LEVEL 17 RAMANUJAN-SATO SERIES

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Abstract. Two level 17 modular functions

$$r = q^2 \prod_{n=1}^{\infty} (1-q^n)^{\left(\frac{n}{17}\right)}, \qquad s = q^2 \prod_{n=1}^{\infty} \frac{(1-q^{17n})^3}{(1-q^n)^3},$$

are used to construct a new class of Ramanujan-Sato series for $1/\pi$. The expansions are induced by modular identities similar to those level of 5 and 13 appearing in Ramanujan's Notebooks. A complete list of rational and quadratic series corresponding to singular values of the parameters is derived.

1. Introduction

Let τ be a complex number with positive imaginary part and set $q = e^{2\pi i \tau}$. Define

$$r(\tau) = q^2 \prod_{n=1}^{\infty} (1-q^n)^{\left(\frac{n}{17}\right)}, \quad s(\tau) = q^2 \prod_{n=1}^{\infty} \frac{(1-q^{17n})^3}{(1-q^n)^3}.$$

In this paper, we derive level 17 Ramanujan-Sato expansions for $1/\pi$ of the form

$$q\frac{d}{dq} \log s = \sum_{n=0}^{\infty} A_n \left(\frac{r(r^2 s + 8rs - r - s)}{8r^3 s - 3r^2 s + r - s} \right)^n, \qquad \frac{1}{s} = r + \frac{1}{r} - 2\sqrt{\frac{4}{r} - 4r - 15}, \tag{1.1}$$

where A_n is defined recursively. These relations are analogous to those at level 13 and 5 [5, 12],

$$q\frac{d}{dq} \log \mathcal{S} = \sum_{n=0}^{\infty} \mathcal{A}(n) \left(\frac{\mathcal{R}(1-3\mathcal{R}-\mathcal{R}^2)}{(1+\mathcal{R}^2)^2} \right)^n, \qquad \frac{1}{\mathcal{S}} = \frac{1}{\mathcal{R}} - 3 - \mathcal{R}, \tag{1.2}$$

$$q\frac{d}{dq} \log S = \sum_{n=0}^{\infty} a(n) \left(\frac{R^5(1-11R^5-R^{10})}{(1+R^{10})^2} \right)^n, \qquad \frac{1}{S} = \frac{1}{R^5} - 11 - R^5. \tag{1.3}$$

Here $a(n)$, $\mathcal{A}(n)$ are recursively defined sequences induced from differential equations and

$$\mathcal{R}(\tau) = q \prod_{n=1}^{\infty} (1-q^n)^{\left(\frac{n}{13}\right)}, \qquad R(\tau) = q^{1/5} \prod_{n=1}^{\infty} (1-q^n)^{\left(\frac{n}{5}\right)}. \tag{1.4}$$

Identity (1.3) and explicit evaluations for $R(\tau)$ were used to formulate expansions for $1/\pi$ including

$$\frac{1}{\pi} = \frac{1705}{81\sqrt{47}} \sum_{n=0}^{\infty} a(n) \left(n + \frac{71}{682}\right) \left(\frac{-1}{15228}\right)^n, \qquad a(n) = \binom{2n}{n} \sum_{j=0}^{n} \binom{n}{j}^2 \binom{n+j}{j}. \tag{1.5}$$

This expression is a generalization of 17 such formulas stated by Ramanujan [3, 18]. In each formula, the algebraic constants come from explicit evaluations for a modular function. The sequence A_k are coefficients in a series solution to a differential equation satisfied by a relevant modular form. Generalizing such relations to higher levels requires finding differential equations for modular parameters and relevant identities. The series [5, 9, 10, 12] have common construction for primes $p - 1 \mid 24$, where $X_0(p)$ has genus zero. More work remains to unify constructions for other levels.

The purpose of this paper is to construct level 17 Ramanujan-Sato series as a prototype for levels such that $X_0(N)$ has positive genus. Central to the construction is the fact that r and s generate the field of functions invariant under action by an index two subgroup of $\Gamma_0(17)$. These constructions and singular value evaluations yield new Ramanujan-Sato expansions, including the rational series

$$\frac{1}{\pi} = \frac{1}{\sqrt{11}} \sum_{k=0}^{\infty} A_k \frac{307 + 748k}{(-21)^{k+2}}. \tag{1.6}$$

Following [5, 6], an expansion for $1/\pi$ is said to be rational or quadratic if C/π can be expressed as a series of algebraic numbers of degree 1 or 2, respectively, for some algebraic number C. We derive a complete list of series of rational and quadratic series from singular values of parameters in (1.1).

In the next section, we given an overview of results at levels 5 and 13 that motivate the approach of the paper. Section 3 includes an analogous construction of level 17 modular functions. This construction is used to motivate the differential equation satisfied by

$$z(\tau) = \theta_q \log s, \qquad \theta_q := q\frac{d}{dq}, \tag{1.7}$$

with coefficients in the field $\mathbb{C}(x)$, where

$$x(\tau) = \frac{r(r^2 s + 8rs - r - s)}{8r^3 s - 3r^2 s + r - s}. \tag{1.8}$$

We conclude with Section 4 in which singular values are derived for x and used to construct a new class of series approximations for $1/\pi$ of level 17. We derive a complete list of values of $x(\tau)$ with $[\mathbb{Q}(x(\tau)) : \mathbb{Q}] \leq 2$ within the radius of convergence for z as a powers series in x and therefore provide a complete list of linear and quadratic Ramanujan-Sato series corresponding to $x(\tau)$.

2. Level 5 and 13 Series

The product $R(\tau)$, defined by (1.4), is the Rogers-Ramanujan continued fraction [19]. Together, $R(\tau)$ and $S(\tau)$, defined by

$$R(\tau) = \cfrac{q^{1/5}}{1 + \cfrac{q}{q + \cfrac{q^2}{1 + \cfrac{q^3}{1 + \cdots}}}}, \qquad S(\tau) = q \prod_{n=1}^{\infty} \frac{(1-q^{5n})^6}{(1-q^n)^6}, \tag{2.1}$$

generate the field of functions invariant under the congruence subgroup

$$\Gamma_0^2(d) = \{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z}) \mid c \equiv 0 \bmod p \text{ and } \chi(d) = 1 \}.$$

This motivates Ramanujan's reciprocal identity [17], [1, p. 267]

$$\frac{1}{R^5} - 11 - R^5 = \frac{1}{S}. \tag{2.2}$$

Equation (1.3) expresses the logarithmic derivative $Z = \theta_q \log S(\tau)$ in terms of a series solution to a third order linear differential equation [5] satisfied by S and the $\Gamma_0(5)$ invariant function $T = T(\tau)$

$$(16T^2 + 44T - 1)Z_{TTT} + (48T^2 + 66T)Z_{TT} + (44T^2 + 34T)Z_T + (12T^2 + 6T)Z = 0, \tag{2.3}$$

$$Z_T = T\frac{d}{dT}Z, \qquad T = \frac{R^5(1 - 11R^5 - R^{10})}{(1 + R^{10})^2}. \tag{2.4}$$

The form of the equation may be anticipated from a general theorem [22] (c.f. [24]).

