SYSTEMS OF EQUATIONS

Linear 2x2 systems

- To solve the linear system

$$\begin{cases} a_1 x + b_1 y = c_1 \\ a_2 x + b_2 y = c_2 \end{cases}$$

we calculate:

$$D = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1 b_2 - a_2 b_1$$

$$D_x = \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix} = c_1 b_2 - c_2 b_1$$

$$D_y = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} = a_1 c_2 - a_2 c_1$$

a) If $D \neq 0 \Rightarrow$ unique solution $\begin{cases} x = D_x / D \\ y = D_y / D \end{cases}$

b) If $D = 0$ and ($D_x \neq 0$ or $D_y \neq 0$), then the system is inconsistent.

c) Otherwise the system can be reduced to one equation or shown to be inconsistent.

EXAMPLES

a) $\begin{cases} 2x+3y=8 \\ 5x-2y=1 \end{cases}$

Solution

$$D=\begin{vmatrix} 2 & 3 \\ 5 & -2 \end{vmatrix}=2\cdot(-2)-3\cdot 5=-4-15=-19$$

$$D_x=\begin{vmatrix} 8 & 3 \\ 1 & -2 \end{vmatrix}=8\cdot(-2)-3\cdot 1=-16-3=-19$$

$$D_y=\begin{vmatrix} 2 & 8 \\ 5 & 1 \end{vmatrix}=2\cdot 1-5\cdot 8=2-40=-38$$

thus there is a unique solution:

$$\left.\begin{aligned} x&=\frac{D_x}{D}=\frac{-19}{-19}=1 \\ y&=\frac{D_y}{D}=\frac{-38}{-19}=2 \end{aligned}\right\} \Rightarrow S=\{(1,2)\}.$$

b) $\begin{cases} (a+1)x+(a-1)y=4a+2 \\ 2ax+(a-1)y=7a-1 \end{cases}$

Solution

$$D=\begin{vmatrix} a+1 & a-1 \\ 2a & a-1 \end{vmatrix}=(a+1)(a-1)-2a(a-1)=$$

$$=(a-1)(a+1-2a)=(a-1)(-a+1)=-(a-1)(a-1)$$

$$=-(a-1)^2.$$

Distinguish two cases:

<u>Case 1</u>: $D \neq 0 \Leftrightarrow -(a-1)^2 = 0 \Leftrightarrow a-1 \neq 0 \Leftrightarrow \underline{a \neq 1}$

$$D_x = \begin{vmatrix} 4a+2 & a-1 \\ 7a-1 & a-1 \end{vmatrix} = (4a+2)(a-1) - (7a-1)(a-1) =$$

$$= (a-1)\left[(4a+2) - (7a-1)\right] =$$

$$= (a-1)(4a+2-7a+1) = (a-1)(-3a+3) =$$

$$= -3(a-1)(a-1) = -3(a-1)^2.$$

and

$$D_y = \begin{vmatrix} a+1 & 4a+2 \\ 2a & 7a-1 \end{vmatrix} = (a+1)(7a-1) - 2a(4a+2) =$$

$$= 7a^2 - a + 7a - 1 - 8a^2 - 4a =$$

$$= (7-8)a^2 + (-1+7-4)a - 1 =$$

$$= -a^2 + 2a - 1 = -(a^2 - 2a + 1) = -(a-1)^2$$

thus we have a unique solution:

$$x = \frac{D_x}{D} = \frac{-3(a-1)^2}{-(a-1)^2} = 3$$

$$y = \frac{D_y}{D} = \frac{-(a-1)^2}{-(a-1)^2} = 1.$$

<u>Case 2</u>: $D = 0 \Leftrightarrow (a-1)^2 = 0 \Leftrightarrow a-1 = 0 \Leftrightarrow a = 1$

For $a=1$, the system reads

$$\begin{cases} (1+1)x + (1-1)y = 4\cdot 1 + 2 \\ 2\cdot 1\cdot x + (1-1)y = 7\cdot 1 - 1 \end{cases} \Leftrightarrow \begin{cases} 2x = 6 \\ 2x = 6 \end{cases} \Leftrightarrow 2x = 6$$

$\Leftrightarrow x = 3. \Leftrightarrow (x,y) = (3,y)$

Solution set $S = \{(3,y) \mid y \in \mathbb{R}\}$.

