Time-Dependent Photon Statistics in Variable Media

Supplementary material: Solutions of Heisenberg Equations of Motion (with DPAII Example)

(This notebook has been created by Sergei Suslov.)

Mathematica Tools

Definitions from the book “Glimpses of Soliton Theory” by Alex Kasman; http://kasmana.people.cofc.edu/GOST/

```
odoapply[ll_, f_] := Module[{i},
  Simplify[
   Sum[ Coefficient[ll, Dd, i] D[f, {x, i}] ,
    {i, 0, Exponent[Collect[ll, Dd], Dd]}]]]
```

```
odomult[L_, M_] := Module[{i, f, rem},
  rem = odoapply[L, odoapply[M, f'[x]]];
  Sum[Simplify[Coefficient[rem, D[f'[x], {x, i}]]] Dd^i,
   {i, 0, Exponent[Collect[L*M, Dd], Dd]}]]
```

```
odosimp[LL_] := Module[{n, i, L, outL}, (
   L = Collect[LL, Dd];
   n = Exponent[L, Dd];
   For[i = 0; outL = 0, i < n + 1, i = i + 1,
    outL = outL + Simplify[Coefficient[L, Dd, i]] Dd^i];
   outL)]
```

Position and momentum operators in coordinate representation

```
{X = x, P = -I*Dd}
```

{x, -ⅈ Dd}

```
XtimesP = odomult[X, P]
```

-ⅈ Dd x

```
PtimesX = odomult[P, X]
```

$-i - i\,\mathrm{Dd}\,x$

```
CommutatorXandP = odosimp[XtimesP - PtimesX]
```

i

An Example: DPAII

Time-dependent creation and annihilation operators (in capital Greek letters) for DPAII

$$\left\{A = \frac{\omega * X - I * P}{\sqrt{2\,\omega}} * \left(\frac{e^{-t(\lambda + i\,\omega)}\left(2\,i\,\alpha - \beta^2 + e^{2\,t\,\lambda}\,\omega\right)}{2\,\beta\,\sqrt{\omega}}\right) + \frac{\omega * X + I * P}{\sqrt{2\,\omega}} * \left(\frac{e^{-t(\lambda + i\,\omega)}\left(2\,i\,\alpha + \beta^2 + e^{2\,t\,\lambda}\,\omega\right)}{2\,\beta\,\sqrt{\omega}}\right) + \frac{i\,e^{-t(\lambda + i\,\omega)}\left(\beta\,\delta - 2\,\alpha\,\varepsilon + i\,e^{2\,t\,\lambda}\,\varepsilon\,\omega\right)}{\sqrt{2}\,\beta\,\sqrt{\omega}},\right.$$

$$\left.B = \frac{\omega * X + I * P}{\sqrt{2\,\omega}} * \left(\frac{e^{-t(\lambda - i\,\omega)}\left(-2\,i\,\alpha - \beta^2 + e^{2\,t\,\lambda}\,\omega\right)}{2\,\beta\,\sqrt{\omega}}\right) + \frac{\omega * X - I * P}{\sqrt{2\,\omega}} * \left(\frac{e^{-t(\lambda - i\,\omega)}\left(-2\,i\,\alpha + \beta^2 + e^{2\,t\,\lambda}\,\omega\right)}{2\,\beta\,\sqrt{\omega}}\right) - \frac{i\,e^{-t(\lambda - i\,\omega)}\left(\beta\,\delta - 2\,\alpha\,\varepsilon - i\,e^{2\,t\,\lambda}\,\varepsilon\,\omega\right)}{\sqrt{2}\,\beta\,\sqrt{\omega}}\right\}$$

$$\left\{\frac{e^{-t(\lambda + i\,\omega)}\left(2\,i\,\alpha - \beta^2 + e^{2\,t\,\lambda}\,\omega\right)\left(-\mathrm{Dd} + x\,\omega\right)}{2\,\sqrt{2}\,\beta\,\omega} + \frac{e^{-t(\lambda + i\,\omega)}\left(2\,i\,\alpha + \beta^2 + e^{2\,t\,\lambda}\,\omega\right)\left(\mathrm{Dd} + x\,\omega\right)}{2\,\sqrt{2}\,\beta\,\omega} + \frac{i\,e^{-t(\lambda + i\,\omega)}\left(\beta\,\delta - 2\,\alpha\,\varepsilon + i\,e^{2\,t\,\lambda}\,\varepsilon\,\omega\right)}{\sqrt{2}\,\beta\,\sqrt{\omega}},\right.$$

$$\left.\frac{e^{-t(\lambda - i\,\omega)}\left(-2\,i\,\alpha + \beta^2 + e^{2\,t\,\lambda}\,\omega\right)\left(-\mathrm{Dd} + x\,\omega\right)}{2\,\sqrt{2}\,\beta\,\omega} + \frac{e^{-t(\lambda - i\,\omega)}\left(-2\,i\,\alpha - \beta^2 + e^{2\,t\,\lambda}\,\omega\right)\left(\mathrm{Dd} + x\,\omega\right)}{2\,\sqrt{2}\,\beta\,\omega} - \frac{i\,e^{-t(\lambda - i\,\omega)}\left(\beta\,\delta - 2\,\alpha\,\varepsilon - i\,e^{2\,t\,\lambda}\,\varepsilon\,\omega\right)}{\sqrt{2}\,\beta\,\sqrt{\omega}}\right\}$$

```
% /. t → 0
```

$$\left\{\frac{\left(2\, i\, \alpha - \beta^2 + \omega\right)\,(-Dd + x\,\omega)}{2\sqrt{2}\,\beta\,\omega} + \frac{\left(2\, i\, \alpha + \beta^2 + \omega\right)\,(Dd + x\,\omega)}{2\sqrt{2}\,\beta\,\omega} + \frac{i\,(\beta\,\delta - 2\,\alpha\,\varepsilon + i\,\varepsilon\,\omega)}{\sqrt{2}\,\beta\,\sqrt{\omega}},\ \frac{\left(-2\, i\, \alpha + \beta^2 + \omega\right)\,(-Dd + x\,\omega)}{2\sqrt{2}\,\beta\,\omega} + \frac{\left(-2\, i\, \alpha - \beta^2 + \omega\right)\,(Dd + x\,\omega)}{2\sqrt{2}\,\beta\,\omega} - \frac{i\,(\beta\,\delta - 2\,\alpha\,\varepsilon - i\,\varepsilon\,\omega)}{\sqrt{2}\,\beta\,\sqrt{\omega}}\right\}$$

```
AtimesB = odomult[A, B]
```

$$-\frac{Dd^2\, e^{-2\,t\,\lambda}\,\beta^2}{2\,\omega^2} - \frac{i\,Dd\, e^{-2\,t\,\lambda}\left(\beta\,\delta - 2\,\alpha\,\varepsilon + 2\,x\,\alpha\sqrt{\omega}\right)}{\omega^{3/2}} + \frac{1}{2\,\beta^2\,\omega}\, e^{-2\,t\,\lambda}\left(4\,\alpha^2\left(\varepsilon - x\sqrt{\omega}\right)^2 + 2\,\alpha\,\beta\left(-i\,\beta - 2\,\delta\,\varepsilon + 2\,x\,\delta\sqrt{\omega}\right) + e^{4\,t\,\lambda}\left(\varepsilon - x\sqrt{\omega}\right)^2\omega^2 + \beta^2\left(\delta^2 + e^{2\,t\,\lambda}\,\omega\right)\right)$$

```
BtimesA = odomult[B, A]
```

$$-\frac{Dd^2\, e^{-2\,t\,\lambda}\,\beta^2}{2\,\omega^2} - \frac{i\,Dd\, e^{-2\,t\,\lambda}\left(\beta\,\delta - 2\,\alpha\,\varepsilon + 2\,x\,\alpha\sqrt{\omega}\right)}{\omega^{3/2}} + \frac{1}{2\,\beta^2\,\omega}\, e^{-2\,t\,\lambda}\left(4\,\alpha^2\left(\varepsilon - x\sqrt{\omega}\right)^2 + 2\,\alpha\,\beta\left(-i\,\beta - 2\,\delta\,\varepsilon + 2\,x\,\delta\sqrt{\omega}\right) + e^{4\,t\,\lambda}\left(\varepsilon - x\sqrt{\omega}\right)^2\omega^2 + \beta^2\left(\delta^2 - e^{2\,t\,\lambda}\,\omega\right)\right)$$

```
CommutatorAandB = odosimp[AtimesB - BtimesA]
```

