Set Theory and Logic

Fundamentals

- A set is a well-defined collection of objects called elements
- We represent sets with upper-case letters. We usually represent elements with lower-case letters.

  Note: A set can be an element of another set.
- A statement is an expression that can be either true or false
- The fundamental statements are

  a) Equality: $x=y$ ← The elements x,y are the same element.

  b) Belonging: $x \in A$ ← The element x belongs to the set A.

  Every statement of mathematics can be constructed from fundamental statements.

- Note: The statement $x \notin A$ means that x does not belong to A. It is not however a fundamental statement.

Set representation

1) Listing : Give the elements of a set as a list

$A = \{1,2,3,4\}$ ← example

$A = \{3,\{4,5\},6\}$

2) Descriptive : Give a rule that decides if an element belongs to a set

example : $x \in A \iff 3x+2=8$

$A = \{x \mid 3x+2=8\}$

3) Subset/Descriptive : All elements of set A that satisfy some additional rule.

example : If $A = \{1,2,3,4,5,6\}$

then

$B = \{x \in A \mid x \geqslant 3\}$

$= \{3,4,5,6\}$

Remarkable sets

1) The empty set $\emptyset$ → It is a set that has no elements.

e.g. If $A = \{1, 4, 6\}$
then $B = \{x \in A \mid x < 0\} = \emptyset$.

We also write $\emptyset = \{\}$

2) The ~~integers~~ natural numbers

$1 = \{\emptyset\}$
$2 = \{\emptyset, 1\} = \{\emptyset, \{\emptyset\}\}$
$3 = \{\emptyset, 1, 2\}$
$4 = \{\emptyset, 1, 2, 3\}$ etc.

Every natural number can be created as a set

3) Natural numbers

$\mathbb{N} = \{0, 1, 2, \ldots\}$
$\mathbb{N}^* = \{1, 2, 3, \ldots\}$

4) Integers

$\mathbb{Z} = \{x, -x \mid x \in \mathbb{N}\}$
$= \{0, \pm 1, \pm 2, \pm 3, \dots\}$
$\mathbb{Z}^* = \{x \in \mathbb{Z} \mid x \neq 0\}$

5) Rational numbers

$\mathbb{Q} = \{a/b \mid a, b \in \mathbb{Z}\}$
$\mathbb{Q}^* = \{a \in \mathbb{Q} \mid a \neq 0\}$

6) Real numbers

$\mathbb{R} = \{x \mid x \text{ is a real number}\}$.

example $\sqrt{2} \notin \mathbb{Q}$ but $\sqrt{2} \in \mathbb{R}$. | $\pi \notin \mathbb{Q}$ but $\pi \in \mathbb{R}$.

Operations on

1) Statements

Let p, q be two statements.
We define compound statements:

$p \wedge q$: p and q are both true (conjunction)

$p \vee q$: p or q or both are true (disjunction)

$p \veebar q$: p or q but not both are true (exclusive disjunction)

$\bar{p}$: p is false. (negation)

These definitions can be represented using truth tables:

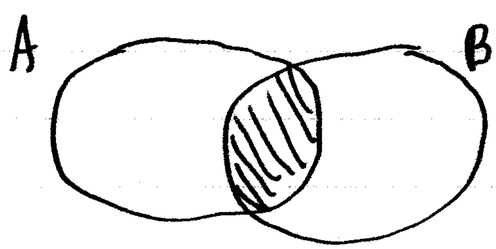
p	q	$p \wedge q$	$p \vee q$	$p \veebar q$	$\bar{p}$
T	T	T	T	F	F
T	F	F	T	T	F
F	T	F	T	T	T
F	F	F	F	F	T

② <u>On Sets</u> : We use operations on statements to define operations on sets.