It follows that:

$$S = \begin{cases} \{(1,2)\} & , \text{ if } a \neq 1 \\ \{(3,y) \mid y \in \mathbb{R}\} & , \text{ if } a = 1. \end{cases}$$

EXERCISES

(1) Solve the following systems:

a) $\begin{cases} 5x - 7 = -y \\ 10x + 2y = 13 \end{cases}$ b) $\begin{cases} x - 2y = 1 \\ 3x + y = 0 \end{cases}$

c) $\begin{cases} 2x - 3y = 15 \\ -6x + 9y = -45 \end{cases}$

(2) Solve the systems with respect to x and y:

a) $\begin{cases} ax + (a+1)y = 3a+2 \\ 2x + (2a-1)y = 8 \end{cases}$ b) $\begin{cases} 2ax + ay = 4 \\ ax + (a-1)y = 2 \end{cases}$

c) $\begin{cases} 2ax + (a-3)y = a-1 \\ (a-3)x + 2ay = a - a^2 \end{cases}$

d) $\begin{cases} (a-1)x - y = a+1 \\ (8a+5)x + (a+5)y = -5 \end{cases}$

e) $\begin{cases} (a^2-1)x - (a-1)y = a \\ (a-1)^2 x + (a-1)y = a+1 \end{cases}$

▼ Linear nxn systems

Consider an nxn linear system of equations of the form

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \vdots \qquad \vdots \qquad \vdots \qquad \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n \end{cases}$$

The prefered method for solving this system is the method of determinants.

Definition of nxn determinants

- An $n \times n$ matrix $A \in M_n(\mathbb{R})$ is a collection of n^2 numbers $A_{ab} \in \mathbb{R}$ arranged in n rows and n columns as follows:

$$A = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{n1} & A_{n2} & \cdots & A_{nn} \end{bmatrix}$$

> Remember:
> A_{rc} : row, column
> A_{vh} : vertical, horizontal

- A_{ab} = element at row a and column b.

- For a 2×2 matrix $A \in M_2(\mathbb{R})$ with

$$A = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$$

we have defined the determinant of A as

$$\det A = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1 b_2 - a_2 b_1$$

For more general $n\times n$ matrices $A \in M_n(\mathbb{R})$ we define the determinant $\det A$ recursively as follows:

- Let $A \in M_n(\mathbb{R})$ be an $n\times n$ matrix. The <u>minor</u> $M_{ab}(A)$ is defined as an $(n-1)\times(n-1)$ matrix obtained from A by deleting
  a) The "a" row of A AND
  b) The "b" row of A.
  The determinant $\det A$ can then be expanded in terms of the determinants of the minors of A, using either:
  a) Expansion accross row "a"; for $a = 1, 2, \dots, n$

$$\boxed{\det A = \sum_{b=1}^{n} (-1)^{a+b} \det(M_{ab}(A))\, A_{ab}}$$

b) Expansion accross column "b" for $b = 1, 2, 3, \dots, n$

$$\boxed{\det A = \sum_{a=1}^{n} (-1)^{a+b} \det(M_{ab}(A))\, A_{ab}}$$

- Each expansion yields determinants of smaller matrices, so we keep expanding until we obtain 2x2 determinants.
- It can be shown that any one of the above expansions gives the same result.

EXAMPLES

- Definition of minors.