1

DPAII quadratic Hamiltonian and required commutators (in capital Greeks). Heisenberg's equations of motion

```
HAB = 1/2 ω * (odomult[A, B] + odomult[B, A]) -
    λ/2 * I * (Exp[2 * I ω t] * odomult[A, A] - Exp[-2 * I ω t] * odomult[B, B]);
{AtimesH = odomult[A, HAB], HtimesA = odomult[HAB, A]};
```

```
CommutatorAandH = odosimp[AtimesH - HtimesA]
```

$$\frac{\text{Dd}\, e^{-t(\lambda+i\omega)}\,\beta\,(-i\lambda+\omega)}{\sqrt{2}\,\omega}+\frac{1}{\sqrt{2}\,\beta\sqrt{\omega}}e^{-t(\lambda+i\omega)}\left(\beta\,\delta\,(\lambda+i\omega)-\left(\varepsilon-\mathbf{x}\sqrt{\omega}\right)\left(2\,\alpha\,(\lambda+i\omega)+e^{2t\lambda}\omega\,(i\lambda+\omega)\right)\right)$$

```
{BtimesH = odomult[B, HAB], HtimesB = odomult[HAB, B]};
CommutatorBandH = odosimp[BtimesH - HtimesB]
```

$$\frac{\text{Dd}\, e^{-t(\lambda-i\omega)}\,\beta\,(i\lambda+\omega)}{\sqrt{2}\,\omega}+\frac{1}{\sqrt{2}\,\beta\sqrt{\omega}}e^{-t(\lambda-i\omega)}\left(-\beta\,\delta\,(\lambda-i\omega)+\left(\varepsilon-\mathbf{x}\sqrt{\omega}\right)\left(2\,\alpha\,(\lambda-i\omega)+e^{2t\lambda}\omega\,(-i\lambda+\omega)\right)\right)$$

```
D[A, t] + I * CommutatorAandH
```

$$-\frac{\sqrt{2}\,e^{2t\lambda-t(\lambda+i\omega)}\,\varepsilon\,\lambda\sqrt{\omega}}{\beta}+\frac{e^{2t\lambda-t(\lambda+i\omega)}\,\lambda\,(-\text{Dd}+\mathbf{x}\,\omega)}{\sqrt{2}\,\beta}+$$
$$\frac{1}{2\sqrt{2}\,\beta\,\omega}e^{-t(\lambda+i\omega)}\,(-\lambda-i\omega)\left(2\,i\,\alpha-\beta^2+e^{2t\lambda}\omega\right)(-\text{Dd}+\mathbf{x}\,\omega)+$$
$$\frac{e^{2t\lambda-t(\lambda+i\omega)}\,\lambda\,(\text{Dd}+\mathbf{x}\,\omega)}{\sqrt{2}\,\beta}+\frac{1}{2\sqrt{2}\,\beta\,\omega}$$
$$e^{-t(\lambda+i\omega)}\,(-\lambda-i\omega)\left(2\,i\,\alpha+\beta^2+e^{2t\lambda}\omega\right)(\text{Dd}+\mathbf{x}\,\omega)+$$
$$\frac{i\,e^{-t(\lambda+i\omega)}\,(-\lambda-i\omega)\left(\beta\,\delta-2\,\alpha\,\varepsilon+i\,e^{2t\lambda}\varepsilon\,\omega\right)}{\sqrt{2}\,\beta\sqrt{\omega}}+$$
$$i\left(\frac{\text{Dd}\,e^{-t(\lambda+i\omega)}\,\beta\,(-i\lambda+\omega)}{\sqrt{2}\,\omega}+\frac{1}{\sqrt{2}\,\beta\sqrt{\omega}}\right.$$
$$\left.e^{-t(\lambda+i\omega)}\left(\beta\,\delta\,(\lambda+i\omega)-\left(\varepsilon-\mathbf{x}\sqrt{\omega}\right)\left(2\,\alpha\,(\lambda+i\omega)+e^{2t\lambda}\omega\,(i\lambda+\omega)\right)\right)\right)$$

```
FullSimplify[%]
```

0

```
D[B, t] + I * CommutatorBandH
```

$$-\frac{\sqrt{2}\, e^{2 t \lambda - t (\lambda - i \omega)}\, \varepsilon \lambda \sqrt{\omega}}{\beta} + \frac{e^{2 t \lambda - t (\lambda - i \omega)}\, \lambda\, (-Dd + x\, \omega)}{\sqrt{2}\, \beta} +$$

$$\frac{1}{2 \sqrt{2}\, \beta\, \omega} e^{-t (\lambda - i \omega)} (-\lambda + i\, \omega) \left(-2\, i\, \alpha + \beta^2 + e^{2 t \lambda}\, \omega\right) (-Dd + x\, \omega) +$$

$$\frac{e^{2 t \lambda - t (\lambda - i \omega)}\, \lambda\, (Dd + x\, \omega)}{\sqrt{2}\, \beta} + \frac{1}{2 \sqrt{2}\, \beta\, \omega}$$

$$e^{-t (\lambda - i \omega)} (-\lambda + i\, \omega) \left(-2\, i\, \alpha - \beta^2 + e^{2 t \lambda}\, \omega\right) (Dd + x\, \omega) -$$

$$\frac{i\, e^{-t (\lambda - i \omega)} (-\lambda + i\, \omega) \left(\beta\, \delta - 2\, \alpha\, \varepsilon - i\, e^{2 t \lambda}\, \varepsilon\, \omega\right)}{\sqrt{2}\, \beta \sqrt{\omega}} +$$

$$i \left(\frac{Dd\, e^{-t (\lambda - i \omega)}\, \beta\, (i\, \lambda + \omega)}{\sqrt{2}\, \omega} + \frac{1}{\sqrt{2}\, \beta \sqrt{\omega}}\right.$$

$$\left. e^{-t (\lambda - i \omega)} \left(-\beta\, \delta\, (\lambda - i\, \omega) + \left(\varepsilon - x \sqrt{\omega}\right) \left(2\, \alpha\, (\lambda - i\, \omega) + e^{2 t \lambda}\, \omega\, (-i\, \lambda + \omega)\right)\right)\right)$$

```
FullSimplify[%]
```

0

An Extension: The General Quadratic Hamiltonian

Time-dependent coordinate and momentum operators (in capital Greeks) in terms of solutions of Ermakov-type system

$$\Big\{\varphi = (2\,\hat{}\,(-1/2)/\beta[t]) * \left(\mathrm{Exp}[2*I*\gamma[t]] * \frac{\omega * X + I * P}{\sqrt{2\,\omega}} + \mathrm{Exp}[-2*I*\gamma[t]] * \frac{\omega * X - I * P}{\sqrt{2\,\omega}}\right) - \varepsilon[t]/\beta[t],\ \mathrm{P} = (\alpha[t]\ 2\,\hat{}\,(1/2)/\beta[t]) * \left(\mathrm{Exp}[2*I*\gamma[t]] * \frac{\omega * X + I * P}{\sqrt{2\,\omega}} + \mathrm{Exp}[-2*I*\gamma[t]] * \frac{\omega * X - I * P}{\sqrt{2\,\omega}}\right) + (\beta[t]/(I*\ 2\,\hat{}\,(1/2))) * \left(\mathrm{Exp}[2*I*\gamma[t]] * \frac{\omega * X + I * P}{\sqrt{2\,\omega}} - \mathrm{Exp}[-2*I*\gamma[t]] * \frac{\omega * X - I * P}{\sqrt{2\,\omega}}\right) + \delta[t] - 2*(\alpha[t]\ \varepsilon[t]/\beta[t])\Big\}$$

$$\left\{\frac{\frac{e^{-2\,i\,\gamma[t]}\,(-\mathrm{Dd}+x\,\omega)}{\sqrt{2}\,\sqrt{\omega}} + \frac{e^{2\,i\,\gamma[t]}\,(\mathrm{Dd}+x\,\omega)}{\sqrt{2}\,\sqrt{\omega}}}{\sqrt{2}\,\beta[t]} - \frac{\varepsilon[t]}{\beta[t]},\right.$$

$$\frac{\sqrt{2}\left(\frac{e^{-2\,i\,\gamma[t]}\,(-\mathrm{Dd}+x\,\omega)}{\sqrt{2}\,\sqrt{\omega}} + \frac{e^{2\,i\,\gamma[t]}\,(\mathrm{Dd}+x\,\omega)}{\sqrt{2}\,\sqrt{\omega}}\right)\alpha[t]}{\beta[t]} -$$

$$\left.\frac{i\left(-\frac{e^{-2\,i\,\gamma[t]}\,(-\mathrm{Dd}+x\,\omega)}{\sqrt{2}\,\sqrt{\omega}} + \frac{e^{2\,i\,\gamma[t]}\,(\mathrm{Dd}+x\,\omega)}{\sqrt{2}\,\sqrt{\omega}}\right)\beta[t]}{\sqrt{2}} + \delta[t] - \frac{2\,\alpha[t]\,\varepsilon[t]}{\beta[t]}\right\}$$