Theorem 2.1. *Let Γ be subgroup of $SL_2(\mathbb{R})$ commensurable with $SL_2(\mathbb{Z})$. If $t(\tau)$ is a nonconstant meromorphic modular function and $F(\tau)$ is a meromorphic modular form of weight k with respect to Γ, then $F, \tau F, \ldots, \tau^k F$, as functions of t, are linearly independent solutions to a $(k+1)$st order differential linear equation with coefficients that are algebraic functions of t. The coefficients are polynomials when $\Gamma \setminus \mathfrak{H}$ has genus zero and t generates the field of modular functions on Γ.*

Therefore, from (2.3),

$$Z = \sum_{k=0}^{\infty} a(n) T^n, \qquad |T| < \frac{5\sqrt{5}-11}{8}, \tag{2.5}$$

where $a(n)$ is recursively determined from (2.3), and expressible in closed form [5] in terms of the summand appearing in (1.5). The final ingredient needed for Ramanujan-Sato series at level 5 are explicit evaluations for the Rogers-Ramanujan continued fraction within the radius of convergence of the power series. Such singular values for $R(\tau)$ were given by Ramanujan in his first letter to Hardy [2] and can be derived from modular equations satisfied by $T(\tau)$ and $T(n\tau)$. We provide a general approach in Section 4.

To formulate the analogous construction at level 13, define $\mathcal{R} = \mathcal{R}(\tau)$ by (1.4) and

$$\mathcal{S}(\tau) = q \prod_{n=1}^{\infty} \frac{(1-q^{13n})^2}{(1-q^n)^2}, \qquad \mathcal{T}(\tau) = \frac{\mathcal{R}(1-3\mathcal{R}-\mathcal{R}^2)}{(1+\mathcal{R}^2)^2}. \tag{2.6}$$

A third order linear differential equation [12] is satisfied by the Eisenstein series $\mathcal{Z}(\tau) = \theta_q \log \mathcal{S}$ with coefficients that are polynomials in the $\Gamma_0(13)$ invariant function $\mathcal{T}(\tau)$. For both the level 5 and 13 cases, the weight zero functions T and $\mathcal{T}$ may be uniformly presented as the quotient of a weight 4 cusp form and the square of a weight 2 Eisenstein series

$$\mathcal{T} = \frac{\mathcal{U}\mathcal{V}}{\mathcal{Z}^2}, \qquad T = \frac{UV}{Z^2}, \tag{2.7}$$

where $\mathcal{U}(\tau) = \theta_q \log \mathcal{R}$, $U(\tau) = \theta_q \log R$,

$$\mathcal{V}(\tau) = \sum_{n=1}^{\infty} \left(\frac{n}{13}\right) \frac{q^n}{(1-q^n)^2}, \quad V(\tau) = \sum_{n=1}^{\infty} \left(\frac{n}{5}\right) \frac{q^n}{(1-q^n)^2}. \tag{2.8}$$

Both levels require singular values for $\mathcal{T}, T$ [11]. Explicit evaluations for $\mathcal{Z}$ and Z follow from

$$\mathcal{W} = \frac{\theta_q \log \mathcal{T}}{\mathcal{Z}} = \sqrt{1-12\mathcal{T}-16\mathcal{T}^2}, \qquad W = \frac{\theta_q \log T}{Z} = \sqrt{1-44T+16T^2}. \tag{2.9}$$

The pairs $(\mathcal{T}, \mathcal{W})$, (T, W), respectively, generate the field of invariant functions for $\Gamma_0(13)$, $\Gamma_0(5)$, and r and s generate invariant function fields for the congruence subgroup $\Gamma_0(17)$.

Proposition 2.2. *Let $A_0(\Gamma)$ denote the field of functions invariant under Γ and denote by χ the real quadratic character modulo p. Then*

1. $A_0(\Gamma_0(5)+) = \mathbb{C}(T)$ *and* $A_0(\Gamma_0(5)) = \mathbb{C}(T, W)$.
2. $A_0(\Gamma_0(13)+) = \mathbb{C}(\mathcal{T})$ *and* $A_0(\Gamma_0(13)) = \mathbb{C}(\mathcal{T}, \mathcal{W})$.
3. *For prime* $p \equiv 1 \pmod 4$, $A_0(\Gamma_0^2(p)) = \mathbb{C}(R_p, S_p)$, *where*

$$R_p = q^{\ell_p} \prod_{n=1}^{\infty} (1-q^n)^{\chi(n)}, \qquad S_p = q^{a_p} \prod_{n=1}^{\infty} \frac{(1-q^{np})^{b_p}}{(1-q^n)^{b_p}}, \tag{2.10}$$

$$\ell_p = \sum_{n=1}^{\frac{p-1}{2}} \frac{n(n-p)}{2p} \chi(n), \qquad \frac{p-1}{24} = \frac{a_p}{b_p}, \qquad \gcd(a_p, b_p) = 1. \tag{2.11}$$

A proof of the first two parts of Proposition 2.2 may be given along the lines of the proof of Proposition 3.2. The third part of the Proposition is a main result of [15]. The results of [15] explain Ramanujan's level 5 reciprocal relation (2.2) and his level 13 reciprocal relation [1, Equation (8.4)]

$$\frac{1}{\mathcal{R}} - 3 - \mathcal{R} = \frac{1}{\mathcal{S}}. \tag{2.12}$$

For our present work at level 17, we apply a new identity proven in [15]

$$r + \frac{1}{r} - 2\sqrt{\frac{4}{r} - 4r - 15} = \frac{1}{s}. \tag{2.13}$$

Our next task is to construct functions analogous to T and W in terms of r, s and Eisenstein series.

3. Functions invariant under $\Gamma_0(17)$ and a differential equation

In this Section we prove an analogue to Proposition 2.2 and derive a second order linear differential equation for z defined by (1.7) with coefficents in $\mathbb{C}(x)$, where x is defined by (1.8). In order to construct functions that are invariant under $\Gamma_0(17)$, we introduce sums of eight Eisenstein series considered in [14]. Set

$$\mathcal{E}_1(\tau) := \frac{1}{8}\sum_{\chi(-1)=-1} E_{\chi,k}(\tau), \qquad E_{\chi,k}(\tau) = 1 + \frac{2}{L(1-k,\chi)}\sum_{n=1}^{\infty}\chi(n)\frac{n^{k-1}q^n}{1-q^n}, \tag{3.1}$$

where the sum in (3.1) is over the odd primitive Dirichlet characters modulo 17 and $L(1-k,\chi)$ is the analytic continuation of the associated Dirichlet L-series and $\chi(-1) = (-1)^k$. For $a \in (\mathbb{Z}/17\mathbb{Z})^*$, apply the diamond operator [13] to define, for $1 \le k \le 8$,

$$\langle a\rangle\mathcal{E}_1(\tau) = \frac{1}{8}\sum_{\chi(-1)=-1}\chi(a)E_{\chi,1}(\tau), \qquad \mathcal{E}_k(\tau) = \pm\langle 3\rangle^{k-1}\mathcal{E}_1(\tau). \tag{3.2}$$