For $A = \begin{bmatrix} 2 & 4 & 3 & 1 \\ 1 & 5 & 7 & 2 \\ 3 & 1 & 5 & 2 \\ 1 & 4 & 7 & 3 \end{bmatrix} \Rightarrow$

Note that $A_{23} = 7$

$$\Rightarrow M_{23}(A) = \begin{bmatrix} 2 & 4 & 1 \\ 3 & 1 & 2 \\ 1 & 4 & 3 \end{bmatrix}$$

- Evaluation of 3x3 determinants

$$\begin{vmatrix} 3 & 1 & 2 \\ 1 & 5 & 1 \\ 2 & 3 & 1 \end{vmatrix} \rightarrow =$$

$$\boxed{\text{sign of } (-1)^{a+b} \longleftrightarrow \begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}}$$

$$= \underset{\uparrow\atop(-1)^{2+1}}{-1} \begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix} \underset{\uparrow\atop(-1)^{2+2}}{+5} \begin{vmatrix} 3 & 2 \\ 2 & 1 \end{vmatrix} \underset{\uparrow\atop(-1)^{2+3}}{-1} \begin{vmatrix} 3 & 1 \\ 2 & 3 \end{vmatrix} =$$

$$= -1(1\cdot 1 - 2\cdot 3) + 5(3\cdot 1 - 2\cdot 2) - 1(3\cdot 3 - 2\cdot 1) =$$

$$= -(1-6) + 5(3-4) - (9-2) =$$

$$= -(-5) + 5\cdot(-1) - 7 = 5 - 5 - 7 = -7$$

- Take advantage of zeroes.

$$\begin{vmatrix} 4 & 1 & 3 & 2 \\ 0 & 1 & 2 & 3 \\ 0 & 2 & 0 & 0 \\ 0 & 1 & 3 & 1 \end{vmatrix} = 4 \begin{vmatrix} 1 & 2 & 3 \\ 2 & 0 & 0 \\ 1 & 3 & 1 \end{vmatrix} \rightarrow =$$

$$= 4\cdot(-2) \begin{vmatrix} 2 & 3 \\ 3 & 1 \end{vmatrix} = 4(-2)(2\cdot 1 - 3\cdot 3)$$

$$= 4(-2)(2-9) = 4(-2)(-7) = (-8)(-7)$$

$$= 56.$$

Cramer's rule

Given the $n \times n$ linear system of equations

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n \end{cases}$$

we define the determinant D given by:

$$D = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

If $D \neq 0$, then the system has a unique solution that can be evaluated as follows:

- Let D_a be the determinants in which the "a" column of D is replaced with $b_1, b_2, \ldots, b_n$, so that

$$D_1 = \begin{vmatrix} b_1 & a_{12} & \cdots & a_{1n} \\ b_2 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_n & a_{n2} & \cdots & a_{nn} \end{vmatrix},$$

$$D_2 = \begin{vmatrix} a_{11} & b_1 & \cdots & a_{1n} \\ a_{21} & b_2 & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & b_n & \cdots & a_{nn} \end{vmatrix}, \ \ldots$$

and $$D_n = \begin{vmatrix} a_{11} & a_{12} & \cdots & b_1 \\ a_{21} & a_{22} & \cdots & b_2 \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & b_n \end{vmatrix}$$

•2 The unique solution is given by

$$\boxed{x_a = \frac{D_a}{D}}$$

↳ This method does not work when $D=0$.
For that case we use more advanced techiniques that you will learn in Linear Algebra.

EXAMPLE

Solve the system.

$$\begin{cases} 2x+y+z=4 \\ y+2z=2 \\ x-z=0 \end{cases}$$

Solution

We note that

$$\begin{cases} 2x+y+z=4 \\ y+2z=2 \\ x-z=0 \end{cases} \Leftrightarrow \begin{cases} 2x+1y+1z=4 \\ 0x+1y+2z=2 \\ 1x+0y-1z=0 \end{cases}$$

and also that

$$D=\begin{vmatrix} 2 & 1 & 1 \\ 0 & 1 & 2 \\ 1 & 0 & -1 \end{vmatrix} = 2\begin{vmatrix} 1 & 2 \\ 0 & -1 \end{vmatrix} - 0 + 1\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} =$$

$$= 2(1\cdot(-1)-0\cdot 2)+1(1\cdot 2-1\cdot 1)=$$

$= 2(-1)+1(2-1) = -2+1=-1 \neq 0 \Rightarrow$ the system has a unique solution.