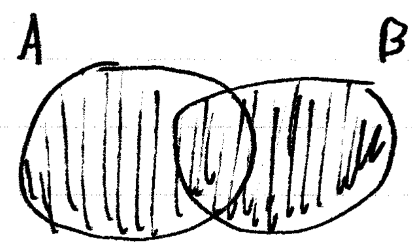
Let A, B be sets

1) $A \cap B = \{x \mid x \in A \wedge x \in B\}$ (intersection)
2) $A \cup B = \{x \mid x \in A \vee x \in B\}$ (union)
3) $A - B = \{x \mid x \in A \wedge x \notin B\}$ (difference).

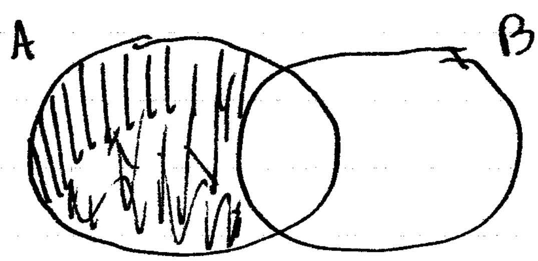
↳ <u>Venn Diagramms</u>



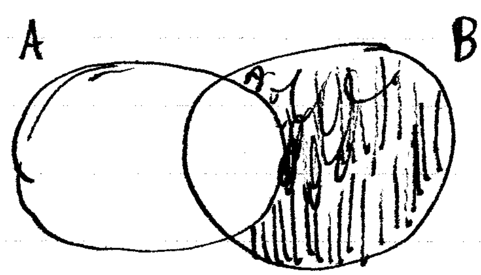
$A \cap B$



$A \cup B$



$A - B$



$B - A$

▶ <u>examples</u> : ...

Implication

① Simple implication $\to$ $p \to q$

$p \to q$: p implies q
if p is true then q is true
p is sufficient condition for q
q necessary condition for p.

▸ Definition

$$\boxed{p \to q \equiv \bar{p} \vee q}$$

p	q	$p \to q$
T	T	T
T	F	F !!
F	T	T
F	F	T

If p false we don't care if q is true or not.

② Double implication $\to$ $p \leftrightarrow q$

$p \leftrightarrow q$: p equivalent to q
p if and only if q
p necessary and sufficient condition for q.

▸ Definition :

$$\boxed{(p \leftrightarrow q) \equiv (p \to q) \wedge (q \to p)}$$

p	q	$p \to q$	$q \to p$	$p \leftrightarrow q$
T	T	T	T	T
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

Order of precedence

1) Negation $\bar{\ }$
2) $\wedge$, $\vee$, $\veebar$
3) $\to$, $\leftarrow$, $\leftrightarrow$

e.g. $p \wedge q \wedge r \to \bar{s} \veebar t$

means: $(p \wedge q \wedge r) \to (\bar{s} \veebar t)$

Reasoning

- A tautology is a compound statement which is always true

example : $(p \wedge q) \rightarrow p$ is a tautology.

p	q	$p \wedge q$	p	$(p \wedge q) \rightarrow p$
T	T	T	T	T
T	F	F	T	T
F	T	F	F	T
F	F	F	F	T

!!!

- If a statement of the form $P \rightarrow Q$ is a tautology, we write $P \Rightarrow Q$.
- If a statement of the form $P \leftrightarrow Q$ is a tautology, we write $P \Leftrightarrow Q$.

METHOD : Tautologies are RULES OF LOGIC that we may use to prove that other more complicated statements are also tautologies.

Rules of Logic

1) Identity $\quad p \Leftrightarrow p$

2) Double Negation $\quad p \Leftrightarrow \overline{\overline{p}}$

3) Exclusion $\quad p \underline{\vee} \bar{p}$

4) Contradiction $\quad \overline{p \wedge \bar{q}}$

5) Contrapositive $\quad (p \Rightarrow q) \Leftrightarrow (\bar{q} \Rightarrow \bar{p})$

6) Syllogism $\quad [(p \Rightarrow q) \wedge (q \Rightarrow r)] \Rightarrow (p \Rightarrow r)$

