Vector Spaces

Let V be a set where we define

1) Addition

   $\forall u,v \in V \longrightarrow u+v \in V$

2) Scalar Multiplication.

   $\forall a \in \mathbb{R} : \forall u \in V \longrightarrow au \in V$

Def: $(V,+,\cdot)$ is a vector space if it satisfies:

1) Closure

   $\forall u,v \in V : u+v \in V$

   $\forall a \in \mathbb{R} : \forall u \in V : au \in V$

2) $(V,+)$ abelian group

   $\forall u,v,w \in V : (u+v)+w = u+(v+w)$

   $\forall u,v \in V : u+v = v+u$

   $\exists \mathbf{0} \in V : \mathbf{0}+u = u+\mathbf{0} = u, \forall u \in V$

   $\forall u \in V : \exists v \in V : u+v = \mathbf{0}$

3) The scalar multiplication satisfies

   $\forall a \in \mathbb{R} : \forall u,v \in V : \lambda(u+v) = \lambda u + \lambda v$

   $\forall a,b \in \mathbb{R} : \forall u \in V : (a+b)u = au + bv$

   $\forall a,b \in \mathbb{R} : \forall u \in V : a(bu) = (ab)u$

   $\forall u \in V : 1u = u$

Def: Let $(V, +, \cdot)$ be a vector space.

a) An inner product is a mapping
$$\forall u, v \in V \longrightarrow (u, v) \in \mathbb{R}$$
such that
$$\forall u, v \in V : (u, v) = (v, u)$$
$$\forall u, v, w \in V : (u, v+w) = (u, v) + (u, w)$$
$$\forall a \in \mathbb{R} : \forall u, v \in V : a(u, v) = (au, v)$$
$$\forall u \in V - \{\mathbf{0}\} : (u, u) > 0$$

b) A norm is a mapping
$$\forall u \in V \longrightarrow \|u\| \in \mathbb{R}.$$
such that
$$u = \mathbf{0} \Rightarrow \|u\| = 0$$
$$u \neq \mathbf{0} \Rightarrow \|u\| > 0$$
$$\forall a \in \mathbb{R} : \forall u \in V : \|au\| = |a| \cdot \|u\|$$
$$\forall u, v \in V : \|u+v\| \leq \|u\| + \|v\|$$

Thm: If (u, v) is an inner product

a) $\forall u, v \in V : |(u, v)|^2 \leq (u, u) \cdot (v, v)$

b) $\|u\| = (u, u)$ defines a norm.

Linear independence and orthogonality.

The

Def: The vectors $\{v_1, v_2, \dots, v_K\}$ are linearly independent iff
$$a_1 v_1 + a_2 v_2 + \dots + a_K v_K \Rightarrow a_1 = a_2 = \dots = a_K = 0$$

Def: The set $B = \{v_1, v_2, \dots, v_K\} \subset V$ is a basis of V iff.

a) $v_1, \dots, v_K$ linearly independent

b) $V = \{a_1 v_1 + a_2 v_2 + \dots + a_n v_n \mid a_1, a_2, \dots, a_n \in \mathbb{R}\}$

$\rightarrow$ All basis of a vector space V have the same number of elements, and we call that number n the dimension of V: $n = \dim V$.

Def: a) Two vectors $u, v \in V$ are orthogonal iff:
$$u \perp v \Leftrightarrow (u, v) = 0.$$

b) The vectors $\{u_1, u_2, \dots, u_n\}$ are orthogonal iff: $\alpha \neq \beta \Rightarrow u_\alpha \perp u_\beta, \ \forall \alpha, \beta \in \{1, 2, \dots, n\}$

c) The vectors $\{u_1, u_2, \dots, u_n\}$ are orthonormal iff $\{u_1, u_2, \dots, u_n\}$ orthogonal and
$$(u_K, u_K) = 0, \ \forall K \in \{1, 2, \dots, n\}$$

↳ If $\{u_1, u_2, \dots, u_k\}$ orthogonal $\Rightarrow$
$\Rightarrow \{u_1, u_2, \dots, u_k\}$ linearly independent.

If $\left.\begin{array}{l}\{u_1, u_2, \dots, u_n\} \text{ orthogonal} \\ \dim V = n\end{array}\right\} \Rightarrow \{u_1, u_2, \dots, u_n\}$ basis of V.