Furthermore:

$$D_1=\begin{vmatrix} 4 & 1 & 1 \\ 2 & 1 & 2 \\ 0 & 0 & -1 \end{vmatrix} = (-1)\begin{vmatrix} 4 & 1 \\ 2 & 1 \end{vmatrix} =$$

$$= (-1)(4\cdot 1 - 2\cdot 1) = (-1)(4-2) = (-1)\cdot 2 = -2 \,,$$

and

$$D_2 = \begin{vmatrix} 2 & 4 & 1 \\ 0 & 2 & 2 \\ 1 & 0 & -1 \end{vmatrix} = 2 \begin{vmatrix} 2 & 2 \\ 0 & -1 \end{vmatrix} - 0 + 1 \begin{vmatrix} 4 & 1 \\ 2 & 2 \end{vmatrix} =$$

$$= 2(2\cdot(-1) - 0) + 1(4\cdot 2 - 2\cdot 1) = 2(-2) + 1(8-2) =$$

$$= -4 + 6 = 2 \,,$$

and

$$D_3 = \begin{vmatrix} 2 & 1 & 4 \\ 0 & 1 & 2 \\ 1 & 0 & 0 \end{vmatrix} = 1 \begin{vmatrix} 1 & 4 \\ 1 & 2 \end{vmatrix} = (1\cdot 2 - 1\cdot 4) =$$

$$= 2 - 4 = -2 .$$

It follows that

$$\left.\begin{aligned} x &= \frac{D_1}{D} = \frac{-2}{-1} = 2 \\ y &= \frac{D_2}{D} = \frac{2}{-1} = -2 \\ z &= \frac{D_3}{D} = \frac{-2}{-1} = 2 \end{aligned}\right\} \Rightarrow \underline{(x,y,z) = (2,-2,2).}$$

Thus $S = \{(2,2,-2)\}$

EXERCISES

(3) Solve the following linear systems of equations:

a) $\begin{cases} 4x - 2y + 3z = -2 \\ 2x + 2y + 5z = 16 \\ 8x - 5y - 2z = 4 \end{cases}$

b) $\begin{cases} x + 2y + 3z = -3 \\ -2x + y - z = 6 \\ 3x - 3y + 2z = -11 \end{cases}$

c) $\begin{cases} 14 + 3x + z = 4y - 2x \\ 2y = 10 + x + 2z \\ x + y + z = 1 - 2x \end{cases}$

d) $\begin{cases} 3(x+y+z) = 1 - 2z \\ 3(x+3z) = 2 - 5y \\ 5(x+2y) = 4 - 17z + y \end{cases}$

▼ 2nd order systems

1) Linear + Quadratic equation

Method: Solve the linear equation first and substitute the solution to the quadratic equation.

EXAMPLE

$$\begin{cases} 2x^2+xy-y^2=0 \\ x+3y=7 \end{cases}$$

Solution

$$\begin{cases} 2x^2+xy-y^2=0 \\ x+3y=7 \end{cases} \Leftrightarrow \begin{cases} 2x^2+xy-y^2=0 \\ x=7-3y \end{cases}$$

$$\Leftrightarrow \begin{cases} 2(7-3y)^2+(7-3y)y-y^2=0 \quad (1) \\ x=7-3y \end{cases}$$

We note that

$$(1) \Leftrightarrow 2(49-42y+9y^2)+(7-3y)y-y^2=0 \Leftrightarrow$$

$$\Leftrightarrow 98-84y+18y^2+7y-3y^2-y^2=0 \Leftrightarrow$$

$$\Leftrightarrow (18-3-1)y^2+(-84+7)y+98=0$$

$$\Leftrightarrow 14y^2-77y+98=0$$

$$\Delta=b^2-4ac=(-77)^2-4\cdot 14\cdot 98=5929-5488$$

$$=441=21^2 \Rightarrow$$

$$\Rightarrow y_1 = \frac{-b+\sqrt{\Delta}}{2a} = \frac{-(-77)+21}{2\cdot 14} = \frac{77+21}{2\cdot 14} =$$