The sign in Equation (3.2) is chosen so that the first coefficient in the $q-$series expansion is 1. The parameters $\mathcal{E}_k(\tau)$ have the product representations [14, Theorems 3.1-3.5]

$$\begin{aligned}
\mathcal{E}_1(\tau) &= \left(\begin{matrix} q^8, q^9, q^{17}, q^{17} \\ q^2, q^3, q^{14}, q^{15}\end{matrix}\ ; q^{17}\right)_\infty, & \mathcal{E}_2(\tau) &= q\left(\begin{matrix} q^3, q^{14}, q^{17}, q^{17} \\ q, q^5, q^{12}, q^{16}\end{matrix}\ ; q^{17}\right)_\infty, \\
\mathcal{E}_3(\tau) &= q^3\left(\begin{matrix} q, q^{16}, q^{17}, q^{17} \\ q^4, q^6, q^{11}, q^{13}\end{matrix}\ ; q^{17}\right)_\infty, & \mathcal{E}_4(\tau) &= q\left(\begin{matrix} q^6, q^{11}, q^{17}, q^{17} \\ q^2, q^7, q^{10}, q^{15}\end{matrix}\ ; q^{17}\right)_\infty, \\
\mathcal{E}_5(\tau) &= q^3\left(\begin{matrix} q^2, q^{15}, q^{17}, q^{17} \\ q^5, q^8, q^9, q^{12}\end{matrix}\ ; q^{17}\right)_\infty, & \mathcal{E}_6(\tau) &= q\left(\begin{matrix} q^5, q^{12}, q^{17}, q^{17} \\ q^3, q^4, q^{13}, q^{14}\end{matrix}\ ; q^{17}\right)_\infty, \\
\mathcal{E}_7(\tau) &= q\left(\begin{matrix} q^4, q^{13}, q^{17}, q^{17} \\ q, q^7, q^{10}, q^{16}\end{matrix}\ ; q^{17}\right)_\infty, & \mathcal{E}_8(\tau) &= q^2\left(\begin{matrix} q^7, q^{10}, q^{17}, q^{17} \\ q^6, q^8, q^9, q^{11}\end{matrix}\ ; q^{17}\right)_\infty,
\end{aligned} \tag{3.3}$$

$$\left(\begin{matrix} a_1, \ldots, a_m \\ b_1, \ldots, b_n\end{matrix}\ ; z\right)_\infty = \prod_{n=1}^{\infty}\frac{(a_1;z)_\infty\cdots(a_m;z)_\infty}{(b_1;z)_\infty\cdots(b_n;z)_\infty}, \quad (a;z)_\infty = \prod_{n=0}^{\infty}(1-az^n).$$

A function Ω is now introduced as a level 17 analogue to the level 5 cusp form UV. Define

$$\Omega(\tau) = \mathcal{E}_1\mathcal{E}_2 - \mathcal{E}_2\mathcal{E}_3 + \mathcal{E}_3\mathcal{E}_4 - \mathcal{E}_4\mathcal{E}_5 + \mathcal{E}_5\mathcal{E}_6 - \mathcal{E}_6\mathcal{E}_7 - \mathcal{E}_7\mathcal{E}_8 - \mathcal{E}_8\mathcal{E}_1. \tag{3.4}$$

Proposition 3.1 demonstrates that the weight two parameters z and Ω, respectively, play a role at level seventeen analogous to that played by the parameters Z and UV at level 5.

Proposition 3.1. *Let $z = z(\tau)$ be defined by (1.7). Then*

(1) *The Eisenstein space of weight two $E_2(\Gamma_0(17))$ is generated by z.*

(2) *The space of cusp forms of weight two* $S_2(\Gamma_0(17))$ *is generated by* Ω.
(3) *Both* z *and* Ω *change sign under* $|_{W_{17},2}$, *where* $f|_{W_{17},k}(\tau) = 17^{-k/2}\tau^{-k}f(-1/17\tau)$.
(4) *Both* z *and* Ω *have zeros at the elliptic points* $\rho_\pm$, *and in the case of* Ω, *the zeros are simple.*

Proof. From (3.2) and the definition of the $\mathcal{E}_i$, if arithmetic is performed modulo 8 on the subscripts,

$$\langle 3\rangle\mathcal{E}_k = \epsilon_k\mathcal{E}_{k+1}, \qquad \epsilon_1,\ldots,\epsilon_8 = +1,-1,+1,-1,+1,-1,-1,-1. \tag{3.5}$$

This, coupled with the transformation formula for Eisenstein series,

$$\mathcal{E}_k\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)\cdot\langle a\rangle\mathcal{E}_k(\tau), \qquad \begin{pmatrix} a & b \\ c & d\end{pmatrix}\in\Gamma_0(17) \tag{3.6}$$

implies that Ω and z are modular forms of weight two with respect to $\Gamma_0(17)$. From their q-expansions, we deduce that Ω and z are linearly independent over $\mathbb{C}$. Therefore, from dimension formulas for the respective vector spaces [13], we see that these parameters generate the vector space of weight two forms for $\Gamma_0(17)$. Thus, we obtain the first two claims of Proposition 3.1. The third claim follows from the fact that W_{17} normalizes $\Gamma_0(17)$.

As fundamental domain for $\mathbb{H}/\Gamma_0(17)$ we take $\bigcup_{k=-8}^{8}F_k(D)\cup D$, where D is the usual fundamental domain for the full modular group and $F_k(\tau) = \frac{-1}{\tau+k}$. The two elliptic points of order 2 are $\rho_\pm = F_{\pm 4}(i)$. Since $\mathbb{H}/\Gamma_0(17)$ has two elliptic points of order 2 and two cusps, the valence formula for a weight k modular form f on $\Gamma_0(17)$ reads as

$$\mathrm{ord}_\infty f + \mathrm{ord}_0 f + \frac{\mathrm{ord}_{\rho_+} f}{2} + \frac{\mathrm{ord}_{\rho_-} f}{2} + \sum_{\tau\in\mathbb{H}-\{\rho_\pm\}}\mathrm{ord}_\tau f = \frac{k}{12}\cdot 18$$

From the q-expansion and the fact that Ω changes sign under $|_{W_{17},2}$, we know that the two cusps are zeros of Ω, so the valence formula for $f=\Omega$ reads as

$$1+1+\frac{\mathrm{ord}_{\rho_+} f}{2} + \frac{\mathrm{ord}_{\rho_-} f}{2} + \sum_{\tau\in\mathbb{H}-\{\rho_\pm\}}\mathrm{ord}_\tau f = 3.$$

Since Ω changes sign under $|_{W_{17},2}$ and the fixed point of W_{17} is not a zero (as one may check numerically), the zeros must come in pairs. Accordingly, the two other zeros must be the two elliptic points, and these are simple zeros. A similar argument gives the result for z. □

The cusp form and Eisenstein series from Proposition 3.1 can now be used in the construction of a $\Gamma_0(17)$ invariant function of the same form as $T, \mathcal{T}$ given by (2.7). Although the representation for $x(\tau)$ given here appears to differ from that given in the introduction, we ultimately demonstrate agreement of the two representations in Proposition 3.3. The parameters $x(\tau)$ and $w(\tau)$, defined in Proposition 3.2, play roles analogous to corresponding parameters T and W in Proposition 2.2.