7) Modus ponens $\quad [p \wedge (p \Rightarrow q)] \Rightarrow q$

8) Modus tolens $\quad [(p \Rightarrow q) \wedge \bar{q}] \Rightarrow \bar{p}$

9) $\overline{p \vee q} \Leftrightarrow \bar{p} \wedge \bar{q}$ De Morgan

$\overline{p \wedge q} \Leftrightarrow \bar{p} \vee \bar{q}$

10) Idempotent : $\quad p \vee p \Leftrightarrow p$

$p \wedge p \Leftrightarrow p$

11) Commutative : $(p \vee q) \Leftrightarrow (q \vee p)$
$(p \wedge q) \Leftrightarrow (q \wedge p)$

12) Distributive : $[p \wedge (q \vee r)] \Leftrightarrow [(p \wedge q) \vee (p \wedge r)]$
$[p \vee (q \wedge r)] \Leftrightarrow [(p \vee q) \wedge (p \vee r)]$

13) Associative: $[(p \vee q) \vee r] \Leftrightarrow [p \vee (q \vee r)]$
$[(p \wedge q) \wedge r] \Leftrightarrow [p \wedge (q \wedge r)]$

Contrapositive

The fact that

$$(p \to q) \Leftrightarrow (\bar{q} \to \bar{p})$$

is a tautology means that:

$$\boxed{\begin{array}{l}\text{If} \quad P \Rightarrow Q \quad \text{is true} \\ \text{then} \quad \bar{Q} \Rightarrow \bar{P} \quad \text{is true.}\end{array}}$$ ← contrapositive statement.

Examples from algebra

1) $ab = 0 \Rightarrow a = 0 \vee b = 0$
   $a \neq 0 \wedge b \neq 0 \Rightarrow ab \neq 0$ ← contrapositive

2) $a^2 + b^2 = 0 \Rightarrow a = 0 \wedge b = 0$
   $a \neq 0 \vee b \neq 0 \Rightarrow a^2 + b^2 \neq 0.$

↳ Contrapositive is used usually in conjunction with De Morgan's rules

↳ The statement $Q \Rightarrow P$ is the converse of $P \Rightarrow Q$ and may or may not be true. If it is also true then we write $P \Leftrightarrow Q$.

<u>Methodology:</u> To show that a statement is a tautology

• 1 Eliminate $\Rightarrow$ and $\Leftrightarrow$ using the properties

$$(p \Rightarrow q) \equiv (\bar{p} \vee q)$$
$$(p \Leftrightarrow q) \equiv (p \Rightarrow q) \wedge (q \Rightarrow p)$$
$$\equiv (\bar{p} \vee q) \wedge (p \vee \bar{q})$$

• 2 Carry out negations using De Morgan rule

$$\overline{(p \vee q)} \equiv \bar{p} \wedge \bar{q}$$
$$\overline{(p \wedge q)} \equiv \bar{p} \vee \bar{q}$$

• 3 Use associative, distributive, commutative laws AND the following, to simplify:

$$p \vee F \equiv p \qquad p \wedge T \equiv p \qquad p \vee \bar{p} \equiv T$$
$$p \wedge F \equiv F \qquad p \vee T \equiv T \qquad p \wedge \bar{p} \equiv F$$

<u>example</u> : Show that

$$[p \wedge (p \to q)] \to q$$

is a tautology:

<u>Solution</u>

$$S \equiv [p \wedge (p \to q)] \to q \equiv \overline{[p \wedge (p \to q)]} \vee q$$

$$\equiv \overline{[p \wedge (\bar{p} \vee q)]} \vee q \equiv$$

$$\equiv \overline{[(p \wedge \bar{p}) \vee (p \wedge q)]} \vee q$$

$$\equiv \overline{[F \vee (p \wedge q)]} \vee q \equiv \overline{(p \wedge q)} \vee q$$

$$\equiv (\bar{p} \vee \bar{q}) \vee q \equiv \bar{p} \vee (\bar{q} \vee q) \equiv$$

$$\equiv \bar{p} \vee T \equiv T \quad \checkmark .$$

Application of tautologies to set theory.