$$= \frac{98}{2\cdot 14} = \frac{7}{2} \quad \text{and}$$

$$y_2 = \frac{-b-\sqrt{\Delta}}{2a} = \frac{-(-77)-21}{2\cdot 14} = \frac{77-21}{2\cdot 14} =$$

$$= \frac{56}{2\cdot 14} = \frac{4}{2} = 2.$$

It follows that the system gives:

$$\begin{cases} x = 7-3y \\ y = 2 \vee y = 7/2 \end{cases} \Leftrightarrow \begin{cases} x = 7-3y \\ y = 2 \end{cases} \vee \begin{cases} x = 7-3y \\ y = 7/2 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} x = 7-6 = 1 \\ y = 2 \end{cases} \vee \begin{cases} x = 7-21/2 = -7/2 \\ y = 7/2 \end{cases}$$

thus

$$S = \{(1,2), (-7/2, 7/2)\}.$$

EXERCISES

④ Solve the following systems.

a) $\begin{cases} 3x^2+4y^2+12x=7 \\ x+2y=3 \end{cases}$

b) $\begin{cases} 2x^2-3xy+5y^2=1 \\ 3x-2y=2 \end{cases}$

c) $\begin{cases} 2x^2+y^2=17 \\ 6x-4y=0 \end{cases}$

d) $\begin{cases} x^2+xy+2y^2=4 \\ x+3y=4. \end{cases}$

2) The Fundamental system

$$\boxed{\begin{cases} x+y=a \\ xy=b \end{cases} \Leftrightarrow \begin{cases} x=\rho_1 \\ y=\rho_2 \end{cases} \vee \begin{cases} x=\rho_2 \\ y=\rho_1 \end{cases}}$$

where ρ_1, ρ_2 are the zeroes of

$$\boxed{f(x) = x^2 - ax + b}$$

If $\rho_1 = \rho_2$, then the system has a unique solution $(x, y) = (\rho, \rho)$.

EXAMPLES

$$\begin{cases} x+y=5 \\ xy=6 \end{cases} \quad (1)$$

Solution

Let $f(x) = z^2 - 5z + 6$

$\Delta = b^2 - 4ac = (-5)^2 - 4 \cdot 1 \cdot 6 = 25 - 24 = 1 \Rightarrow$

$$\Rightarrow z_{1,2} = \frac{-b \pm \sqrt{\Delta}}{2a} = \frac{-(-5) \pm 1}{2 \cdot 1} = \frac{5 \pm 1}{2} =$$

$$= \begin{cases} 6/2 = 3 \\ 4/2 = 2 \end{cases}, \text{ therefore}$$

$$(1) \Leftrightarrow \begin{cases} x=3 \\ y=2 \end{cases} \vee \begin{cases} x=2 \\ y=3 \end{cases}, \text{ thus } S = \{(3,2), (2,3)\}$$

EXERCISES

(5) Solve the following systems

a) $\begin{cases} x+y=-5 \\ xy=4 \end{cases}$

d) $\begin{cases} x+y=1 \\ xy=1 \end{cases}$

b) $\begin{cases} x+y=3 \\ xy=2 \end{cases}$

e) $\begin{cases} x+y=7 \\ xy=-2 \end{cases}$

c) $\begin{cases} x+y=4 \\ xy=4 \end{cases}$

f) $\begin{cases} x+y=2 \\ xy=4 \end{cases}$

3) Symmetric systems

A symmetric system is a system of the form

$$\begin{cases} f_1(x,y) = 0 \\ f_2(x,y) = 0 \end{cases}$$

such that $f_1(x,y) = f_1(y,x)$ and
$f_2(x,y) = f_2(y,x)$.

Method: We use the Cauchy identities:

$$\boxed{\begin{aligned} a^2+b^2 &= (a+b)^2 - 2ab \\ a^3+b^3 &= (a+b)^3 - 3ab(a+b) \end{aligned}}$$

to rewrite the system in terms of $x+y$ and xy. Then let $a = x+y$ and $b = xy$ to solve for a, b. Then we solve the resulting fundamental systems to find x, y.