Ptimes♀ = odomult[P, ♀]

$$\frac{1}{4\,\omega\,\beta[t]^2}\,\mathrm{Dd}^2\,e^{-4 i \gamma[t]}\left(2\left(-1+e^{4 i \gamma[t]}\right)^2\alpha[t]-i\left(-1+e^{8 i \gamma[t]}\right)\beta[t]^2\right)+\frac{1}{2\sqrt{\omega}\,\beta[t]^2}\,\mathrm{Dd}\,e^{-4 i \gamma[t]}\left(2\left(-1+e^{4 i \gamma[t]}\right)\alpha[t]\left(\left(1+e^{4 i \gamma[t]}\right)x\sqrt{\omega}-2\,e^{2 i \gamma[t]}\,\varepsilon[t]\right)+\beta[t]\left(e^{2 i \gamma[t]}\left(-1+e^{4 i \gamma[t]}\right)\delta[t]-i\,\beta[t]\left(\left(1+e^{8 i \gamma[t]}\right)x\sqrt{\omega}-e^{2 i \gamma[t]}\left(1+e^{4 i \gamma[t]}\right)\varepsilon[t]\right)\right)\right)+\frac{1}{4\,\beta[t]^2}\,e^{-4 i \gamma[t]}\left(2\,\alpha[t]\left(\left(1+e^{4 i \gamma[t]}\right)\left(-1+x^2\omega+e^{4 i \gamma[t]}\left(1+x^2\omega\right)\right)-4\,e^{2 i \gamma[t]}\left(1+e^{4 i \gamma[t]}\right)x\sqrt{\omega}\,\varepsilon[t]+4\,e^{4 i \gamma[t]}\,\varepsilon[t]^2\right)+\beta[t]\left(2\,e^{2 i \gamma[t]}\,\delta[t]\left(\left(1+e^{4 i \gamma[t]}\right)x\sqrt{\omega}-2\,e^{2 i \gamma[t]}\,\varepsilon[t]\right)-i\,\beta[t]\left(1+2\,e^{4 i \gamma[t]}-x^2\omega+e^{8 i \gamma[t]}\left(1+x^2\omega\right)-2\,e^{2 i \gamma[t]}\left(-1+e^{4 i \gamma[t]}\right)x\sqrt{\omega}\,\varepsilon[t]\right)\right)\right)$$

♀timesP = odomult[♀, P]

$$\frac{1}{4\,\omega\,\beta[t]^2}\,\mathrm{Dd}^2\,e^{-4 i \gamma[t]}\left(2\left(-1+e^{4 i \gamma[t]}\right)^2\alpha[t]-i\left(-1+e^{8 i \gamma[t]}\right)\beta[t]^2\right)+\frac{1}{2\sqrt{\omega}\,\beta[t]^2}\,\mathrm{Dd}\,e^{-4 i \gamma[t]}\left(2\left(-1+e^{4 i \gamma[t]}\right)\alpha[t]\left(\left(1+e^{4 i \gamma[t]}\right)x\sqrt{\omega}-2\,e^{2 i \gamma[t]}\,\varepsilon[t]\right)+\beta[t]\left(e^{2 i \gamma[t]}\left(-1+e^{4 i \gamma[t]}\right)\delta[t]-i\,\beta[t]\left(\left(1+e^{8 i \gamma[t]}\right)x\sqrt{\omega}-e^{2 i \gamma[t]}\left(1+e^{4 i \gamma[t]}\right)\varepsilon[t]\right)\right)\right)+\frac{1}{4\,\beta[t]^2}\,e^{-4 i \gamma[t]}\left(2\,\alpha[t]\left(\left(1+e^{4 i \gamma[t]}\right)\left(-1+x^2\omega+e^{4 i \gamma[t]}\left(1+x^2\omega\right)\right)-4\,e^{2 i \gamma[t]}\left(1+e^{4 i \gamma[t]}\right)x\sqrt{\omega}\,\varepsilon[t]+4\,e^{4 i \gamma[t]}\,\varepsilon[t]^2\right)+\beta[t]\left(2\,e^{2 i \gamma[t]}\,\delta[t]\left(\left(1+e^{4 i \gamma[t]}\right)x\sqrt{\omega}-2\,e^{2 i \gamma[t]}\,\varepsilon[t]\right)-i\left(-1+e^{4 i \gamma[t]}\right)\beta[t]\left(-1+x^2\omega+e^{4 i \gamma[t]}\left(1+x^2\omega\right)-2\,e^{2 i \gamma[t]}\,x\sqrt{\omega}\,\varepsilon[t]\right)\right)\right)$$

Commutator♀andP = odosimp[♀timesP - Ptimes♀]

i

General quadratic Hamiltonian (in Heisenberg's picture) and required commutators

```
H = a[t] * odomult[P, P] + b[t] * odomult[φ, φ] +
  c[t] * odomult[φ, P] - I * d[t] - f[t] * φ - g[t] * P
```

$$
\begin{aligned}
&-i\, d[t] - f[t]\left(\frac{\frac{e^{-2 i \gamma[t]}\,(-Dd+x\,\omega)}{\sqrt{2}\,\sqrt{\omega}}+\frac{e^{2 i \gamma[t]}\,(Dd+x\,\omega)}{\sqrt{2}\,\sqrt{\omega}}}{\sqrt{2}\,\beta[t]}-\frac{\varepsilon[t]}{\beta[t]}\right)- \\
&g[t]\left(\frac{\sqrt{2}\left(\frac{e^{-2 i \gamma[t]}\,(-Dd+x\,\omega)}{\sqrt{2}\,\sqrt{\omega}}+\frac{e^{2 i \gamma[t]}\,(Dd+x\,\omega)}{\sqrt{2}\,\sqrt{\omega}}\right)\alpha[t]}{\beta[t]}-\right. \\
&\left.\quad\frac{i\left(-\frac{e^{-2 i \gamma[t]}\,(-Dd+x\,\omega)}{\sqrt{2}\,\sqrt{\omega}}+\frac{e^{2 i \gamma[t]}\,(Dd+x\,\omega)}{\sqrt{2}\,\sqrt{\omega}}\right)\beta[t]}{\sqrt{2}}+\delta[t]-\frac{2\,\alpha[t]\,\varepsilon[t]}{\beta[t]}\right)+ \\
&b[t]\left(\frac{Dd^2\, e^{-4 i \gamma[t]}\left(-1+e^{4 i \gamma[t]}\right)^2}{4\,\omega\,\beta[t]^2}+\left(Dd\, e^{-4 i \gamma[t]}\left(-1+e^{4 i \gamma[t]}\right)\right.\right. \\
&\quad\left.\left(\left(1+e^{4 i \gamma[t]}\right)x\sqrt{\omega}-2\, e^{2 i \gamma[t]}\,\varepsilon[t]\right)\right)\Big/\left(2\sqrt{\omega}\,\beta[t]^2\right)+ \\
&\quad\frac{1}{4\,\beta[t]^2}e^{-4 i \gamma[t]}\left(\left(1+e^{4 i \gamma[t]}\right)\left(-1+x^2\,\omega+e^{4 i \gamma[t]}\left(1+x^2\,\omega\right)\right)-\right. \\
&\quad\left.\left.4\, e^{2 i \gamma[t]}\left(1+e^{4 i \gamma[t]}\right)x\sqrt{\omega}\,\varepsilon[t]+4\, e^{4 i \gamma[t]}\,\varepsilon[t]^2\right)\right)+c[t] \\
&\left(\frac{1}{4\,\omega\,\beta[t]^2}Dd^2\, e^{-4 i \gamma[t]}\left(2\left(-1+e^{4 i \gamma[t]}\right)^2\alpha[t]-i\left(-1+e^{8 i \gamma[t]}\right)\beta[t]^2\right)+\right. \\
&\quad\frac{1}{2\sqrt{\omega}\,\beta[t]^2}\,Dd\, e^{-4 i \gamma[t]}\left(2\left(-1+e^{4 i \gamma[t]}\right)\alpha[t]\left(\left(1+e^{4 i \gamma[t]}\right)x\sqrt{\omega}-\right.\right. \\
&\quad\quad\left.2\, e^{2 i \gamma[t]}\,\varepsilon[t]\right)+\beta[t]\left(e^{2 i \gamma[t]}\left(-1+e^{4 i \gamma[t]}\right)\delta[t]-\right. \\
&\quad\quad\left.\left.i\,\beta[t]\left(\left(1+e^{8 i \gamma[t]}\right)x\sqrt{\omega}-e^{2 i \gamma[t]}\left(1+e^{4 i \gamma[t]}\right)\varepsilon[t]\right)\right)\right)+ \\
&\quad\frac{1}{4\,\beta[t]^2}\,e^{-4 i \gamma[t]}\left(2\,\alpha[t]\left(\left(1+e^{4 i \gamma[t]}\right)\left(-1+x^2\,\omega+e^{4 i \gamma[t]}\left(1+x^2\,\omega\right)\right)-\right.\right. \\
&\quad\quad\left.4\, e^{2 i \gamma[t]}\left(1+e^{4 i \gamma[t]}\right)x\sqrt{\omega}\,\varepsilon[t]+4\, e^{4 i \gamma[t]}\,\varepsilon[t]^2\right)+ \\
&\quad\quad\beta[t]\left(2\, e^{2 i \gamma[t]}\,\delta[t]\left(\left(1+e^{4 i \gamma[t]}\right)x\sqrt{\omega}-2\, e^{2 i \gamma[t]}\,\varepsilon[t]\right)-\right. \\
&\quad\quad\quad i\left(-1+e^{4 i \gamma[t]}\right)\beta[t] \\
&\quad\quad\quad\left.\left.\left.\left(-1+x^2\,\omega+e^{4 i \gamma[t]}\left(1+x^2\,\omega\right)-2\, e^{2 i \gamma[t]}\,x\sqrt{\omega}\,\varepsilon[t]\right)\right)\right)\right)+a[t]
\end{aligned}
$$