↳ If $\{u_1, u_2, \dots, u_n\}$ orthogonal basis of V then

$$\boxed{\forall v \in V: \; v = \sum_{k=1}^{n} \frac{(v, u_k)}{(u_k, u_k)} u_k}.$$

From a linearly independent basis $\{u_1, u_2, \dots, u_n\}$ one may construct an orthogonal basis $\{v_1, v_2, \dots, v_n\}$ by the Gram-Schmidt iteration:

$$v_1 = u_1$$
$$v_{k+1} = u_{k+1} - \sum_{a=1}^{k} \frac{(u_{k+1}, v_a)}{(\;}$$

$$\boxed{\begin{array}{l} v_1 = u_1 \\ v_{k+1} = u_{k+1} - \displaystyle\sum_{a=1}^{k} \frac{(u_{k+1}, v_a)}{(v_a, v_a)} v_a \end{array}}$$

Gram-Schmidt is NOT numerically stable.

The vector space $\mathbb{R}^n$

Let $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ and $y = (y_1, y_2, \dots, y_n) \in \mathbb{R}^n$ and $a \in \mathbb{R}$. We define

$$x + y = (x_1 + y_1, x_2 + y_2, \dots, x_n + y_n)$$
$$ax = (ax_1, ax_2, \dots, ax_n)$$

This makes $(\mathbb{R}^n, +, \cdot)$ a vector space.
We can also define an inner product

$$(x, y) = x_1 y_1 + x_2 y_2 + \dots + x_n y_n.$$

Vector norms in $\mathbb{R}^n$

Let $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$. We define:

a) p-norm: $$\|x\|_p = \left(\sum_{a=1}^{n} |x_a|^p \right)^{1/p}, \quad p \geq 1$$

b) ∞-norm: $$\|x\|_\infty = \lim_{p \to +\infty} \|x\|_p = \max_{a \in [n]} |x_a|$$

It can be shown that these definitions satisfy the required properties of a norm.

Hölder Inequality

$$\boxed{\frac{1}{p}+\frac{1}{q}=1 \Rightarrow \forall x,y\in\mathbb{R}^n: |\langle x,y\rangle| \leqslant \|x\|_p\|y\|_q.}$$

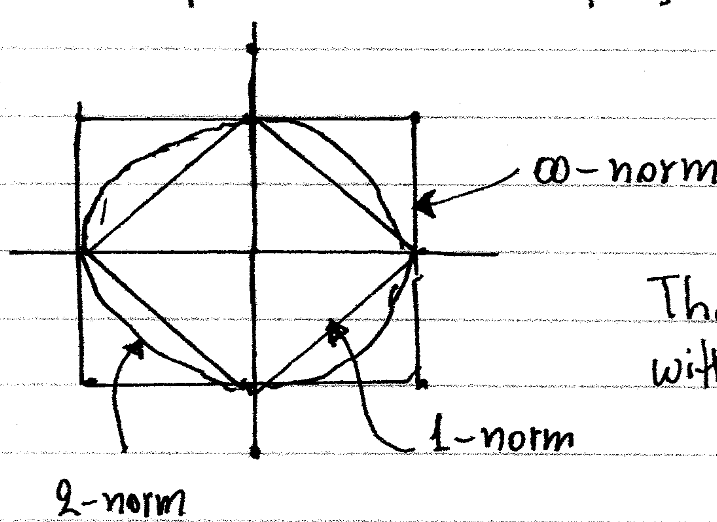
When $p=q=2$ we obtain the Schwarz-Cauchy inequality:

$$\boxed{\forall x,y\in\mathbb{R}^n: |\langle x,y\rangle| \leqslant \|x\|_2\,\|y\|_2.}$$

The p-norm ball

$\to$ The set of all vectors x whose p-norm equals the radius ρ of the ball.

$$\mathcal{B}_p(\rho)=\{x\in\mathbb{R}^n \mid \|x\|_p=\rho\}$$



The "ball" will "puff up" with increasing p.

- **Norm Equivalence** : All p-norms are roughly-speaking equivalent.

$$\forall a, b \in (0, +\infty) : \exists\, c_1, c_2 \in (0, +\infty) : \; c_1 \|x\|_a \leqslant \|x\|_b \leqslant c_2 \|x\|_a.$$

For example:

a) $\|x\|_2 \leqslant \|x\|_1 \leqslant \sqrt{n}\, \|x\|_2$

b) $\|x\|_\infty \leqslant \|x\|_2 \leqslant \sqrt{n}\, \|x\|_\infty$

c) $\|x\|_\infty \leqslant \|x\|_1 \leqslant n \|x\|_\infty$