EXAMPLES

a) $\begin{cases} x^3+y^3 = 9 \\ xy(x+y) = 6 \end{cases}$

Solution

$$\begin{cases} x^3+y^3=9 \\ xy(x+y)=6 \end{cases} \Leftrightarrow \begin{cases} (x+y)^3-3xy(x+y)=9 \\ xy(x+y)=6 \end{cases} \quad (1)$$

Let $a=x+y$ and $b=xy$. Then

$$(1) \Leftrightarrow \begin{cases} a^3-3ab=9 \\ ab=6 \end{cases} \Leftrightarrow \begin{cases} a^3-3\cdot 6=9 \\ ab=6 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} a^3=18+9=27 \\ ab=6 \end{cases} \Leftrightarrow \begin{cases} a=3 \\ ab=6 \end{cases} \Leftrightarrow \begin{cases} a=3 \\ 3b=6 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} a=3 \\ b=2 \end{cases} \Leftrightarrow \begin{cases} x+y=3 \\ xy=2 \end{cases} \quad (2)$$

Let $f(z)=z^2-3z+2=(z-2)(z-1)=0 \Leftrightarrow$

$\Leftrightarrow z_1=2 \vee z_2=1$, thus

$$(2) \Leftrightarrow \begin{cases} x=2 \\ y=1 \end{cases} \vee \begin{cases} x=1 \\ y=2 \end{cases}.$$

b) $\begin{cases} 3x^2+3y^2-xy=33 \\ x^2+y^2+xy=19 \end{cases}$

Solution

$$\begin{cases} 3x^2+3y^2-xy=33 \\ x^2+y^2+xy=19 \end{cases} \Leftrightarrow \begin{cases} 3(x^2+y^2)-xy=33 \\ x^2+y^2+xy=19 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} 3[(x+y)^2-2xy]-xy=33 \\ (x+y)^2-2xy+xy=19 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} 3(x+y)^2 - 6xy - xy = 33 \\ (x+y)^2 - xy = 19 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} 3(x+y)^2 - 7xy = 33 \\ (x+y)^2 - xy = 19 \end{cases} \qquad (1)$$

Let $a = (x+y)^2$ and $b = xy$. Then

$$(1) \Leftrightarrow \begin{cases} 3a - 7b = 33 \\ a - b = 19 \end{cases} \Leftrightarrow \begin{cases} 3a - 7b = 33 \\ -3a + 3b = -57 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} a - b = 19 \\ -4b = -24 \end{cases} \Leftrightarrow \begin{cases} a - 6 = 19 \\ b = 6 \end{cases} \Leftrightarrow \begin{cases} a = 25 \\ b = 6 \end{cases}$$

$$\Leftrightarrow \begin{cases} (x+y)^2 = 25 \\ xy = 6 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} x+y = 5 \\ xy = 6 \end{cases} \vee \begin{cases} x+y = -5 \\ xy = 6 \end{cases} \qquad (2)$$

Since $f_1(z) = z^2 - 5z + 6 = (z-2)(z-3) = 0 \Leftrightarrow$

$\Leftrightarrow z = 2 \vee z = 3$ and

$f_2(z) = z^2 + 5z + 6 = (z+2)(z+3) = 0 \Leftrightarrow z = -2 \vee z = -3$

it follows that

$$(2) \Leftrightarrow \begin{cases} x = 2 \\ y = 3 \end{cases} \vee \begin{cases} x = 3 \\ y = 2 \end{cases} \vee \begin{cases} x = -2 \\ y = -3 \end{cases} \vee \begin{cases} x = -3 \\ y = -2. \end{cases}$$