$$
\Bigg(-\frac{1}{4\,\omega\,\beta[t]^2}\text{Dd}^2\,e^{-4\,i\,\gamma[t]}\left(2\,i\left(-1+e^{4\,i\,\gamma[t]}\right)\alpha[t]+\left(1+e^{4\,i\,\gamma[t]}\right)\beta[t]^2\right)^2+
\frac{1}{2\,\sqrt{\omega}\,\beta[t]^2}\,\text{Dd}\,e^{-4\,i\,\gamma[t]}\left(2\left(-1+e^{4\,i\,\gamma[t]}\right)\alpha[t]-i\left(1+e^{4\,i\,\gamma[t]}\right)\beta[t]^2\right)
\left(\beta[t]\left(-i\left(-1+e^{4\,i\,\gamma[t]}\right)x\sqrt{\omega}\,\beta[t]+2\,e^{2\,i\,\gamma[t]}\,\delta[t]\right)+
2\,\alpha[t]\left(\left(1+e^{4\,i\,\gamma[t]}\right)x\sqrt{\omega}-2\,e^{2\,i\,\gamma[t]}\,\varepsilon[t]\right)\right)+\frac{1}{4\,\beta[t]^2}
e^{-4\,i\,\gamma[t]}\Bigg(-\beta[t]^2\left(\left(-1+e^{4\,i\,\gamma[t]}\right)\left(1-x^2\,\omega+e^{4\,i\,\gamma[t]}\left(1+x^2\,\omega\right)\right)\beta[t]^2+
4\,i\,e^{2\,i\,\gamma[t]}\left(-1+e^{4\,i\,\gamma[t]}\right)x\sqrt{\omega}\,\beta[t]\,\delta[t]-4\,e^{4\,i\,\gamma[t]}\,\delta[t]^2\right)+
4\,\alpha[t]^2\left(\left(1+e^{4\,i\,\gamma[t]}\right)\left(-1+x^2\,\omega+e^{4\,i\,\gamma[t]}\left(1+x^2\,\omega\right)\right)-
4\,e^{2\,i\,\gamma[t]}\left(1+e^{4\,i\,\gamma[t]}\right)x\sqrt{\omega}\,\varepsilon[t]+4\,e^{4\,i\,\gamma[t]}\,\varepsilon[t]^2\right)+
4\,\alpha[t]\,\beta[t]\Bigg(2\,e^{2\,i\,\gamma[t]}\,\delta[t]\left(\left(1+e^{4\,i\,\gamma[t]}\right)x\sqrt{\omega}-2\,e^{2\,i\,\gamma[t]}\,\varepsilon[t]\right)-
i\,\beta[t]\Big(1-x^2\,\omega+e^{8\,i\,\gamma[t]}\left(1+x^2\,\omega\right)-
2\,e^{2\,i\,\gamma[t]}\left(-1+e^{4\,i\,\gamma[t]}\right)x\sqrt{\omega}\,\varepsilon[t]\Big)\Bigg)\Bigg)\Bigg)
$$

```
Timing[{♀timesH = odomult[♀, H], Htimes♀ = odomult[H, ♀]};]
```

{162.865, Null}

```
Commutator♀andH = odosimp[♀timesH - Htimes♀]
```

$$
\frac{1}{2\,\sqrt{\omega}\,\beta[t]}\text{Dd}\,e^{-2\,i\,\gamma[t]}\Big(i\left(-1+e^{4\,i\,\gamma[t]}\right)c[t]+
2\,a[t]\left(2\,i\left(-1+e^{4\,i\,\gamma[t]}\right)\alpha[t]+\left(1+e^{4\,i\,\gamma[t]}\right)\beta[t]^2\right)\Big)+\frac{1}{2\,\beta[t]}\,e^{-2\,i\,\gamma[t]}
\Big(-2\,i\,e^{2\,i\,\gamma[t]}\,g[t]\,\beta[t]+i\,c[t]\left(\left(1+e^{4\,i\,\gamma[t]}\right)x\sqrt{\omega}-2\,e^{2\,i\,\gamma[t]}\,\varepsilon[t]\right)+
2\,a[t]\Big(\beta[t]\left(\left(-1+e^{4\,i\,\gamma[t]}\right)x\sqrt{\omega}\,\beta[t]+2\,i\,e^{2\,i\,\gamma[t]}\,\delta[t]\right)+
2\,i\,\alpha[t]\left(\left(1+e^{4\,i\,\gamma[t]}\right)x\sqrt{\omega}-2\,e^{2\,i\,\gamma[t]}\,\varepsilon[t]\right)\Big)\Big)
$$

```
Timing[{PtimesH = odomult[P, H], HtimesP = odomult[H, P]};]
```

{179.51, Null}

```
CommutatorPandH = odosimp[PtimesH - HtimesP]
```

$$-\frac{1}{2\sqrt{\omega}\,\beta[t]}\mathrm{i}\,\mathrm{Dd}\,e^{-2\,\mathrm{i}\,\gamma[t]}\left(2\left(-1+e^{4\,\mathrm{i}\,\gamma[t]}\right)b[t]+c[t]\left(2\left(-1+e^{4\,\mathrm{i}\,\gamma[t]}\right)\alpha[t]-\mathrm{i}\left(1+e^{4\,\mathrm{i}\,\gamma[t]}\right)\beta[t]^2\right)\right)+\frac{1}{2\,\beta[t]}e^{-2\,\mathrm{i}\,\gamma[t]}\left(2\,\mathrm{i}\,e^{2\,\mathrm{i}\,\gamma[t]}f[t]\,\beta[t]-2\,\mathrm{i}\,b[t]\left(\left(1+e^{4\,\mathrm{i}\,\gamma[t]}\right)x\sqrt{\omega}-2\,e^{2\,\mathrm{i}\,\gamma[t]}\varepsilon[t]\right)+c[t]\left(-\beta[t]\left(\left(-1+e^{4\,\mathrm{i}\,\gamma[t]}\right)x\sqrt{\omega}\,\beta[t]+2\,\mathrm{i}\,e^{2\,\mathrm{i}\,\gamma[t]}\delta[t]\right)-2\,\mathrm{i}\,\alpha[t]\left(\left(1+e^{4\,\mathrm{i}\,\gamma[t]}\right)x\sqrt{\omega}-2\,e^{2\,\mathrm{i}\,\gamma[t]}\varepsilon[t]\right)\right)\right)$$

Checking Heisenberg's equations of motion for general quadratic Hamiltonian (in capital Greeks)

```
D[♀, t] + I * Commutator♀andH
```

$$\mathrm{i}\left(\frac{1}{2\sqrt{\omega}\,\beta[t]}\mathrm{Dd}\,e^{-2\,\mathrm{i}\,\gamma[t]}\left(\mathrm{i}\left(-1+e^{4\,\mathrm{i}\,\gamma[t]}\right)c[t]+2\,a[t]\left(2\,\mathrm{i}\left(-1+e^{4\,\mathrm{i}\,\gamma[t]}\right)\alpha[t]+\left(1+e^{4\,\mathrm{i}\,\gamma[t]}\right)\beta[t]^2\right)\right)+\frac{1}{2\,\beta[t]}e^{-2\,\mathrm{i}\,\gamma[t]}\left(-2\,\mathrm{i}\,e^{2\,\mathrm{i}\,\gamma[t]}g[t]\,\beta[t]+\mathrm{i}\,c[t]\left(\left(1+e^{4\,\mathrm{i}\,\gamma[t]}\right)x\sqrt{\omega}-2\,e^{2\,\mathrm{i}\,\gamma[t]}\varepsilon[t]\right)+2\,a[t]\left(\beta[t]\left(\left(-1+e^{4\,\mathrm{i}\,\gamma[t]}\right)x\sqrt{\omega}\,\beta[t]+2\,\mathrm{i}\,e^{2\,\mathrm{i}\,\gamma[t]}\delta[t]\right)+2\,\mathrm{i}\,\alpha[t]\left(\left(1+e^{4\,\mathrm{i}\,\gamma[t]}\right)x\sqrt{\omega}-2\,e^{2\,\mathrm{i}\,\gamma[t]}\varepsilon[t]\right)\right)\right)\right)-$$

$$\frac{\left(\frac{e^{-2\,\mathrm{i}\,\gamma[t]}(-\mathrm{Dd}+x\,\omega)}{\sqrt{2}\sqrt{\omega}}+\frac{e^{2\,\mathrm{i}\,\gamma[t]}(\mathrm{Dd}+x\,\omega)}{\sqrt{2}\sqrt{\omega}}\right)\beta'[t]}{\sqrt{2}\,\beta[t]^2}+\frac{\varepsilon[t]\,\beta'[t]}{\beta[t]^2}+\frac{-\frac{\mathrm{i}\sqrt{2}\,e^{-2\,\mathrm{i}\,\gamma[t]}(-\mathrm{Dd}+x\,\omega)\,\gamma'[t]}{\sqrt{\omega}}+\frac{\mathrm{i}\sqrt{2}\,e^{2\,\mathrm{i}\,\gamma[t]}(\mathrm{Dd}+x\,\omega)\,\gamma'[t]}{\sqrt{\omega}}}{\sqrt{2}\,\beta[t]}-\frac{\varepsilon'[t]}{\beta[t]}$$

```
% /. {α'[t] → -b[t] - 2 c[t] α[t] - 4 a[t] α[t]^2 + a[t] β[t]^4,
  β'[t] → -(c[t] + 4 a[t] α[t]) β[t], γ'[t] → -a[t] β[t]^2, δ'[t] →
   f[t] + 2 g[t] α[t] - c[t] δ[t] - 4 a[t] α[t] δ[t] + 2 a[t] β[t]^3 ε[t],
  ε'[t] → g[t] β[t] - 2 a[t] β[t] δ[t]}
```