Proposition 3.2. *If the Fricke involution is denoted* $W_{17} = W_{17,0}$ *and* x *and* w *are defined by*

$$x(\tau) = \frac{\Omega}{z}, \qquad w(\tau) = \frac{2}{z}\theta_q\log x, \tag{3.7}$$

(1) x *is invariant under* $\Gamma_0(17)$ *as well as* W_{17}*; and*
(2) x *has two simple zeros on* $\mathbb{H}/\Gamma_0(17)$ *at the two cusps.*
(3) *The field of functions invariant under* $\Gamma_0(17)$ *and* W_{17} *is* $A_0(\langle\Gamma_0(17)+\rangle) = \mathbb{C}(x)$.
(4) *The field of functions invariant under* $\Gamma_0(17)$ *is given by* $A_0(\Gamma_0(17)) = \mathbb{C}(x,w)$.
(5) *The relation* $w^2 = -127x^4 - 48x^3 - 66x^2 - 16x + 1$ *holds.*

Proof. The first two assertions follow directly from Proposition 3.1. The third assertion is then a direct consequence of the first two. For the fourth assertion, the functions $x(\tau)$ and $w(\tau)$ are invariant under $\Gamma_0(17)$, so it suffices to show that they generate the whole field. Since x has order 2, we have $[A_0(\Gamma_0(17)) : \mathbb{C}(x)] = 2$. Since $w \notin \mathbb{C}(x)$ because it changes sign under W_{17}, we must have $[A_0(\Gamma_0(17)) : \mathbb{C}(x, w)] = 1$, that is, the second assertion holds. For the final assertion, the function w^2 is fixed under W_{17} and has the same set of poles as x, hence it is a polynomial in x. We bound the degree of this polynomial by 4 and find its coefficients by comparing q−expansions. □

The parameter x is expressible as the rational function of r and s appearing in the Introduction and in terms of the McKay-Thompson series $17A$ [8, Table 4A].

Proposition 3.3. *Define $\eta(\tau) = q^{1/24}(q;q)_\infty$, and let x be defined as in Proposition 3.2. Then*

$$x = \frac{r(r^2 s + 8rs - r - s)}{8r^3 s - 3r^2 s + r - s}, \tag{3.8}$$

$$\frac{1-x}{2x} = \frac{1}{4\eta(\tau)^2\eta(17\tau)^2}\left(\sum_{m,n=-\infty}^{\infty}(e^{\pi i m} - e^{\pi i n})q^{\frac{1}{4}n^2 + \frac{17}{4}m^2}\right)^2. \tag{3.9}$$

Proof. From the product representation for r and those for the Eisenstein sums $\mathcal{E}_i$, from (3.3)

$$r = \frac{\mathcal{E}_1\mathcal{E}_3\mathcal{E}_5\mathcal{E}_7}{\mathcal{E}_2\mathcal{E}_4\mathcal{E}_6\mathcal{E}_8}. \tag{3.10}$$

Therefore, r is the quotient of weight four modular forms for $\Gamma_1(17)$, and $x = \Omega/z$ is the quotient of weight two modular forms for $\Gamma_1(17)$. Hence, the quadratic relation between x and r,

$$\frac{4}{r} - 4r - 15 = \frac{(xr-1)^2(4r-1)^2}{(x+r)^2} \tag{3.11}$$

may be transcribed as a relation between modular forms of weight 20 for $\Gamma_1(17)$ and proved from the Sturm bound by verifying the q-expansion to order $481 = 1 + 20 \cdot 288/12$. Then

$$\sqrt{\frac{4}{r} - 4r - 15} = \frac{(xr-1)(4r-1)}{(x+r)}, \tag{3.12}$$

where the branch of the square root is determined using the definition of x and r. Therefore,

$$x = \frac{4r - 1 + r\beta(r)}{4r^2 - r - \beta(r)}, \qquad \beta(r) = \sqrt{\frac{4}{r} - 4r - 15}. \tag{3.13}$$

The first equation of (3.13) is seen to be equivalent to (3.8) by applying (2.13). Equation (3.9) may be derived from respective q-expansions since each side is a Hauptmodul for $\Gamma_0(17)+$. □

It follows from the first part of Proposition 3.2 and Theorem 2.1 that z satisfies a third order linear homogeneous differential equation with coefficients in $\mathbb{C}(x)$. In order to formulate the differential equation, we state the following preliminary nonlinear differential equation in terms of the differential operator $\theta_q := q\frac{q}{dq}$. This is written even more succinctly as $f_q := \theta_q f$.

Lemma 3.4.

$$\frac{2zz_{qq} - 3z_q^2}{3z^4} = \frac{x\left(127x^5 - 222x^4 + 126x^3 + 4x^2 + 27x + 2\right)}{4(x-1)^2}$$

Proof. Let $f(\tau)$ denote the function on the left hand side of the proposed equality. If z satsifies the functional equation

$$z\left(\frac{a\tau+b}{c\tau+d}\right)=\epsilon\frac{(c\tau+d)^2}{ad-bc}z(\tau)$$

one can compute that

$$\frac{2zz_{qq}-3z_q^2}{3z^4}\left(\frac{a\tau+b}{c\tau+d}\right)=\frac{1}{\epsilon^2}\frac{2zz_{qq}-3z_q^2}{3z^4}(\tau).$$

By Proposition 3.1, we have $\epsilon=1$ for elements of $\Gamma_0(17)$ and $\epsilon=-1$ for W_{17}. Thus we see that $f(\tau)$ is invariant under $\Gamma_0(17)$ and W_{17} in weight 0. According to Theorem 4.4, we see that x does not have a pole at the two elliptic points, i.e. $x(\rho_\pm)=1$. This means that the two zeros of z at these elliptic points are both simple. Hence, z has two other simple zeros p_1 and $p_2=W_{17}(p_1)$, which are also the poles of x, modulo $\Gamma_0(17)$, as observed in the proof of Proposition 3.1. Since all of the poles are z are simple, we can take the expansion

$$z(\tau)=c(\tau-r)+\ldots$$

at the zeros $r=\rho_+,\rho_-,p_1,p_2$, where c is non-zero. Each of these zeros contributes a quadruple pole to $f(\tau)$ since

$$\frac{2zz_{qq}-3z_q^2}{3z^4}(\tau)=\frac{3}{(2\pi c)^2(\tau-r)^4}+\cdots$$

In the fundamental domain of $\mathbb{H}/\Gamma_0(17)$, the translate $F_4(D)$ is adjacent to itself. Thus $x(\tau)$ must identify the two halves of the corresponding side of $F_4(D)$ (the side that contains $F_4(i)$). Likewise for F_{-4}. Therefore, at the elliptic point $\rho_\pm$, the function $x(\tau)$ is locally a holomorphic function of $((\tau-\rho_\pm)/(\tau-\bar{\rho_\pm}))^2$ so that

$$x(\tau)=1+c_\pm(\tau-\rho_\pm)^2+\cdots$$

We see now that $(x-1)^2f$ has poles only at p_1 and p_2, each of order six. It is therefore a polynomial of degree six in x, and we can compute that

$$4(x-1)^2f-x(127x^5-222x^4+126x^3+4x^2+27x+2)=O(q^7).$$

The left hand side has poles of order 6 at p_1,p_2 and zeros at least order 7 at 0 and ∞. This contradicts the valence formula unless the left hand side is constant. □

We now give the third order linear differential equation for z with rational coefficients in x. The concise formulation of the differential equation in (3.14) is motivated by the general form of such differential equations from [22, 24].