The rules of logic can be used to derive corresponding rules for sets.

Method

To show a set identity

- 1 Use set definitions to consolidate expression to the form
$$\{x \mid p(x)\}$$
- 2 Use rules of logic to manipulate $p(x)$
- 3 Use set definitions in reverse to get to the other side.

example

1) Show $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

$$\begin{aligned}
A \cap (B \cup C) &= A \cap \{x \mid x \in B \vee x \in C\} \\
* \quad &= \{x \mid x \in A \wedge (x \in B \vee x \in C)\} \\
&= \{x \mid (x \in A \wedge x \in B) \vee (x \in A \wedge x \in C)\} \\
&= \{x \mid (x \in A \wedge x \in B)\} \cup \{x \mid x \in A \wedge x \in C\} \\
&= (A \cap B) \cup (A \cap C).
\end{aligned}$$

* key step: use Distributive Rule of Logic

2) Prove : $U-(A\cap B)=(U-A)\cup(U-B)$

↳ This follows from De-Morgan on $\overline{p\wedge q}$

$$
\begin{aligned}
U-(A\cap B) &= \{x\in U \mid x\notin A\cap B\} \\
&= \{x\in U \mid \overline{x\in A\cap B}\} \\
&= \{x\in U \mid \overline{x\in A \wedge x\in B}\} \\
&= \{x\in U \mid \overline{x\in A} \vee \overline{x\in B}\} \\
&= \{x\in U \mid x\notin A \vee x\notin B\} \\
&= \{x\in U \mid x\notin A\}\cup\{x\in U \mid x\notin B\} \\
&= (U-A)\cup(U-B).
\end{aligned}
$$

* De-Morgan.

Remark : The other De Morgan law for sets is

$$U-(A\cup B)=(U-A)\cap(U-B)$$

to be shown by you for homework.

Note that these rules work even when U is not a so-called "universal" set!!

Predicates

- A predicate $p(x)$ is a statement about x which may be true ~~or~~ or false depending on x.

- It is assumed that x comes from some "universal" set U.

  $\longrightarrow x \in U$

- The truth set of $p(x)$ is the set A of all $x \in U$ such that $p(x)$ is true. We write

$$A = \{x \mid p(x)\}.$$

- Recall that if

$$A = \{x \mid p(x)\} \text{ and } B = \{x \mid q(x)\}$$

then

$$A \cap B = \{x \mid p(x) \wedge q(x)\}$$
$$A \cup B = \{x \mid p(x) \vee q(x)\}$$

Quantifiers (i.e. Quantified statements)

$$\{x \mid p(x)\} = U \iff \boxed{\forall x \in U : p(x)}$$

→ For all $x \in U$, the statement $p(x)$ is true

$$\{x \mid p(x)\} \neq \emptyset \iff \boxed{\exists x \in U : p(x)}$$

→ There is at least one $x \in U$ such that $p(x)$ is true.

$$\boxed{\begin{array}{l} \forall \equiv \text{for all} \\ \exists \equiv \text{there exists} \end{array}}$$

← inverted A (All)
← inverted E (Exists)

- Negations

$$\boxed{\begin{array}{l} \overline{\forall x \in U : p(x)} \equiv \exists x \in U : \overline{p(x)} \\ \overline{\exists x \in U : p(x)} \equiv \forall x \in U : \overline{p(x)} \end{array}}$$

Examples with Quantifiers

1) The equation $2x+1=0$ has a real solution

$$\exists x \in \mathbb{R} : 2x+1=0$$

2) An integer n is odd if and only if there is another integer such that $n = 2k+1$

$$n \text{ odd} \Leftrightarrow \exists k \in \mathbb{Z} : n = 2k+1$$

3) For any nonzero number a, there is a number b such that $ab=1$

$$\forall a \in \mathbb{R}-\{0\} : \exists b \in \mathbb{R} : ab = 1$$

4) For any integers $a > b > 0$, there are integers q, r (quotient and remainder) such that $a = bq + r$.