EXERCISES

⑥ Solve the following systems

a) $\begin{cases} x^2+y^2=17 \\ xy=14 \end{cases}$ b) $\begin{cases} x+y+xy=23 \\ xy(x+y)=126 \end{cases}$

c) $\begin{cases} x^2+y^2+x+y=44 \\ 3(x^2+y^2)-4xy=87 \end{cases}$ d) $\begin{cases} x+y=1 \\ \dfrac{1}{x}+\dfrac{1}{y}=-\dfrac{1}{6} \end{cases}$

e) $\begin{cases} x+y=13 \\ \dfrac{x}{y}+\dfrac{y}{x}=\dfrac{97}{36} \end{cases}$ f) $\begin{cases} 2x^2+2y^2-xy=32 \\ x^2+y^2+3xy=44 \end{cases}$

⑦ Solve the following systems:

a) $\begin{cases} x^3+y^3=35 \\ x+y=5 \end{cases}$ b) $\begin{cases} x+xy+y=11 \\ x^2y+xy^2=30 \end{cases}$

c) $\begin{cases} x^3+y^3=7 \\ xy(x+y)=-2 \end{cases}$ d) $\begin{cases} (x+y)xy=30 \\ (x+y)(x^2+y^2)=65 \end{cases}$

→ The following systems become symmetric after a change of variables.

⑧ Solve the following systems

a) $\begin{cases} x+y^2=7 \\ xy^2=12 \end{cases}$

b) $\begin{cases} x^2-y=23 \\ x^2y=50 \end{cases}$

c) $\begin{cases} x^2+y^2=(5/2)xy \\ x-y=(1/4)xy \end{cases}$

d) $\begin{cases} x^2-xy+y^2=7 \\ x-y=1. \end{cases}$

4) Homogeneous systems

A homogeneous 2nd-order system is a system of the form

$$\begin{cases} a_1 x^2 + b_1 xy + c_1 y^2 = d_1 \\ a_2 x^2 + b_2 xy + c_2 y^2 = d_2 \end{cases}$$

with $|d_1| + |d_2| \neq 0$. To solve this system:

- 1 Examine if it has solutions $(0,k)$ and $(k,0)$
- 2 Now assume $xy \neq 0$. Define $y = \lambda x$
- 3 Rewrite:

$$a_1 x^2 + b_1 xy + c_1 y^2 = d_1 \Leftrightarrow$$
$$x^2 (a_1 + b_1 \lambda + c_2 \lambda^2) = d_1 \Leftrightarrow$$
$$x^2 = \frac{d_1}{a_1 + b_1 \lambda + c_1 \lambda^2}$$

and similarly

$$a_2 x^2 + b_2 xy + c_2 y^2 = d_2 \Leftrightarrow \cdots$$

$$\Leftrightarrow x^2 = \frac{d_2}{a_2 + b_2 \lambda + c_2 \lambda^2}$$

- 4 Solve for λ:

$$\frac{d_1}{a_1 + b_1 \lambda + c_1 \lambda^2} = \frac{d_2}{a_2 + b_2 \lambda + c_2 \lambda^2}$$

EXAMPLE

$$\begin{cases} x^2+xy+y^2=19 \\ 2x^2+3xy-y^2=17 \end{cases} \quad (1)$$

Solution

Case 1 : For $x=0$:

$$(1) \Leftrightarrow \begin{cases} y^2 = 19 \\ -y^2 = 17 \end{cases} \leftarrow \text{inconsistent.}$$

Case 2 : For $y=0$:

$$(1) \Leftrightarrow \begin{cases} x^2 = 19 \\ 2x^2 = 17 \end{cases} \Leftrightarrow \begin{cases} x^2 = 19 \\ x^2 = 17/2 \end{cases} \leftarrow \text{inconsistent.}$$