Theorem 3.5. *With respect to the function x, the form $f=z$ satisfies the differential equation.*

$$\begin{aligned}0&=3x(254x^6-714x^5+681x^4-250x^3-6x^2-28x-1)f\\&\quad+x(x-1)(1397x^5-2482x^4+1094x^3-28x^2+197x+14)f_x\\&\quad+6x(x-1)^3(127x^3+36x^2+33x+4)f_{xx}\\&\quad+(x-1)^3(127x^4+48x^3+66x^2+16x-1)f_{xxx}.\end{aligned}$$

Proof. The differential equation satisfied by $f=z$ is given as

$$\det\begin{pmatrix} f & f_x & f_{xx} & f_{xxx}\\ (z) & (z)_x & (z)_{xx} & (z)_{xxx}\\ (z\log q) & (z\log q)_x & (z\log q)_{xx} & (z\log q)_{xxx}\\ (z\log^2 q) & (z\log^2 q)_x & (z\log^2 q)_{xx} & (z\log^2 q)_{xxx}\end{pmatrix}=0. \tag{3.14}$$

When expanding this determinant, we make the following substitutions:

(1) For the diffential with respect to x, use the definition (3.7) in the form

$$\theta_x = x\frac{\partial}{\partial x} = \frac{2}{wz}\theta_q.$$

(2) When the first derivative x_q appears, use the definition (3.7) in the form

$$x_q = \frac{1}{2}xwz.$$

(3) When the first derivative w_q appears, use the relation between w and x to obtain

$$w_q = -x(127x^3 + 36x^2 + 33x + 4)z.$$

(4) When the second derivative z_{qq} appears, use Lemma 3.4 in the form

$$z_{qq} = \frac{3z_q^2}{2z} + \frac{3x(127x^5 - 222x^4 + 126x^3 + 4x^2 + 27x + 2)}{8(x-1)^2}z^3$$

When these substitutions are made in (3.14), the claimed differential equation results after clearing denominators by multiplying by $(1-x)^3w^5/16$ and using Proposition 3.2 (5). □

The linear differential equation in Theorem 3.5 induces a series expansion for z in terms of x with coefficients A_n.

Corollary 3.6.

$$z = \sum_{n=0}^{\infty} A_n x^n \qquad |x| < 0.05122\ldots, \tag{3.15}$$

where $A_0 = 2$, $A_{-1,\ldots,-6} = 0$ *and*

$$\begin{aligned} 0 &= (n+1)^3A_{n+1} + (-19n^3 - 24n^2 - 14n - 3)A_n \\ &-3\left(5n^3 + 27n^2 - 8n + 4\right)A_{n-1} + \left(101n^3 - 300n^2 + 213n - 52\right)A_{n-2} \\ &-3\left(55n^3 - 267n^2 + 491n - 305\right)A_{n-3} + 3(n-3)\left(101n^2 - 297n + 253\right)A_{n-4} \\ &-9(n-4)(n-3)(37n-66)A_{n-5} + 127(n-5)(n-4)(n-3)A_{n-6}. \end{aligned}$$

The radius of convergence is the positive root of $127x^4 + 48x^3 + 66x^2 + 16x - 1$.

To make use of the series appearing in Corollary 3.6, we require explicit evaluations for the $x(\tau)$ within the domain of validity. In the next section, we prove that the number of singular values is finite and compile a complete list of quadratic evaluations and expansions.

4. Singular Values and Series for $1/\pi$

In this Section, singular values for $x(\tau)$ are derived and used to formulate Ramanujan-Sato expansions. The work culminates in a proof that there are precisely 11 singular values for $x(\tau)$ of degree at most two over $\mathbb{Q}$ within the radius of convergence of Corollary 3.6. The series given by (1.6) is the only such expansion with a rational singular value for $x(\tau)$. The main challenge in proving the expansions lies in rigorously determining exact evaluations for $x(\tau)$ for given τ and deriving constants appearing in the Ramanujan-Sato series. To do this, we formulate modular equations for $x(\tau)$ and provide an explicit relation between the modular equations and constants appearing in the series.

We demonstrate in the proof of Theorem 4.7 that the following table is a complete list of singular values for $x(\tau)$ in a fundamental domain for $\Gamma_0(17)$ with $[\mathbb{Q}(x(\tau)) : \mathbb{Q}] \leq 2$ within the radius of convergence of Corollary 3.6. Each value τ is listed by the coefficients (a, b, c) of its minimal polynomial, and the values are ordered by discriminant.

$$
\begin{array}{lll}
b^2-4ac & \tau(a,b,c) & x(\tau) \\
-1411 & (17,17,25) & (-1025-252\sqrt{17})^{-1} \\
-1003 & (17,17,19) & (-345-84\sqrt{17})^{-1} \\
-595 & (17,-17,13) & (-90-21\sqrt{17})^{-1} \\
-427 & (17,-27,17) & (30+33i\sqrt{7})^{-1} \\
-427 & (17,-41,31) & (30-33i\sqrt{7})^{-1} \\
-408 & (17,-34,23) & (55+24\sqrt{2})^{-1} \\
-408 & (34,-68,37) & (55-24\sqrt{2})^{-1} \\
-340 & (17,-34,22) & (29+4\sqrt{85})^{-1} \\
-323 & (17,-17,9) & (-22-7\sqrt{17})^{-1} \\
-187 & (17,-17,7) & -1/21 \\
-136 & (17,-34,19) & (12+3\sqrt{17})^{-1}
\end{array}
\tag{4.1}
$$

TABLE 1. Complete list of singular values of $x(\tau)$ of degree at most 2 within the radius of convergence of Corollary 3.6, ordered by discriminant.

By Proposition 3.3, finding singular values for $x(\tau)$ is equivalent to finding singular values for the normalized Thompson series $17A$. The fundamental results needed for such evaluations are presented in [7]. Our work below is a detailed rendition of the general presentation in [7] tailored to the modular function $x(\tau)$. We begin with the set of matrices

$$\Delta_n^*(17) = \{(\begin{smallmatrix}\alpha & \beta\\ \gamma & \delta\end{smallmatrix}) \in \mathbb{Z}^{2\times 2} |\, \gcd(\alpha,\beta,\gamma,\delta)=1 \text{ and } \alpha\delta-\beta\gamma=n \text{ and } \gamma\equiv 0 \bmod 17\}.$$

Lemma 4.1. *If* $\gcd(n,17)=1$*, then* $\Delta_n^*(17)$ *has the coset decomposition*

$$\Delta_n^*(17) = \bigcup_{\substack{\alpha\delta=n\\ 0\le\beta<\delta\\ \gcd(\alpha,\beta,\delta)=1}} \Gamma_0(17)\begin{pmatrix}\alpha & \beta\\ 0 & \delta\end{pmatrix},$$

and the double coset representation

$$\Delta_n^*(17) = \Gamma_0(17)\begin{pmatrix}1 & 0\\ 0 & n\end{pmatrix}\Gamma_0(17).$$