$$\forall a, b \in \mathbb{N}-\{0\} : \exists q, r \in \mathbb{N} : a = bq + r$$

5) The natural number b divides a (notation: $b|a$) if and only if there is a natural number c such that $a = bc$

$$b|a \Leftrightarrow \exists c \in \mathbb{N} : a = bc$$

<u>Negation</u> : b does not divide a

$$\overline{(b|a)} \Leftrightarrow \overline{\exists c \in \mathbb{N} : a = bc}$$

$$\Leftrightarrow \forall c \in \mathbb{N} : \overline{a = bc}$$

$$\Leftrightarrow \forall c \in \mathbb{N} : a \neq bc$$

6) Definition of prime number

A natural number n is a prime number if and only if any number $p \neq 1$ and $p \neq n$ does not divide n.

$$n \in \mathbb{N} \text{ prime number} \Leftrightarrow \forall p \in \mathbb{N} - \{1, n\} : \overline{p|n}$$

$$\Leftrightarrow \forall p \in \mathbb{N} - \{1, n\} : \forall c \in \mathbb{N} : n \neq pc$$

Negation :

$n \in \mathbb{N}$ not a prime number $\Leftrightarrow$

$$\Leftrightarrow \overline{\forall p \in \mathbb{N} - \{1, n\} : \forall c \in \mathbb{N} : n \neq pc}$$

$$\Leftrightarrow \exists p \in \mathbb{N} - \{1, n\} : \overline{\forall c \in \mathbb{N} : n \neq pc}$$

$$\Leftrightarrow \exists p \in \mathbb{N} - \{1, n\} : \exists c \in \mathbb{N} : \overline{n \neq pc}$$

$$\Leftrightarrow \exists p \in \mathbb{N} - \{1, n\} : \exists c \in \mathbb{N} : n = pc$$

Application of Logic to set theory.

Set Relationships

1) Equality: $A = B \Leftrightarrow \forall x : [x \in A \Leftrightarrow x \in B]$

2) Subset: $A \subseteq B \Leftrightarrow \forall x : [x \in A \Rightarrow x \in B]$

3) Strict subset (also called "proper" subset) $A \subset B \Leftrightarrow A \subseteq B \wedge A \neq B$.

Properties:

1) $A \subseteq B \wedge B \subseteq A \Leftrightarrow A = B$
2) $A \subseteq A$
3) $A \subseteq B \wedge B \subseteq C \Rightarrow A \subseteq C$

Power set

If A is a set, then the set of all subsets of A is the power-set. $P(A)$

$$P(A) = \{ B \mid B \subseteq A \}.$$

Equivalently

$$B \in P(A) \Leftrightarrow B \subseteq A.$$

example

For $A = \{1,2,3\}$

$P(A) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{2,3\}, \{1,3\}, \{1,2,3\}\}$.

Ordered pairs and cartesian product

- An ordered pair is an expression (a,b) in which a is the first component ~~and~~ b is the second component

- Equality among ordered pairs is defined as follows:

$$\boxed{(a_1,b_1) = (a_2,b_2) \Leftrightarrow a_1 = a_2 \land b_1 = b_2}$$

- Let A, B be two sets. The cartesian product $A \times B$ is defined as

$$A \times B = \{(a,b) \mid a \in A \land b \in B\}$$

or ~~xe~~ $(a,b) \in A \times B \Leftrightarrow a \in A \land b \in B$.

• A mapping $\varphi: A \to B$ is a subset $\varphi \subseteq A \times B$ such that

$$\begin{cases} \forall (a_1, b_1), (a_2, b_2) \in \varphi : a_1 = a_2 \Rightarrow b_1 = b_2. \\ \forall a \in A : \exists b \in B : (a, b) \in \varphi \end{cases}$$

example : For $A = \{1, 2, 3\}$, $B = \{4, 5\}$

$\varphi_1 = \{(1,4), (3,5), (2,4)\}$ is a mapping $\varphi: A \to B$.