$$-\frac{1}{\sqrt{2}\,\beta[t]}\left(\frac{e^{-2 i \gamma[t]}\,(-Dd+x\,\omega)}{\sqrt{2}\,\sqrt{\omega}}+\frac{e^{2 i \gamma[t]}\,(Dd+x\,\omega)}{\sqrt{2}\,\sqrt{\omega}}\right)(-c[t]-4\,a[t]\,\alpha[t])+\frac{1}{\sqrt{2}\,\beta[t]}\left(\frac{i\sqrt{2}\,e^{-2 i \gamma[t]}\,(-Dd+x\,\omega)\,a[t]\,\beta[t]^2}{\sqrt{\omega}}-\frac{i\sqrt{2}\,e^{2 i \gamma[t]}\,(Dd+x\,\omega)\,a[t]\,\beta[t]^2}{\sqrt{\omega}}\right)-\frac{g[t]\,\beta[t]-2\,a[t]\,\beta[t]\,\delta[t]}{\beta[t]}+\frac{(-c[t]-4\,a[t]\,\alpha[t])\,\varepsilon[t]}{\beta[t]}+i\left(\frac{1}{2\sqrt{\omega}\,\beta[t]}Dd\,e^{-2 i \gamma[t]}\left(i\left(-1+e^{4 i \gamma[t]}\right)c[t]+2\,a[t]\left(2\,i\left(-1+e^{4 i \gamma[t]}\right)\alpha[t]+\left(1+e^{4 i \gamma[t]}\right)\beta[t]^2\right)\right)+\frac{1}{2\,\beta[t]}e^{-2 i \gamma[t]}\left(-2\,i\,e^{2 i \gamma[t]}\,g[t]\,\beta[t]+i\,c[t]\left(\left(1+e^{4 i \gamma[t]}\right)x\sqrt{\omega}-2\,e^{2 i \gamma[t]}\,\varepsilon[t]\right)+2\,a[t]\left(\beta[t]\left(\left(-1+e^{4 i \gamma[t]}\right)x\sqrt{\omega}\,\beta[t]+2\,i\,e^{2 i \gamma[t]}\,\delta[t]\right)+2\,i\,\alpha[t]\left(\left(1+e^{4 i \gamma[t]}\right)x\sqrt{\omega}-2\,e^{2 i \gamma[t]}\,\varepsilon[t]\right)\right)\right)\right)$$

```
FullSimplify[%]
```

0

```
D[P, t] + I * CommutatorPandH;
% /. {α'[t] → -b[t] - 2 c[t] α[t] - 4 a[t] α[t]^2 + a[t] β[t]^4,
  β'[t] → -(c[t] + 4 a[t] α[t]) β[t], γ'[t] → -a[t] β[t]^2, δ'[t] →
   f[t] + 2 g[t] α[t] - c[t] δ[t] - 4 a[t] α[t] δ[t] + 2 a[t] β[t]^3 ε[t],
  ε'[t] → g[t] β[t] - 2 a[t] β[t] δ[t]};
FullSimplify[
 %]
```

0

Creation-annihilation operators in Heisenberg's equations of motion for general quadratic Hamiltonian (in capital Latin)

```
{A = Exp[2 * I * γ[t]] / (2 * β[t] * (ω) ^ (1 / 2)) *
      (ω + β[t]^2 + 2 * I * α[t]) * (ω * X + I * P) / (2 * ω) ^ (1 / 2) +
    Exp[-2 * I * γ[t]] / (2 * β[t] * ω^ (1 / 2)) * (ω - β[t]^2 + 2 * I * α[t]) *
      (ω * X - I * P) / (2 * ω) ^ (1 / 2) - (2 * ω) ^ (-1 / 2) *
      ((ω * ε[t]) / β[t] - I * (δ[t] - (2 * α[t] * ε[t]) / β[t])),
  B = Exp[2 * I * γ[t]] / (2 * β[t] * (ω) ^ (1 / 2)) *
      (ω - β[t]^2 - 2 * I * α[t]) * (ω * X + I * P) / (2 * ω) ^ (1 / 2) +
    Exp[-2 * I * γ[t]] / (2 * β[t] * ω^ (1 / 2)) * (ω + β[t]^2 - 2 * I * α[t]) *
      (ω * X - I * P) / (2 * ω) ^ (1 / 2) - (2 * ω) ^ (-1 / 2) *
      ((ω * ε[t]) / β[t] + I * (δ[t] - (2 * α[t] * ε[t]) / β[t]))};

AtimesB = odomult[A, B];

BtimesA = odomult[B, A];

CommutatorAandB = odosimp[AtimesB - BtimesA]

1

Timing[{AtimesH = odomult[A, H], HtimesA = odomult[H, A]};]

{240.304, Null}

Timing[{BtimesH = odomult[B, H], HtimesB = odomult[H, B]};]

{344.684, Null}

CommutatorAandH = odosimp[AtimesH - HtimesA];

CommutatorBandH = odosimp[BtimesH - HtimesB];

D[A, t] + I * CommutatorAandH;
% /. {α′[t] → -b[t] - 2 c[t] α[t] - 4 a[t] α[t]^2 + a[t] β[t]^4,
   β′[t] → - (c[t] + 4 a[t] α[t]) β[t], γ′[t] → -a[t] β[t]^2, δ′[t] →
    f[t] + 2 g[t] α[t] - c[t] δ[t] - 4 a[t] α[t] δ[t] + 2 a[t] β[t]^3 ε[t],
   ε′[t] → g[t] β[t] - 2 a[t] β[t] δ[t]};

FullSimplify[%]

0

D[B, t] + I * CommutatorBandH;
% /. {α′[t] → -b[t] - 2 c[t] α[t] - 4 a[t] α[t]^2 + a[t] β[t]^4,
   β′[t] → - (c[t] + 4 a[t] α[t]) β[t], γ′[t] → -a[t] β[t]^2, δ′[t] →
    f[t] + 2 g[t] α[t] - c[t] δ[t] - 4 a[t] α[t] δ[t] + 2 a[t] β[t]^3 ε[t],
   ε′[t] → g[t] β[t] - 2 a[t] β[t] δ[t]};

FullSimplify[%]

0
```

Derivation of Ermakov-type System from the Heisenberg Equations for Time-dependent Quadratic Hamiltonian

Annihilation operator

```
Timing[HeisenbergA = FullSimplify[D[A, t] + I * CommutatorAandH]]
```

$$\left\{83.3669,\ \frac{1}{2\sqrt{2}\,\omega\,\beta[t]^2} e^{-2 i \gamma[t]} \Big(-2 i\, Dd\, b[t]\, \beta[t] + 2 i\, x\, \omega\, b[t]\, \beta[t] + 4\, Dd\, \omega\, a[t]\, \alpha[t]\, \beta[t] - 4\, x\, \omega^2\, a[t]\, \alpha[t]\, \beta[t] + 2 i\, Dd\, \omega\, a[t]\, \beta[t]^3 - 2 i\, x\, \omega^2\, a[t]\, \beta[t]^3 + c[t]\, \beta[t] \Big((Dd - x\, \omega) \left(\omega - 2 i\, \alpha[t] + \beta[t]^2\right) + e^{4 i \gamma[t]} (Dd + x\, \omega) \left(-\omega + 2 i\, \alpha[t] + \beta[t]^2\right) + 2\, e^{2 i \gamma[t]} \sqrt{\omega}\, (i\, \beta[t]\, \delta[t] + (\omega - 2 i\, \alpha[t])\, \varepsilon[t])\Big) - 2 i\, Dd\, \beta[t]\, \alpha'[t] + 2 i\, x\, \omega\, \beta[t]\, \alpha'[t] + Dd\, \omega\, \beta'[t] - x\, \omega^2\, \beta'[t] + 2 i\, Dd\, \alpha[t]\, \beta'[t] - 2 i\, x\, \omega\, \alpha[t]\, \beta'[t] + Dd\, \beta[t]^2\, \beta'[t] - x\, \omega\, \beta[t]^2\, \beta'[t] + 2 i\, Dd\, \omega\, \beta[t]\, \gamma'[t] - 2 i\, x\, \omega^2\, \beta[t]\, \gamma'[t] - 4\, Dd\, \alpha[t]\, \beta[t]\, \gamma'[t] + 4\, x\, \omega\, \alpha[t]\, \beta[t]\, \gamma'[t] - 2 i\, Dd\, \beta[t]^3\, \gamma'[t] + 2 i\, x\, \omega\, \beta[t]^3\, \gamma'[t] + i\, e^{4 i \gamma[t]} (Dd + x\, \omega) \Big(2\, b[t]\, \beta[t] + 2\, \omega\, a[t]\, \beta[t] \left(2 i\, \alpha[t] + \beta[t]^2\right) + i\, (\omega + 2 i\, \alpha[t])\, \beta'[t] + \beta[t] \Big(2\, \alpha'[t] - i\, \beta[t]\, \beta'[t] + 2 \left(\omega + 2 i\, \alpha[t] + \beta[t]^2\right) \gamma'[t]\Big)\Big) + 2\, e^{2 i \gamma[t]} \sqrt{\omega} \Big(-i\, f[t]\, \beta[t]^2 + \omega\, g[t]\, \beta[t]^2 - 2\, \omega\, a[t]\, \beta[t]^2\, \delta[t] - 2 i\, b[t]\, \beta[t]\, \varepsilon[t] + 4\, \omega\, a[t]\, \alpha[t]\, \beta[t]\, \varepsilon[t] - 2 i\, \beta[t]\, \varepsilon[t]\, \alpha'[t] + \omega\, \varepsilon[t]\, \beta'[t] + 2 i\, \alpha[t]\, \varepsilon[t]\, \beta'[t] + i\, \beta[t]^2\, \delta'[t] - (\omega + 2 i\, \alpha[t])\, \beta[t]\, \varepsilon'[t]\Big)\Big)\right\}$$

```
heisenbergA = HeisenbergA * (Sqrt[2] ω β[t]^2);
```

```
Expand[heisenbergA]
```