Proof. Any $(\begin{smallmatrix}\alpha & \beta\\ \gamma & \delta\end{smallmatrix}) \in \Delta_n^*$ can be converted to an upper triangular matrix by multiplying on the left by a matrix of the form

$$\begin{pmatrix} * & * \\ \frac{\gamma}{\gcd(\alpha,\gamma)} & \frac{-\alpha}{\gcd(\alpha,\gamma)}\end{pmatrix} \in \Gamma_0(17).$$

It is then easy to see that the claimed representatives are distinct modulo $\Gamma_0(17)$. This proves the decomposition formula. Next, by performing elementary row and column operations on the matrix $m \in \Delta_n^*(17)$, we find matrices $\gamma_1, \gamma_2 \in \Gamma(1)$ such that $m = \gamma_1(\begin{smallmatrix}1 & 0\\ 0 & n\end{smallmatrix})\gamma_2$. Since

$$\begin{pmatrix}1 & 0\\ 0 & n\end{pmatrix} = \begin{pmatrix}a & b\\ nc & d\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & n\end{pmatrix}\begin{pmatrix}a & -bn\\ -c & d\end{pmatrix},$$

we may find appropriate a, b, c, and d such that $\gamma_1' = \gamma_1(\begin{smallmatrix}a & b\\ nc & d\end{smallmatrix}) \in \Gamma_0(17)$. This results in an equality of the form $m = \gamma_1'(\begin{smallmatrix}1 & 0\\ 0 & n\end{smallmatrix})\gamma_2'$ where $\gamma_1' \in \Gamma_0(17)$, which forces $\gamma_2' \in \Gamma_0(17)$ as well. This establishes the double coset representation. □

We now establish modular equations central to our explicit evaluations for $x(\tau)$.

Proposition 4.2. *For any integer $n \geq 2$ with $\gcd(n, 17) = 1$, there is a polynomial $\Psi_n(X, Y)$ of degree $\psi(n) = n \prod_{\substack{q|n \\ q \text{ prime}}} (1 + \frac{1}{q})$ in X and Y such that:*

(1) *$\Psi_n(X, Y)$ is irreducible and has degree $\psi(n)$ in X and Y.*
(2) *$\Psi_n(X, Y)$ is symmetric in X and Y.*
(3) *The roots of $\Psi_n(x(\tau), Y) = 0$ are precisely the numbers $Y = x((\alpha\tau + \beta)/\delta)$ for integers α, β and δ such that $\alpha\delta = n$, $0 \leq \beta < \delta$, and $\gcd(\alpha, \beta, \delta) = 1$.*

Proof. The polynomial Ψ_n satisfies

$$(XY)^{-\psi(n)}\Psi_n(X, Y) = \prod_{\substack{\alpha\delta = n \\ 0 \leq \beta < \delta \\ (\alpha,\beta,\delta)=1}} \left(Y^{-1} - x\left(\frac{\alpha\tau + \beta}{\delta}\right)^{-1}\right), \tag{4.2}$$

where the coefficients of Y^{-k} on the right hand side should expressed as polynomials in $1/X$ for $X = x(\tau)$ as demonstrated in the proof of Corollary 4.3. This relies on the fact that $\Gamma_0(17)$ and W_{17} permute the set of functions $x\left((\alpha\tau + \beta)/\delta\right)$ where $\alpha\delta = n$, $0 \leq \beta < \delta$, and $\gcd(\alpha, \beta, \delta) = 1$. The double coset representation in Lemma 4.1 shows that every orbit contains $x(\tau/n)$, and hence the action of $\Gamma_0(17)$ on the roots must be transitive. Since

$$\begin{pmatrix} 0 & -1 \\ 17 & 0 \end{pmatrix}^{-1} \begin{pmatrix} \alpha & \beta \\ 0 & \delta \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 17 & 0 \end{pmatrix} = \begin{pmatrix} \delta & 0 \\ -17\beta & \alpha \end{pmatrix} \in \Delta_n^*(17),$$

it is clear that W_{17} permutes these functions as well by the decomposition in Lemma 4.1. The coefficient of $X^{\psi(n)}Y^{\psi(n)}$ in $\Psi_n(X, Y)$ is the constant term of the product on the right hand side of (4.2), which is clearly non-zero because the function $x(\tau)$ does not have poles at the cusps of $\mathbb{H}/\Gamma_0(17)$. Therefore, $\Psi_n(X, X)$ has the claimed degree $2\psi(n)$. The symmetry can be proven by noting that $\tau \to -1/(17n\tau)$ interchanges $x(\tau)$ and $x(n\tau)$. ☐

In Corollary 4.3, modular equations $\Psi_n(X, Y) = 0$ are derived for $X = x(\tau)$ and $y = x((\alpha\tau+\beta)/\delta)$ satisfying the conditions of Proposition 4.2. The proof indicates how modular equations for larger n may be derived and involves techniques analogous to those used to deduce classical modular equations of level n satisfied by the j invariant [16, 21].

Corollary 4.3. *We have*

$$\begin{aligned}
\Psi_2(X, Y) &= -9X^3Y^3 - 12X^3Y^2 + X^3Y + 2X^3 - 12X^2Y^3 \\
&\quad + 8X^2Y^2 + 10X^2Y + XY^3 + 10XY^2 - XY + 2Y^3, \\
\Psi_3(X, Y) &= 435X^4Y^4 + 231X^4Y^3 + 231X^3Y^4 + 45X^4Y^2 - 385X^3Y^3 + 45X^2Y^4 \\
&\quad - 39X^4Y - 63X^3Y^2 - 63X^2Y^3 - 39XY^4 + 4X^4 + 9X^3Y + 123X^2Y^2 + 9XY^3 \\
&\quad + 4Y^4 + 15X^2Y + 15XY^2 - XY.
\end{aligned}$$

Proof. The level $n = 2$ result is representative of $n = 3$ and other cases. For $n = 2$, we have

$$\{(\alpha, \beta, \delta) \mid \gcd(\alpha, \beta, \delta) = 1, 0 \leq \beta < \delta, \alpha\delta = n\} = \{(1, 0, 2), (2, 0, 1), (1, 1, 2)\}.$$

Let $x(\tau)$ be defined as in Proposition 3.2. Then

$$x_1 = x(\tau/2), \quad x_2 = x(2\tau), \quad x_3 = x\left(\frac{\tau + 1}{2}\right).$$

By (4.2),

$$\Psi_2(1/X, 1/Y) = Y^{-3} - \left(\frac{x_1x_2 + x_1x_3 + x_2x_3}{x_1x_2x_3}\right)Y^{-2} + \left(\frac{x_1 + x_2 + x_3}{x_1x_2x_3}\right)Y^{-1} - \frac{1}{x_1x_2x_3}.$$