Case 3 : For $xy \neq 0$. Let $y=ax$.
We note that

$$x^2+xy+y^2=19 \Leftrightarrow x^2(1+a+a^2)=19 \Leftrightarrow$$

$$\Leftrightarrow x^2 = \frac{19}{1+a+a^2} \quad \text{and}$$

$$2x^2+3xy-y^2=17 \Leftrightarrow x^2(2+3a-a^2)=17 \Leftrightarrow$$

$$\Leftrightarrow x^2 = \frac{17}{2+3a-a^2}$$

Solve:

$$\frac{19}{1+a+a^2} = \frac{17}{2+3a-a^2} \Leftrightarrow$$

$$\Leftrightarrow 19(2+3a-a^2) - 17(1+a+a^2) = 0 \Leftrightarrow$$

$$\Leftrightarrow 38 + 57a - 19a^2 - 17 - 17a - 17a^2 = 0 \Leftrightarrow$$

$$\Leftrightarrow (-19-17)a^2 + (57-17)a + (38-17) = 0 \Leftrightarrow$$

$$\Leftrightarrow -36a^2 + 40a + 21 = 0 \Leftrightarrow$$

$$\Leftrightarrow 36a^2 - 40a - 21 = 0.$$

$$\Delta = b^2 - 4ac = (-40)^2 - 4 \cdot 36 \cdot (-21) =$$

$$= 1600 + 3024 = 4624 = 68^2 \Rightarrow$$

$$\Rightarrow \alpha_1 = \frac{-b+\sqrt{\Delta}}{2a} = \frac{-(-40)+68}{2 \cdot 36} = \frac{40+68}{72} =$$

$$= \frac{108}{72} = \frac{3}{2} \quad \text{and}$$

$$\alpha_2 = \frac{-b-\sqrt{\Delta}}{2a} = \frac{-(-40)-68}{2 \cdot 36} = \frac{40-68}{72} =$$

$$= \frac{-28}{72} = \frac{-7}{18}$$

It follows that:

$$(1) \Leftrightarrow \begin{cases} x^2+xy+y^2 = 19 \\ y = (3/2)x \end{cases} \vee \begin{cases} x^2+xy+y^2 = 19 \\ y = -(7/18)x. \end{cases} \quad (2)$$

We note that; for $y = (3/2)x$:

$$x^2+xy+y^2 = 19 \Leftrightarrow x^2 + (3/2)x^2 + (9/4)x^2 = 19$$

$\Leftrightarrow 4x^2 + 6x^2 + 9x^2 = 19 \Leftrightarrow 19x^2 = 19 \Leftrightarrow x^2 = 1$

and for $y = -(7/18)x$:

$x^2 + xy + y^2 = 19 \Leftrightarrow x^2 - (7/18)x^2 + (7/18)^2 x^2 = 19$

$\Leftrightarrow 18^2 x^2 - 7 \cdot 18 x^2 + 7^2 x^2 = 19 \cdot 18^2 \Leftrightarrow$

$\Leftrightarrow 324x^2 - 126x^2 + 49x^2 = 6156 \Leftrightarrow$

$\Leftrightarrow 247x^2 = 6156 \Leftrightarrow 13x^2 = 324 \Leftrightarrow 13x^2 = 18^2$

and therefore:

$$(2) \Leftrightarrow \begin{cases} x^2 = 1 \\ y = (3/2)x \end{cases} \vee \begin{cases} 13x^2 = 18^2 \\ y = -(7/18)x \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} x = 1 \\ y = 3/2 \end{cases} \vee \begin{cases} x = -1 \\ y = -3/2 \end{cases} \vee \begin{cases} x = 18/\sqrt{13} \\ y = -7/\sqrt{13} \end{cases} \vee \begin{cases} x = -18/\sqrt{13} \\ y = 7/\sqrt{13} \end{cases}$$

It follows that

$S = \{(1, 3/2), (-1, -3/2), (18/\sqrt{13}, -7/\sqrt{13}), (-18/\sqrt{13}, 7/\sqrt{13})\}$.

EXERCISES

(9) Solve the following systems

a) $\begin{cases} x^2+2xy-y^2=1 \\ 2x^2-xy+3y^2=12 \end{cases}$

b) $\begin{cases} x^2-xy+y^2=1 \\ 3x^2-2xy-2y^2=-3 \end{cases}$

c) $\begin{cases} 2x^2+3xy+5y^2=8 \\ 4x^2-7xy+10y^2=16 \end{cases}$