$$-\mathrm{i}\,Dd\,e^{-2\mathrm{i}\gamma[t]}\,b[t]\,\beta[t]+\mathrm{i}\,Dd\,e^{2\mathrm{i}\gamma[t]}\,b[t]\,\beta[t]+\mathrm{i}\,e^{-2\mathrm{i}\gamma[t]}\,x\,\omega\,b[t]\,\beta[t]+\mathrm{i}\,e^{2\mathrm{i}\gamma[t]}\,x\,\omega\,b[t]\,\beta[t]+\frac{1}{2}Dd\,e^{-2\mathrm{i}\gamma[t]}\,\omega\,c[t]\,\beta[t]-\frac{1}{2}Dd\,e^{2\mathrm{i}\gamma[t]}\,\omega\,c[t]\,\beta[t]-\frac{1}{2}e^{-2\mathrm{i}\gamma[t]}\,x\,\omega^2\,c[t]\,\beta[t]-\frac{1}{2}e^{2\mathrm{i}\gamma[t]}\,x\,\omega^2\,c[t]\,\beta[t]+2\,Dd\,e^{-2\mathrm{i}\gamma[t]}\,\omega\,a[t]\,\alpha[t]\,\beta[t]-2\,Dd\,e^{2\mathrm{i}\gamma[t]}\,\omega\,a[t]\,\alpha[t]\,\beta[t]-2\,e^{-2\mathrm{i}\gamma[t]}\,x\,\omega^2\,a[t]\,\alpha[t]\,\beta[t]-2\,e^{2\mathrm{i}\gamma[t]}\,x\,\omega^2\,a[t]\,\alpha[t]\,\beta[t]-\mathrm{i}\,Dd\,e^{-2\mathrm{i}\gamma[t]}\,c[t]\,\alpha[t]\,\beta[t]+\mathrm{i}\,Dd\,e^{2\mathrm{i}\gamma[t]}\,c[t]\,\alpha[t]\,\beta[t]+\mathrm{i}\,e^{-2\mathrm{i}\gamma[t]}\,x\,\omega\,c[t]\,\alpha[t]\,\beta[t]+\mathrm{i}\,e^{2\mathrm{i}\gamma[t]}\,x\,\omega\,c[t]\,\alpha[t]\,\beta[t]-\mathrm{i}\sqrt{\omega}\,f[t]\,\beta[t]^2+\omega^{3/2}\,g[t]\,\beta[t]^2+\mathrm{i}\,Dd\,e^{-2\mathrm{i}\gamma[t]}\,\omega\,a[t]\,\beta[t]^3+\mathrm{i}\,Dd\,e^{2\mathrm{i}\gamma[t]}\,\omega\,a[t]\,\beta[t]^3-\mathrm{i}\,e^{-2\mathrm{i}\gamma[t]}\,x\,\omega^2\,a[t]\,\beta[t]^3+\mathrm{i}\,e^{2\mathrm{i}\gamma[t]}\,x\,\omega^2\,a[t]\,\beta[t]^3+\frac{1}{2}Dd\,e^{-2\mathrm{i}\gamma[t]}\,c[t]\,\beta[t]^3+\frac{1}{2}Dd\,e^{2\mathrm{i}\gamma[t]}\,c[t]\,\beta[t]^3-\frac{1}{2}e^{-2\mathrm{i}\gamma[t]}\,x\,\omega\,c[t]\,\beta[t]^3+\frac{1}{2}e^{2\mathrm{i}\gamma[t]}\,x\,\omega\,c[t]\,\beta[t]^3-2\,\omega^{3/2}\,a[t]\,\beta[t]^2\,\delta[t]+\mathrm{i}\sqrt{\omega}\,c[t]\,\beta[t]^2\,\delta[t]-2\,\mathrm{i}\sqrt{\omega}\,b[t]\,\beta[t]\,\varepsilon[t]+\omega^{3/2}\,c[t]\,\beta[t]\,\varepsilon[t]+4\,\omega^{3/2}\,a[t]\,\alpha[t]\,\beta[t]\,\varepsilon[t]-2\,\mathrm{i}\sqrt{\omega}\,c[t]\,\alpha[t]\,\beta[t]\,\varepsilon[t]-\mathrm{i}\,Dd\,e^{-2\mathrm{i}\gamma[t]}\,\beta[t]\,\alpha'[t]+\mathrm{i}\,Dd\,e^{2\mathrm{i}\gamma[t]}\,\beta[t]\,\alpha'[t]+\mathrm{i}\,e^{-2\mathrm{i}\gamma[t]}\,x\,\omega\,\beta[t]\,\alpha'[t]+\mathrm{i}\,e^{2\mathrm{i}\gamma[t]}\,x\,\omega\,\beta[t]\,\alpha'[t]-2\,\mathrm{i}\sqrt{\omega}\,\beta[t]\,\varepsilon[t]\,\alpha'[t]+\frac{1}{2}Dd\,e^{-2\mathrm{i}\gamma[t]}\,\omega\,\beta'[t]-\frac{1}{2}Dd\,e^{2\mathrm{i}\gamma[t]}\,\omega\,\beta'[t]-\frac{1}{2}e^{-2\mathrm{i}\gamma[t]}\,x\,\omega^2\,\beta'[t]-\frac{1}{2}e^{2\mathrm{i}\gamma[t]}\,x\,\omega^2\,\beta'[t]+\mathrm{i}\,Dd\,e^{-2\mathrm{i}\gamma[t]}\,\alpha[t]\,\beta'[t]-\mathrm{i}\,Dd\,e^{2\mathrm{i}\gamma[t]}\,\alpha[t]\,\beta'[t]-\mathrm{i}\,e^{-2\mathrm{i}\gamma[t]}\,x\,\omega\,\alpha[t]\,\beta'[t]-\mathrm{i}\,e^{2\mathrm{i}\gamma[t]}\,x\,\omega\,\alpha[t]\,\beta'[t]+\frac{1}{2}Dd\,e^{-2\mathrm{i}\gamma[t]}\,\beta[t]^2\,\beta'[t]+\frac{1}{2}Dd\,e^{2\mathrm{i}\gamma[t]}\,\beta[t]^2\,\beta'[t]-\frac{1}{2}e^{-2\mathrm{i}\gamma[t]}\,x\,\omega\,\beta[t]^2\,\beta'[t]+\frac{1}{2}e^{2\mathrm{i}\gamma[t]}\,x\,\omega\,\beta[t]^2\,\beta'[t]+\omega^{3/2}\,\varepsilon[t]\,\beta'[t]+2\,\mathrm{i}\sqrt{\omega}\,\alpha[t]\,\varepsilon[t]\,\beta'[t]+\mathrm{i}\,Dd\,e^{-2\mathrm{i}\gamma[t]}\,\omega\,\beta[t]\,\gamma'[t]+\mathrm{i}\,Dd\,e^{2\mathrm{i}\gamma[t]}\,\omega\,\beta[t]\,\gamma'[t]-\mathrm{i}\,e^{-2\mathrm{i}\gamma[t]}\,x\,\omega^2\,\beta[t]\,\gamma'[t]+\mathrm{i}\,e^{2\mathrm{i}\gamma[t]}\,x\,\omega^2\,\beta[t]\,\gamma'[t]-2\,Dd\,e^{-2\mathrm{i}\gamma[t]}\,\alpha[t]\,\beta[t]\,\gamma'[t]-2\,Dd\,e^{2\mathrm{i}\gamma[t]}\,\alpha[t]\,\beta[t]\,\gamma'[t]+2\,e^{-2\mathrm{i}\gamma[t]}\,x\,\omega\,\alpha[t]\,\beta[t]\,\gamma'[t]-2\,e^{2\mathrm{i}\gamma[t]}\,x\,\omega\,\alpha[t]\,\beta[t]\,\gamma'[t]-\mathrm{i}\,Dd\,e^{-2\mathrm{i}\gamma[t]}\,\beta[t]^3\,\gamma'[t]+\mathrm{i}\,Dd\,e^{2\mathrm{i}\gamma[t]}\,\beta[t]^3\,\gamma'[t]+\mathrm{i}\,e^{-2\mathrm{i}\gamma[t]}\,x\,\omega\,\beta[t]^3\,\gamma'[t]+\mathrm{i}\,e^{2\mathrm{i}\gamma[t]}\,x\,\omega\,\beta[t]^3\,\gamma'[t]+\mathrm{i}\sqrt{\omega}\,\beta[t]^2\,\delta'[t]-\omega^{3/2}\,\beta[t]\,\varepsilon'[t]-2\,\mathrm{i}\sqrt{\omega}\,\alpha[t]\,\beta[t]\,\varepsilon'[t]$$

```
Coefficient[Collect[heisenbergA, Dd], Dd]
```