By Theorem 3.2, we know that $1/x(\tau)$ is analytic on $X_0(17)$ except for simple poles at the cusps. Therefore, the only poles in $\mathbb{H}/\Gamma_0(17)$ of the coefficients of Y^{-k} are at points equivalent to the cusps 0 and ∞. We can explicitly compute the q-expansion for each of the coefficients and deduce that each has a pole of order at most 3 at $q = 0$. Since each coefficient is invariant under $\Gamma_0(17)$ and W_{17}, the coefficients may be expressed as polynomials of degree at most 3 in $1/x(\tau)$. Explicitly,

$$\begin{aligned}
-\frac{x_1x_2 + x_1x_3 + x_2x_3}{x_1x_2x_3} &= -\frac{2}{q^2} - 15 + O(q),\\
\frac{x_1 + x_2 + x_3}{x_1x_2x_3} &= \frac{20}{q^2} + \frac{108}{q} + 419 + O(q),\\
-\frac{1}{x_1x_2x_3} &= \frac{8}{q^3} + \frac{62}{q^2} + \frac{316}{q} + 1307 + O(q).
\end{aligned}$$

Therefore, with $x = x(\tau)$, we may determine polynomials in $1/x$ such that

$$\begin{aligned}
c_1(\tau) &= -\frac{1}{x_1x_2x_3} - \left(-\frac{9}{2} - 6x^{-1} + \frac{1}{2}x^{-2} + x^{-3}\right) = O(q),\\
c_2(\tau) &= \frac{x_1 + x_2 + x_3}{x_1x_2x_3} - \left(-6 + 4x^{-1} + 5x^{-2}\right) = O(q),\\
c_3(\tau) &= -\frac{x_1x_2 + x_1x_3 + x_2x_3}{x_1x_2x_3} - \left(\frac{1}{2} + 5x^{-1} - \frac{1}{2}x^{-2}\right) = O(q).
\end{aligned}$$

Since the functions $c_j(\tau)$, $j = 1, 2, 3$, are analytic on the upper half plane and at the cusps of $X_0(17)$, each c_j is constant, and $c_j(i\infty) = 0$. □

In Theorem 4.4, we prove each evaluation from Table 1. The evaluations are proven by showing that $x(\tau)$ satisfies a modular equation of degree n for some n.

Theorem 4.4. *Each evaluation for $x(\tau(a, b, c))$ from Table 1 holds, and $x(\tau(17, -8, 1)) = 1$.*

Proof. First note that

$$\tau(17, -34 - 19) = \frac{1}{17}\left(17 + i\sqrt{34}\right).$$

Therefore,

$$x(\tau) = x\left(i\sqrt{\frac{2}{17}}\right)$$

Observe that with

$$\tau = i\sqrt{\frac{2}{17}}, \qquad -\frac{1}{17\tau} = \frac{\tau}{2}.$$

Since $x(\tau)$ is invariant under the Fricke involution W_{17}, we have $x(\tau) = x(\tau/2)$. That is, $X = Y$ in the degree 2 modular equation above. Setting $Y = X$ and simplifying the equation, we get

$$X^2(X - 1)(X + 1)(9X^2 + 24X - 1) = 0. \tag{4.3}$$

Now that we have proven $x(\tau)$ satisfies (4.3), we may numerically deduce $x(\tau)$is a root of $9X^2 + 24X - 1$, and

$$x\left(i\sqrt{\frac{2}{17}}\right) = -\frac{4}{3} + \frac{1}{3}\sqrt{17}.$$

We may similarly prove $x(\tau(17, -8, 1)) = 1$ and the remaining evaluations in Table 1 from modular equations of degree n if we can determine, for each given value of $\tau = \tau(a, b, c)$, an upper upper triangular matrix $(\alpha, \beta; 0, \gamma)$ such that $x(\tau) = x((\alpha, \beta; 0, \gamma)\tau)$ and $\alpha\delta = n$, $0 \le \beta < \delta$, $\gcd(\alpha, \beta, \delta) = 1$, with $\gcd(n, 17) = 1$. For each τ, Table 2 provides a $\gamma \in \Delta_n^*(17)$ such that $\gamma\tau = \tau$ or $W_{17}\tau$ and a $\Gamma_0(17)$ equivalent upper triangular matrix $(\alpha, \beta; 0, \gamma)$. □

$\tau(a,b,c)$	Element of $\Delta_n^*(17)$	$(\alpha,\beta;0,\gamma)$
$(17,17,25)$	$(-1,-2;17,-25)$	$(1,2;0,59)$
$(17,17,19)$	$(1,0;17,19)$	$(1,0;0,19)$
$(17,-17,13)$	$(-1,0;17,-13)$	$(1,0;0,13)$
$(17,-27,17)$	$(13,-17;17,-14)^\dagger$	$(1,81;0,107)$
$(17,-41,31)$	$(20,-31;17,-21)^\dagger$	$(1,68;0,107)$
$(17,-34,23)$	$(-1,0;34,-23)$	$(1,0;0,23)$
$(34,-68,37)$	$(-2,1;51,-37)$	$(1,11;0,23)$
$(17,-34,22)$	$(-1,1;17,-22)$	$(1,4;0,5)$
$(17,-17,9)$	$(-2,1;17,-18)$	$(1,9;0,19)$
$(17,-17,7)$	$(-1,0;17,-7)$	$(1,0;0,7)$
$(17,-34,19)$	$(-1,1;17,-19)$	$(1,1;0,2)$
$(17,-8,1)$	$(3,-1;17,-5)$	$(1,1;0,2)$

(4.4)

TABLE 2. Elements of $\Delta_n^*(17)$ mapping τ to its image under W_{17} or fixing† τ and a corresponding $\Gamma_0(17)$ equivalent upper triangular matrix $(\alpha,\beta;0,\gamma)$.

For values $x(\tau)$ in the domain of validity for series from Corollary 3.6, we may construct Ramanujan-Sato expansions via Theorem 4.5, a specialization of [4, Theorem 2.1].

Theorem 4.5 (Series for $1/\pi$)**.** *Suppose there is a matrix $(a,b;c,d)\in\langle\Gamma_0(17),W_{17}\rangle$ such that*

$$\frac{a\tau+b}{c\tau+d}=\frac{\alpha\tau+\beta}{\delta}$$

for $\alpha\delta=n$ and $0\le\beta<\delta$. Set $X=x(\tau)$, which is determined from $\Psi_n(X,X)=0$, and further set

$$W=w(\tau),\quad \Psi_X=\frac{\partial\Psi_p}{\partial X}(X,X),\quad \Psi_Y=\frac{\partial\Psi_p}{\partial Y}(X,X),$$

and let $\epsilon\in\mathbb{Q}$ and $\eta=\pm1$ satsify for all τ

$$z\left(\frac{a\tau+b}{c\tau+d}\right)=\epsilon(c\tau+d)^2z(\tau),\quad w\left(\frac{a\tau+b}{c\tau+d}\right)=\eta w(\tau).$$

