16X^2-7X+1\right) = 0.$$

By numerically approximating $X(\tau)$ and applying (4.5), we deduce (4.5),

$$X(\tau_0) = 1/16, \qquad \frac{dY}{dX} = -1, \qquad \left.\frac{d^2Y}{dX^2}\right|_{\tau=\tau_0} = -\frac{32}{3}. \tag{4.6}$$

Moreover, we may apply (4.5) to expand $Y$ about $X(\tau_0)$

$$X(M\tau) = \sum_{k=0}^{\infty} Y^{(k)}(X(\tau_0))\frac{(X(\tau)-X(\tau_0))^k}{k!}, \qquad Y^{(k)} = \frac{dY^k}{dX^k}. \tag{4.7}$$

By applying $\frac{1}{2\pi i}\frac{d}{d\tau}$ twice to both sides of (4.7) and applying (3.6),

$$\left.\frac{20i}{\pi W\left(1-\frac{dY}{dX}\right)(20\tau-20)}\right|_{\tau=\tau_0} \tag{4.8}$$
$$= \left.\left(X\frac{dZ}{dX} + \left(1+\frac{X}{W}\frac{dW}{dX} + X\frac{\frac{d^2Y}{dX^2}}{\frac{dY}{dX}\left(1-\frac{dY}{dX}\right)}\right)Z\right)\right|_{\tau=\tau_0},$$

where

$$W := W(X) := \sqrt{(1-4X)(1-12X+16X^2)}. \tag{4.9}$$

By (4.6), (4.8), and the fact that $Z = \sum_{n=0}^{\infty} A_n X^n$, we obtain

$$\frac{1}{\pi} = \frac{3}{8}\sum_{n=0}^{\infty}\frac{A_n}{16^n}\left(n+\frac{1}{6}\right). \tag{4.10}$$

Similarly, for each $\tau_0$ such that $X(\tau_0)$ lies in the radius of convergence for Theorem 3.7, we use the modular equation to deduce the series expansions for $\pi$. The matrix $M$ under which $X$ is invariant, and other parameters for each modular equation appear in Table 3. 3.7. We restrict ourselves to singular values of degree no more than 2 over $\mathbb{Q}$.

In order to formulate a complete list of algebraic $\tau$ such that $X(\tau)$ has algebraic degree two over $\mathbb{Q}$, e use well known facts about the $j$ invariant [11]. First, for algebraic $\tau$, the only algebraic values of $j(\tau)$ occur at $\operatorname{Im}\tau > 0$ satisfying $a\tau^2+b\tau+c=0$ for $a,b,c\in\mathbb{Z}$, with $d = b^2-4ac < 0$. Moreover,

$$[\mathbb{Q}(j(\tau)):\mathbb{Q}] = h(d),$$

where $h(d)$ is the class number. Since there is a polynomial relation $P(X,j)$ between $X$ and $j$ of degree 2 [3, Remark 1.5.3], we have

$$[\mathbb{Q}(j(\tau)):\mathbb{Q}] \le 2[\mathbb{Q}(X(\tau)):\mathbb{Q}],$$

and so values $\tau$ with $[\mathbb{Q}(X(\tau)):\mathbb{Q}] \le 2$ satisfy

$$[\mathbb{Q}(j(\tau)):\mathbb{Q}] = h(d) \le 4.$$

Therefore, the bound $|d| \le 1555$ for $h(d) \le 4$ from [14] implies that Algorithm 4.1 gives a complete list of algebraic $(\tau, X(\tau))$ with $[\mathbb{Q}(X(\tau)):\mathbb{Q}] \le 2$:

| $b^2-4ac$ | $\tau(a,b,c)$ | $X(\tau)$ |
|---|---|---|
| $-16$ | $(4,-8,5)$ | $\frac{1}{8}(7-3\sqrt{5})$ |
| $\begin{bmatrix}-40\\-160\end{bmatrix}$ | $\begin{bmatrix}(10,-20,11)\\(40,-80,41)\end{bmatrix}$ | $-3/2+\sqrt{5/2}$ |
| $-240$ | $(20,-40,23)$ | $1/16$ |
| $-256$ | $(20,-12,5)$ | $i/8\sqrt{2}$ |
| $-256$ | $(20,-28,13)$ | $-i/8\sqrt{2}$ |
| $-320$ | $(20,-20,9)$ | $\frac{1}{16}(1-\sqrt{5})$ |
| $-400$ | $(100,-200,101)$ | $\frac{1}{8}(7-3\sqrt{5})$ |
| $-480$ | $(20,-20,11)$ | $\frac{1}{8}(-7+3\sqrt{5})$ |
| $-640$ | $(20,-20,13)$ | $\frac{1}{8}(3-\sqrt{10})$ |
| $-880$ | $(20,-40,31)$ | $\frac{1}{32}(7-3\sqrt{5})$ |
| $-960$ | $(20,-20,17)$ | $\frac{1}{16}(-4+\sqrt{15})$ |
| $-1120$ | $(20,-20,19)$ | $\frac{1}{8}(-47+21\sqrt{5})$ |
| $-1360$ | $(20,-40,37)$ | $1/18\sqrt{85}+166$ |
| $-2080$ | $(20,20,31)$ | $-161/4+18\sqrt{5}$ |
| $-3040$ | $(20,20,43)$ | $-721/4+57\sqrt{10}$ |

TABLE 2. Complete list of quadratic singular values of $X(\tau)$ within the radius of convergence of Theorem 3.7.

**Algorithm 4.1.** *For each discriminant* $-1555 \le d \le -1$,

*(1) List all primitive reduced* $\tau = \tau(a,b,c)$ *of discriminant* $d$ *in a fundamental domain for* $PSL_2(\mathbb{Z})$. *Translate these values via a set of coset representatives for* $\Gamma_0(20)$ *to a fundamental domain for* $\Gamma_0(20)$.

*(2) Factor the resultant of* $P(X,Y)$ *and the class polynomial*

$$H_d(Y) = \prod_{\substack{(a,b,c) \text{ reduced, primitive} \\ d=b^2-4ac}} \left( Y - j\Big(\frac{-b+\sqrt{d}}{2a}\Big)\right).$$

*Linear and quadratic factors of the resultant correspond to a complete list of* $X(\tau)$, *for* $\tau$ *of discriminant* $d$, *such that* $[\mathbb{Q}(X(\tau)):\mathbb{Q}] \le 2$.

*(3) Associate candidate values* $\tau$ *from Step 1 to* $X$ *by approximating* $X(\tau)$. *For each approximation, prove the evaluation* $X = X(\tau)$ *by deriving a corresponding modular equation for which* $X(\tau)$ *are roots. The proof of each evaluation may be accomplished through a rigorous derivation of the first decimal digits of* $X(\tau)$ *and a comparison of these values with those of the roots of the modular equation.*

Algorithm 4.1 was implemented in Mathematica resulting in Table 4. In the final two tables, we list constants defining level 20 expansions