$$\frac{1}{2}e^{-2\mathrm{i}\gamma[t]}\Big(-2\,\mathrm{i}\,b[t]\,\beta[t]+4\,\omega\,a[t]\,\alpha[t]\,\beta[t]+2\,\mathrm{i}\,\omega\,a[t]\,\beta[t]^3+c[t]\,\beta[t]\,\left(\omega-2\,\mathrm{i}\,\alpha[t]+\beta[t]^2\right)+e^{4\mathrm{i}\gamma[t]}\,c[t]\,\beta[t]\,\left(-\omega+2\,\mathrm{i}\,\alpha[t]+\beta[t]^2\right)-2\,\mathrm{i}\,\beta[t]\,\alpha'[t]+\omega\,\beta'[t]+2\,\mathrm{i}\,\alpha[t]\,\beta'[t]+\beta[t]^2\,\beta'[t]+2\,\mathrm{i}\,\omega\,\beta[t]\,\gamma'[t]-4\,\alpha[t]\,\beta[t]\,\gamma'[t]-2\,\mathrm{i}\,\beta[t]^3\,\gamma'[t]+\mathrm{i}\,e^{4\mathrm{i}\gamma[t]}\Big(2\,b[t]\,\beta[t]+2\,\omega\,a[t]\,\beta[t]\,\left(2\,\mathrm{i}\,\alpha[t]+\beta[t]^2\right)+\mathrm{i}\,(\omega+2\,\mathrm{i}\,\alpha[t])\,\beta'[t]+\beta[t]\,\left(2\,\alpha'[t]-\mathrm{i}\,\beta[t]\,\beta'[t]+2\,\left(\omega+2\,\mathrm{i}\,\alpha[t]+\beta[t]^2\right)\gamma'[t]\right)\Big)\Big)$$

```
Coefficient[Collect[heisenbergA, x], x]
```

$$\frac{1}{2} e^{-2 i \gamma[t]} \Big(2 i \omega b[t] \beta[t] - 4 \omega^2 a[t] \alpha[t] \beta[t] - 2 i \omega^2 a[t] \beta[t]^3 - \omega c[t] \beta[t] \left(\omega - 2 i \alpha[t] + \beta[t]^2\right) + e^{4 i \gamma[t]} \omega c[t] \beta[t] \left(-\omega + 2 i \alpha[t] + \beta[t]^2\right) + 2 i \omega \beta[t] \alpha'[t] - \omega^2 \beta'[t] - 2 i \omega \alpha[t] \beta'[t] - \omega \beta[t]^2 \beta'[t] - 2 i \omega^2 \beta[t] \gamma'[t] + 4 \omega \alpha[t] \beta[t] \gamma'[t] + 2 i \omega \beta[t]^3 \gamma'[t] + i e^{4 i \gamma[t]} \omega \Big(2 b[t] \beta[t] + 2 \omega a[t] \beta[t] \left(2 i \alpha[t] + \beta[t]^2\right) + i (\omega + 2 i \alpha[t]) \beta'[t] + \beta[t] \left(2 \alpha'[t] - i \beta[t] \beta'[t] + 2 \left(\omega + 2 i \alpha[t] + \beta[t]^2\right) \gamma'[t]\right)\Big)\Big)$$

```
heisenbergA /. Dd → 0 /. x → 0
```

$$\frac{1}{2} e^{-2 i \gamma[t]} \Big(2 e^{2 i \gamma[t]} \sqrt{\omega}\, c[t] \beta[t] (i \beta[t] \delta[t] + (\omega - 2 i \alpha[t]) \varepsilon[t]) + 2 e^{2 i \gamma[t]} \sqrt{\omega} \Big(-i f[t] \beta[t]^2 + \omega g[t] \beta[t]^2 - 2 \omega a[t] \beta[t]^2 \delta[t] - 2 i b[t] \beta[t] \varepsilon[t] + 4 \omega a[t] \alpha[t] \beta[t] \varepsilon[t] - 2 i \beta[t] \varepsilon[t] \alpha'[t] + \omega \varepsilon[t] \beta'[t] + 2 i \alpha[t] \varepsilon[t] \beta'[t] + i \beta[t]^2 \delta'[t] - (\omega + 2 i \alpha[t]) \beta[t] \varepsilon'[t]\Big)\Big)$$

```
{Together[Coefficient[Coefficient[Collect[Expand[heisenbergA] /.
      Complex[x_, y_] :> x + i * y, Dd], Dd], i, 0]] == 0,
 Together[Coefficient[Coefficient[Collect[Expand[heisenbergA] /.
      Complex[x_, y_] :> x + i * y, Dd], Dd], i, 1]] == 0,
 Together[Coefficient[Coefficient[Collect[Expand[heisenbergA] /.
      Complex[x_, y_] :> x + i * y, x], x], i, 0]] == 0,
 Together[Coefficient[Coefficient[Collect[Expand[heisenbergA] /.
      Complex[x_, y_] :> x + i * y, x], x], i, 1]] == 0,
 Together[Coefficient[Expand[heisenbergA] /. Dd -> 0 /. x -> 0 /.
    Complex[x_, y_] :> x + i * y, i, 0]] == 0,
 Together[Coefficient[Expand[heisenbergA] /. Dd -> 0 /. x -> 0 /.
    Complex[x_, y_] :> x + i * y, i, 1]] == 0}
```

$$\Big\{-\frac{1}{2} e^{-2 i \gamma[t]} \big(-\omega\, c[t]\, \beta[t] + e^{4 i \gamma[t]}\, \omega\, c[t]\, \beta[t] - 4\, \omega\, a[t]\, \alpha[t]\, \beta[t] + 4\, e^{4 i \gamma[t]}\, \omega\, a[t]\, \alpha[t]\, \beta[t] - c[t]\, \beta[t]^3 - e^{4 i \gamma[t]}\, c[t]\, \beta[t]^3 - \omega\, \beta'[t] + e^{4 i \gamma[t]}\, \omega\, \beta'[t] - \beta[t]^2\, \beta'[t] - e^{4 i \gamma[t]}\, \beta[t]^2\, \beta'[t] + 4\, \alpha[t]\, \beta[t]\, \gamma'[t] + 4\, e^{4 i \gamma[t]}\, \alpha[t]\, \beta[t]\, \gamma'[t]\big) == 0,$$

$$e^{-2 i \gamma[t]} \big(-b[t]\, \beta[t] + e^{4 i \gamma[t]}\, b[t]\, \beta[t] - c[t]\, \alpha[t]\, \beta[t] + e^{4 i \gamma[t]}\, c[t]\, \alpha[t]\, \beta[t] + \omega\, a[t]\, \beta[t]^3 + e^{4 i \gamma[t]}\, \omega\, a[t]\, \beta[t]^3 - \beta[t]\, \alpha'[t] + e^{4 i \gamma[t]}\, \beta[t]\, \alpha'[t] + \alpha[t]\, \beta'[t] - e^{4 i \gamma[t]}\, \alpha[t]\, \beta'[t] + \omega\, \beta[t]\, \gamma'[t] + e^{4 i \gamma[t]}\, \omega\, \beta[t]\, \gamma'[t] - \beta[t]^3\, \gamma'[t] + e^{4 i \gamma[t]}\, \beta[t]^3\, \gamma'[t]\big) == 0,$$

$$-\frac{1}{2} e^{-2 i \gamma[t]}\, \omega \big(\omega\, c[t]\, \beta[t] + e^{4 i \gamma[t]}\, \omega\, c[t]\, \beta[t] + 4\, \omega\, a[t]\, \alpha[t]\, \beta[t] + 4\, e^{4 i \gamma[t]}\, \omega\, a[t]\, \alpha[t]\, \beta[t] + c[t]\, \beta[t]^3 - e^{4 i \gamma[t]}\, c[t]\, \beta[t]^3 + \omega\, \beta'[t] + e^{4 i \gamma[t]}\, \omega\, \beta'[t] + \beta[t]^2\, \beta'[t] - e^{4 i \gamma[t]}\, \beta[t]^2\, \beta'[t] - 4\, \alpha[t]\, \beta[t]\, \gamma'[t] + 4\, e^{4 i \gamma[t]}\, \alpha[t]\, \beta[t]\, \gamma'[t]\big) == 0,$$