If A_k is the sequence defined in Corollary 3.6 and

$$B=-\frac{iW\left(\delta^2\Psi_X(ad-bc)+\alpha^2\eta\epsilon\Psi_Y(c\tau+d)^4\right)}{2\alpha^2c\eta\epsilon\Psi_Y(c\tau+d)^3},$$

$$C=\frac{i\delta^2(bc-ad)W}{2\alpha^2c\eta\epsilon\Psi_Y^3(c\tau+d)^3}\left(\Psi_X\Psi_Y\left(\Psi_X+\Psi_Y\right)(1+\theta_X\log W)\right.$$
$$\left.+\left(\Psi_X^2\Psi_{YY}-2\Psi_X\Psi_{XY}\Psi_Y+\Psi_{XX}\Psi_Y^2\right)X\right),$$

then

$$\frac{1}{\pi}=\sum_{k=0}^{\infty}A_k(Bk+C)X^k.$$

Proof. Differentiate the relation

$$z\left(\frac{a\tau+b}{c\tau+d}\right)=\epsilon(c\tau+d)^2z(\tau)$$

once and the relation

$$\Psi_n\left(x(\tau),x\left(\frac{\alpha\tau+\beta}{\delta}\right)\right)=0$$

twice and then set τ to the value in the hypothesis of the theorem. ☐

Corollary 4.6. *If A_k is the sequence defined in Corollary 3.6,*

$$\frac{\sqrt{11}}{\pi} = \sum_{k=0}^{\infty} A_k \frac{307+748k}{(-21)^{k+2}},$$

$$\frac{2\sqrt{154\sqrt{17}-634}}{\pi} = \sum_{k=0}^{\infty} A_k \frac{1779-195\sqrt{17}+3040k}{(-22-7\sqrt{17})^{k+2}},$$

$$\frac{214\sqrt{119}-882\sqrt{7}}{\pi} = \sum_{k=0}^{\infty} A_k \frac{9241-1047\sqrt{17}+21280k}{(-90-21\sqrt{17})^{k+2}},$$

$$\frac{\sqrt{1041894\sqrt{17}-4295839}}{\pi} = \sum_{k=0}^{\infty} A_k \frac{71065-15096\sqrt{17}+50740k}{(-345-84\sqrt{17})^{k+2}},$$

$$\frac{9\sqrt{2038550094\sqrt{17}-8405157343}}{\pi} = \sum_{k=0}^{\infty} A_k \frac{74004567-11655082\sqrt{17}+178775028k}{(-1025-252\sqrt{17})^{k+2}},$$

$$\frac{\sqrt{14(1267990301 \mp 85084065i\sqrt{7})}}{\pi} = \sum_{k=0}^{\infty} A_k \frac{3370317797 \pm 95119383i\sqrt{7}+12974719520k}{161874(30 \pm 33i\sqrt{7})^{k}},$$

and

$$\frac{\sqrt{9\sqrt{17}-37}}{\pi} = \sum_{k=0}^{\infty} A_k \frac{32-3\sqrt{17}+32k}{(12+3\sqrt{17})^{k+2}},$$

$$\frac{261\sqrt{5}-135\sqrt{17}}{\pi} = \sum_{k=0}^{\infty} A_k \frac{21500-788\sqrt{85}+54720k}{(29+4\sqrt{85})^{k+2}},$$

$$\frac{539\sqrt{6} \mp 735\sqrt{3}}{\pi} 2^{(1\mp 1)/2} = \sum_{k=0}^{\infty} A_k \frac{58962 \mp 7226\sqrt{2}+199920k}{(55 \pm 24\sqrt{2})^{k+2}}.$$

Proof. The first five series may be derived by setting $\tau = \frac{1}{2} + \frac{1}{2}\sqrt{\frac{n}{17}}i$ and using $\frac{17\tau-9}{34\tau-17} = \frac{\tau+(n-1)/2}{n}$ with $\epsilon = -1/17$ and $\eta = -1$ in Theorem 4.5 for the values $n = 11, 19, 35, 59, 83$. The subsequent pair may be derived from Theorem 4.5 by setting $\tau = (\pm 7 + \sqrt{427}i)/34$, using $(11\tau-2)/(17\tau-3) = (\tau+69)/107$ and $(13\tau+3)/(17\tau+4) = (\tau+82)/107$, respectively. The next three arise from setting $\tau = \sqrt{\frac{n}{17}}i$ and using $\frac{-1}{17\tau} = \frac{\tau}{n}$ for $n = 2, 5, 6$. The final series comes from setting $\tau = \sqrt{3/34}i$ and using $\frac{-1}{17\tau} = \frac{2\tau}{3}$. ☐

Theorem 4.7. *There are precisely eleven $\Gamma_0(17)$ inequivalent algebraic τ in the upper half plane such that $[\mathbb{Q}(x(\tau)) : \mathbb{Q}] \le 2$ with $x(\tau)$ in the radius of convergence of Corollary 3.6.*

Proof. We formulate a complete list of algebraic τ such that $[\mathbb{Q}(x(\tau)) : \mathbb{Q}] \le 2$ using well known facts about the j invariant [20]. First, for algebraic τ, the only algebraic values of $j(\tau)$ occur at $\Im\tau > 0$ satisfying $a\tau^2 + b\tau + c = 0$ for $a, b, c \in \mathbb{Z}$, with $d = b^2 - 4ac < 0$. Moreover, $[\mathbb{Q}(j(\tau)) : \mathbb{Q}] = h(d)$, where $h(d)$ is the class number. Since there is a polynomial relation $P(x, j)$ between x and j of degree 2 [7, Remark 1.5.3], we have $[\mathbb{Q}(j(\tau)) : \mathbb{Q}] \le 2[\mathbb{Q}(x(\tau)) : \mathbb{Q}]$, and so values τ with $[\mathbb{Q}(x(\tau)) : \mathbb{Q}] \le 2$ satisfy $[\mathbb{Q}(j(\tau)) : \mathbb{Q}] = h(d) \le 4$. Therefore, the bound $|d| \le 1555$ for $h(d) \le 4$ from [23] implies that the following algorithm results in a complete list of algebraic $(\tau, x(\tau))$ with $[\mathbb{Q}(x(\tau)) : \mathbb{Q}] \le 2$:

For each discriminant $-1555 \le d \le -1$,

(1) List all primitive reduced $\tau = \tau(a, b, c)$ of discriminant d in a fundamental domain for $PSL_2(\mathbb{Z})$. Translate these values via a set of coset representatives for $\Gamma_0(17)$ to a fundamental domain for $\Gamma_0(17)$.

(2) Factor the resultant of $P(X,Y)$ and the class polynomial

$$H_d(Y) = \prod_{\substack{(a,b,c) \text{ reduced, primitive} \\ d=b^2-4ac}} \left(Y - j\Big(\frac{-b+\sqrt{d}}{2a}\Big) \right).$$

The linear and quadratic factors of the resultant correspond to a complete list of $x = x(\tau)$, for τ of discriminant d, such that $[\mathbb{Q}(x(\tau)) : \mathbb{Q}] \le 2$. Associate candidate values τ from Step 1 to x by numerically approximating $x(\tau)$. For each tentative pair, (τ, x), prove the evaluation $x = x(\tau)$ as indicated in the proof of Theorem 4.4.

The algorithm is easy to implement. The resulting values of $\tau(a,b,c)$ with $x(\tau)$ within the radius of convergence of Corollary 3.6 are given in Table 1. □

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