$$\frac{1}{\pi} = A\sum_{n=0}^{\infty} a_n(n+B)C^n. \tag{4.11}$$

| $\tau(a,b,c)$ | $n$ | $e$ | $(\alpha,\beta;0,\delta)$ | $\gamma\in W_e$ |
|---|---|---|---|---|
| $(4,-8,5)$ | 29 | 4 | $(1,5;0,29)$ | $(4,-5;20,-24)$ |
| $(10,-20,11)$ | 13 | 5 | $(1,9;0,13)$ | $(5,-5;20,-30)$ |
| $(40,-80,41)$ | 13 | 5 | $(1,4;0,13)$ | $(15,-17;40,-45)$ |
| $(20,-40,23)$ | 3 | 20 | $(1,2;0,3)$ | $(20,-21,20,-20)$ |
| $(20,-12,5)$ | 17 | 4 | $(1,10,0,17)$ | $(12,-5;20,-8)$ |
| $(20,-28,13)$ | 17 | 4 | $(1,13;0,17)$ | $(16,-13;20,-16)$ |
| $(20,-20,9)$ | 9 | 20 | $(1,0;0,9)$ | $(0,-1;20,-20)$ |
| $(100,-200,101)$ | 29 | 4 | $(1,6;0,29)$ | $(24,-25;100,-104)$ |
| $(20,-20,11)$ | 11 | 20 | $(1,0;0,11)$ | $(0,-1;20,-20)$ |
| $(20,-20,13)$ | 13 | 20 | $(1,0;0,13)$ | $(0,-1;20,-20)$ |
| $(20,-40,31)$ | 11 | 20 | $(1,10;0,11)$ | $(20,-21;20,-20)$ |
| $(20,-20,17)$ | 17 | 20 | $(1,0;0,17)$ | $(0,-1;20,-20)$ |
| $(20,-20,19)$ | 19 | 20 | $(1,0;0,19)$ | $(0,-1;20,-20)$ |
| $(20,-40,37)$ | 17 | 20 | $(1,16;0,17)$ | $(0,-21;20,-20)$ |
| $(20,20,31)$ | 31 | 20 | $(1,1;0,31)$ | $(0,-1;20,-20)$ |
| $(20,20,43)$ | 43 | 20 | $(1,0;0,43)$ | $(0,-1;20,-20)$ |

TABLE 3. Matrices $(\alpha,\beta;0,\delta)$ mapping $\tau$ to the image under an element from $W_e$, demonstrating $X(\tau)$ is a root of $\Psi_n^X(X)$

| $A$ | $B$ | $C$ |
|---|---|---|
| $2\sqrt{9\sqrt{5}-20}$ | $\frac{1}{4}(3-\sqrt{5})$ | $\frac{1}{8}(7-3\sqrt{5})$ |
| $\sqrt{506-160\sqrt{10}}$ | $\frac{1}{6}\left(4-\sqrt{10}\right)$ | $-3/2+\sqrt{5/2}$ |
| $3/8$ | $1/6$ | $1/16$ |
| $\frac{1}{5}\sqrt{8-\frac{31i}{\sqrt{2}}}$ | $\frac{1}{66}\left(31-8i\sqrt{2}\right)$ | $i/8\sqrt{2}$ |
| $\frac{1}{5}\sqrt{8+\frac{31i}{\sqrt{2}}}$ | $\frac{1}{66}\left(31+8i\sqrt{2}\right)$ | $-i/8\sqrt{2}$ |
| $\sqrt{\frac{1}{2}\left(\sqrt{5}+2\right)}$ | $1/2$ | $\frac{1}{16}\left(1-\sqrt{5}\right)$ |
| $\sqrt{\sqrt{10}-1}$ | $\frac{1}{6}\left(5-\sqrt{10}\right)$ | $\frac{1}{8}\left(3-\sqrt{10}\right)$ |
| $11/8$ | $\frac{1}{22}\left(13-4\sqrt{5}\right)$ | $\frac{1}{32}\left(7-3\sqrt{5}\right)$ |
| $\frac{1}{4}\sqrt{636-153\sqrt{15}}$ | $\frac{1}{462}\left(261-40\sqrt{15}\right)$ | $\frac{1}{16}\left(\sqrt{15}-4\right)$ |
| $14\sqrt{805-360\sqrt{5}}$ | $\frac{1}{140}\left(105-34\sqrt{5}\right)$ | $\frac{1}{8}\left(21\sqrt{5}-47\right)$ |
| $12\sqrt{697\sqrt{85}-6426}$ | $\frac{1}{204}\left(153-13\sqrt{85}\right)$ | $\frac{1}{8}\left(83-9\sqrt{85}\right)$ |
| $78\sqrt{14445-6460\sqrt{5}}$ | $\frac{1}{780}\left(585-212\sqrt{5}\right)$ | $\frac{1}{4}\left(72\sqrt{5}-161\right)$ |
| $646\sqrt{27379-8658\sqrt{10}}$ | $\frac{969-259\sqrt{10}}{1292}$ | $\frac{1}{4}\left(228\sqrt{10}-721\right)$ |

TABLE 4. Constants defining level 20 expansions (4.11).

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