$\varphi_2 = \{\underbrace{(1,4), (1,5)}_{\substack{1=1 \text{ true but} \\ 4=5 \text{ false.}}}, (3,5)\}$ is NOT a mapping

$\varphi_3 = \{(1,4)\}$ is not a mapping.
bc. 2 and 3 are not mapped.

A mapping $\varphi: \mathbb{R} \to \mathbb{R}$ is a function

A mapping $\varphi: \mathbb{N} \to \mathbb{R}$ is a sequence

• A mapping $\varphi: A \to B$ is "1-1" if and only if

$$\forall (a_1, b_1), (a_2, b_2) \in \varphi : b_1 = b_2 \Rightarrow a_1 = a_2.$$

Cardinality.

- Let A be a finite set. (to be defined carefully later).
  The number of elements of A is the cardinality of A and it is written.

  - $n(A)$ or $|A|$.

examples : $|\{1,2,3,4,5\}| = 5$
$|\{1,\{2,3,4\},5\}| = 3$!!!

Properties of cardinality

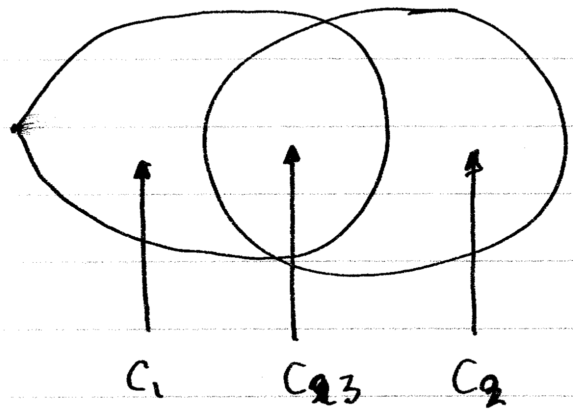
1) Fundamental counting principle

$$A \cap B = \emptyset \Rightarrow |A \cup B| = |A| + |B|$$

2) Exclusion-Inclusion principle

$$|A \cup B| = |A| + |B| - |A \cap B|$$

Proof



Let $C_1 = A - B$, $C_2 = B - A$, $C_3 = A \cap B$

Then

$C_1 \cap C_3 = \emptyset \Rightarrow |C_1 \cup C_3| = |C_1| + |C_3|$

$C_2 \cap C_3 = \emptyset \Rightarrow |C_2 \cup C_3| = |C_2| + |C_3|$

It follows that

$$
\begin{aligned}
|A \cup B| &= |C_1| + |C_2| + |C_3| \\
&= (|C_1| + |C_3|) + (|C_2| + |C_3|) - |C_3| \\
&= |C_1 \cup C_3| + |C_2 \cup C_3| - |C_3| \\
&= |A| + |B| - |A \cap B|.
\end{aligned}
$$

3) $\boxed{|A - B| = |A| - |A \cap B|}$

Proof

Using the same definitions for C_1, C_2, C_3

$|A - B| = |C_1| = (|C_1| + |C_3|) - |C_3| =$

$= |C_1 \cup C_3| - |C_3| = |A| - |A \cap B|.$

Combining 2 + 3 we get

4) $|A \cup B| = |B| + |A - B| = |A| + |B - A|$.

Proof

$$
\begin{aligned}
|A \cup B| &= |A| + |B| - |A \cap B| \\
&= |B| + (|A| - |A \cap B|) \\
&= |B| + |A - B|.
\end{aligned}
$$

Similarly

$$
\begin{aligned}
|A \cup B| &= |A| + |B| - |A \cap B| \\
&= |A| + (|B| - |B \cap A|) \\
&= |A| + |B - A|.
\end{aligned}
$$

5) $|A \times B| = |A| \cdot |B|$.