$$e^{-2 i \gamma[t]}\, \omega \big(b[t]\, \beta[t] + e^{4 i \gamma[t]}\, b[t]\, \beta[t] + c[t]\, \alpha[t]\, \beta[t] + e^{4 i \gamma[t]}\, c[t]\, \alpha[t]\, \beta[t] - \omega\, a[t]\, \beta[t]^3 + e^{4 i \gamma[t]}\, \omega\, a[t]\, \beta[t]^3 + \beta[t]\, \alpha'[t] + e^{4 i \gamma[t]}\, \beta[t]\, \alpha'[t] - \alpha[t]\, \beta'[t] - e^{4 i \gamma[t]}\, \alpha[t]\, \beta'[t] - \omega\, \beta[t]\, \gamma'[t] + e^{4 i \gamma[t]}\, \omega\, \beta[t]\, \gamma'[t] + \beta[t]^3\, \gamma'[t] + e^{4 i \gamma[t]}\, \beta[t]^3\, \gamma'[t]\big) == 0,$$

$$\omega^{3/2}\, g[t]\, \beta[t]^2 - 2\, \omega^{3/2}\, a[t]\, \beta[t]^2\, \delta[t] + \omega^{3/2}\, c[t]\, \beta[t]\, \varepsilon[t] + 4\, \omega^{3/2}\, a[t]\, \alpha[t]\, \beta[t]\, \varepsilon[t] + \omega^{3/2}\, \varepsilon[t]\, \beta'[t] - \omega^{3/2}\, \beta[t]\, \varepsilon'[t] == 0,$$

$$-\sqrt{\omega}\, f[t]\, \beta[t]^2 + \sqrt{\omega}\, c[t]\, \beta[t]^2\, \delta[t] - 2\sqrt{\omega}\, b[t]\, \beta[t]\, \varepsilon[t] - 2\sqrt{\omega}\, c[t]\, \alpha[t]\, \beta[t]\, \varepsilon[t] - 2\sqrt{\omega}\, \beta[t]\, \varepsilon[t]\, \alpha'[t] + 2\sqrt{\omega}\, \alpha[t]\, \varepsilon[t]\, \beta'[t] + \sqrt{\omega}\, \beta[t]^2\, \delta'[t] - 2\sqrt{\omega}\, \alpha[t]\, \beta[t]\, \varepsilon'[t] == 0\Big\}$$

```
Solve[%, {α'[t], β'[t], γ'[t], δ'[t], ε'[t]}]
```

$$\{\{\alpha'[t] \to -b[t] - 2\,c[t]\,\alpha[t] - 4\,a[t]\,\alpha[t]^2 + a[t]\,\beta[t]^4,\ \beta'[t] \to -c[t]\,\beta[t] - 4\,a[t]\,\alpha[t]\,\beta[t],\ \gamma'[t] \to -a[t]\,\beta[t]^2,\ \delta'[t] \to f[t] + 2\,g[t]\,\alpha[t] - c[t]\,\delta[t] - 4\,a[t]\,\alpha[t]\,\delta[t] + 2\,a[t]\,\beta[t]^3\,\varepsilon[t],\ \varepsilon'[t] \to g[t]\,\beta[t] - 2\,a[t]\,\beta[t]\,\delta[t]\}\}$$

```
Sort[First[%]] // TableForm
```

$$\alpha'[t] \to -b[t] - 2\,c[t]\,\alpha[t] - 4\,a[t]\,\alpha[t]^2 + a[t]\,\beta[t]^4$$
$$\beta'[t] \to -c[t]\,\beta[t] - 4\,a[t]\,\alpha[t]\,\beta[t]$$
$$\gamma'[t] \to -a[t]\,\beta[t]^2$$
$$\delta'[t] \to f[t] + 2\,g[t]\,\alpha[t] - c[t]\,\delta[t] - 4\,a[t]\,\alpha[t]\,\delta[t] + 2\,a[t]\,\beta[t]^3\,\varepsilon[t]$$
$$\varepsilon'[t] \to g[t]\,\beta[t] - 2\,a[t]\,\beta[t]\,\delta[t]$$

Creation operator

```
Timing[HeisenbergB = FullSimplify[D[B, t] + I * CommutatorBandH]]
```

$$\Big\{151.118,\ \frac{1}{2\sqrt{2}\,\omega\,\beta[t]^2}\, e^{-2 i \gamma[t]} \Big(2 i\, Dd\, b[t]\,\beta[t] - 2 i\, x\,\omega\, b[t]\,\beta[t] + 4\, Dd\,\omega\, a[t]\,\alpha[t]\,\beta[t] - 4\, x\,\omega^2\, a[t]\,\alpha[t]\,\beta[t] + 2 i\, Dd\,\omega\, a[t]\,\beta[t]^3 - 2 i\, x\,\omega^2\, a[t]\,\beta[t]^3 + c[t]\,\beta[t]\Big((Dd - x\,\omega)\,(\omega + 2 i\,\alpha[t] - \beta[t]^2) - e^{4 i \gamma[t]}\,(Dd + x\,\omega)\,(\omega + 2 i\,\alpha[t] + \beta[t]^2) + 2\, e^{2 i \gamma[t]}\sqrt{\omega}\,(-i\,\beta[t]\,\delta[t] + (\omega + 2 i\,\alpha[t])\,\varepsilon[t])\Big) + 2 i\, Dd\,\beta[t]\,\alpha'[t] - 2 i\, x\,\omega\,\beta[t]\,\alpha'[t] + Dd\,\omega\,\beta'[t] - x\,\omega^2\,\beta'[t] - 2 i\, Dd\,\alpha[t]\,\beta'[t] + 2 i\, x\,\omega\,\alpha[t]\,\beta'[t] - Dd\,\beta[t]^2\,\beta'[t] + x\,\omega\,\beta[t]^2\,\beta'[t] + 2 i\, Dd\,\omega\,\beta[t]\,\gamma'[t] - 2 i\, x\,\omega^2\,\beta[t]\,\gamma'[t] + 4\, Dd\,\alpha[t]\,\beta[t]\,\gamma'[t] - 4\, x\,\omega\,\alpha[t]\,\beta[t]\,\gamma'[t] + 2 i\, Dd\,\beta[t]^3\,\gamma'[t] - 2 i\, x\,\omega\,\beta[t]^3\,\gamma'[t] - e^{4 i \gamma[t]}\,(Dd + x\,\omega)\,\Big(2 i\, b[t]\,\beta[t] + a[t]\,(4\,\omega\,\alpha[t]\,\beta[t] - 2 i\,\omega\,\beta[t]^3) + (\omega - 2 i\,\alpha[t])\,\beta'[t] + \beta[t]\,\big(2 i\,\alpha'[t] + \beta[t]\,\beta'[t] + 2 i\,(-\omega + 2 i\,\alpha[t] + \beta[t]^2)\,\gamma'[t]\big)\Big) + 2\, e^{2 i \gamma[t]}\sqrt{\omega}\,\Big(i\, f[t]\,\beta[t]^2 + \omega\, g[t]\,\beta[t]^2 - 2\,\omega\, a[t]\,\beta[t]^2\,\delta[t] + 2 i\, b[t]\,\beta[t]\,\varepsilon[t] + 4\,\omega\, a[t]\,\alpha[t]\,\beta[t]\,\varepsilon[t] + 2 i\,\beta[t]\,\varepsilon[t]\,\alpha'[t] + \omega\,\varepsilon[t]\,\beta'[t] - 2 i\,\alpha[t]\,\varepsilon[t]\,\beta'[t] - i\,\beta[t]^2\,\delta'[t] - (\omega - 2 i\,\alpha[t])\,\beta[t]\,\varepsilon'[t]\Big)\Big)\Big\}$$

```
heisenbergB = HeisenbergB * (√2 ω β[t]^2);
```

```
{Together[Coefficient[Coefficient[Collect[Expand[heisenbergB] /.
        Complex[x_, y_] :→ x + i * y, Dd], Dd], i, 0]] == 0,
  Together[Coefficient[Coefficient[Collect[Expand[heisenbergB] /.
        Complex[x_, y_] :→ x + i * y, Dd], Dd], i, 1]] == 0,
  Together[Coefficient[Coefficient[Collect[Expand[heisenbergB] /.
        Complex[x_, y_] :→ x + i * y, x], x], i, 0]] == 0,
  Together[Coefficient[Coefficient[Collect[Expand[heisenbergB] /.
        Complex[x_, y_] :→ x + i * y, x], x], i, 1]] == 0,
  Together[Coefficient[Expand[heisenbergB] /. Dd → 0 /. x → 0 /.
      Complex[x_, y_] :→ x + i * y, i, 0]] == 0,
  Together[Coefficient[Expand[heisenbergB] /. Dd → 0 /. x → 0 /.
      Complex[x_, y_] :→ x + i * y, i, 1]] == 0};
Solve[%, {α'[t], β'[t], γ'[t], δ'[t], ε'[t]}] ;
Sort[First[%]] // TableForm
```

$\alpha'[t] \to -b[t] - 2\,c[t]\,\alpha[t] - 4\,a[t]\,\alpha[t]^2 + a[t]\,\beta[t]^4$

$\beta'[t] \to -c[t]\,\beta[t] - 4\,a[t]\,\alpha[t]\,\beta[t]$

$\gamma'[t] \to -a[t]\,\beta[t]^2$

$\delta'[t] \to f[t] + 2\,g[t]\,\alpha[t] - c[t]\,\delta[t] - 4\,a[t]\,\alpha[t]\,\delta[t] + 2\,a[t]\,\beta[t]^3\,\varepsilon[t]$

$\varepsilon'[t] \to g[t]\,\beta[t] - 2\,a[t]\,\beta[t]\,\delta[t]$