EXPONENTIAL AND LOGARITHMIC FUNCTIONS

Definition of powers

- First we recall the following definitions of number sets:

$\mathbb{N} = \{0, 1, 2, 3, \ldots\}$	Natural numbers
$\mathbb{Z} = \{0, \pm 1, \pm 2, \pm 3, \ldots\}$	Integers
$\mathbb{Q} = \{a/b \mid a \in \mathbb{Z} \wedge b \in \mathbb{N} - \{0\}\}$	Rational numbers
$\mathbb{R}$ = set of real numbers.	Real numbers

- Let $a \in \mathbb{R}$. We give the following incremental definitions of powers:

1) Integer powers $\mapsto$ Let $n \in \mathbb{N}$. Then we define

$$\boxed{a^n = \begin{cases} 1 & , n = 0 \\ \underbrace{a \cdot a \cdot a \cdot \ldots \cdot a}_{n \text{ times}} & , n > 0 \end{cases}}$$

$$\boxed{a^{-n} = \frac{1}{a^n}, \quad \text{for } a \neq 0}$$

example: $(-2)^3 = (-2)(-2)(-2) = -8$

$$\left(\frac{-1}{2}\right)^{-2} = \frac{1}{\left(\frac{-1}{2}\right)^2} = \frac{1}{\left(\frac{-1}{2}\right)\left(\frac{-1}{2}\right)} =$$

$$= \frac{1}{\left(\frac{1}{4}\right)} = 4$$

$3^0 = 1$, 0^0 undefined.

2) Rational powers

- First, recall the definition of roots. Let $n \in \mathbb{N} - \{0\}$. Then, we define:

$$\boxed{\begin{array}{ll} x = \sqrt[2n]{a} \iff x^{2n} = a \wedge x > 0 \\ x = \sqrt[2n+1]{a} \iff x^{2n+1} = a \end{array}} \begin{array}{l} \leftarrow a \in (0, +\infty) \\ \leftarrow a \in \mathbb{R}. \end{array}$$

- Note that the equation $x^{2n} = a$ has two solutions and, by convention, we choose the positive solution. The equation $x^{2n+1} = a$ has a unique solution.

examples : $\sqrt{9} = 3$, because $3^2 = 9$
$\sqrt[3]{-8} = 2$, because $(-2)^3 = 8$.

- Let $a \in (0, +\infty)$, $p \in \mathbb{Z}$, and $q \in \mathbb{N} - \{0, 1\}$. Then we define:

$$\boxed{a^{p/q} = (\sqrt[q]{a})^p , \forall a \in (0, +\infty)}$$

<u>example</u> : $4^{3/2} = (\sqrt{4})^3 = 2^3 = 8$

$$27^{5/3} = (\sqrt[3]{27})^5 = 3^5 = 243$$

3) Real powers

Let $x \in \mathbb{R}$ and let $x_1, x_2, x_3, \ldots \in \mathbb{Q}$ be a rational sequence approximating x. We indicate that by writing

$$x = \lim (x_n)$$

Then we define:

$$a^x = \lim (a^{x_n}).$$

<u>example</u>

To approximate $2^{\sqrt{3}}$, we note that

$$\sqrt{3} \approx 1.7320508075\ldots$$

and therefore define $2^{\sqrt{3}}$ via the following sequence of approximations:

$$2^{1.7} = 3.249009585\ldots$$

$$2^{1.73} = 3.317278183\ldots$$

$$2^{1.732} = 3.321880096\ldots$$

$$2^{1.73205} = 3.321995226\ldots$$

Properties of powers

- Let $a, b \in (0, +\infty)$ and $x_1, x_2, x \in \mathbb{R}$. It can be shown that:

$a^x > 0$	$(a^{x_1})^{x_2} = a^{x_1 x_2}$
$a^{x_1} a^{x_2} = a^{x_1 + x_2}$	$(ab)^x = a^x b^x$
$\dfrac{a^{x_1}}{a^{x_2}} = a^{x_1 - x_2}$	$\left(\dfrac{a}{b}\right)^x = \dfrac{a^x}{b^x}$

- To compare a^x with b^x:

$\left.\begin{array}{l} a > b \\ x > 0 \end{array}\right\} \Rightarrow a^x > b^x$	$\left.\begin{array}{l} a > b \\ x < 0 \end{array}\right\} \Rightarrow a^x < b^x$

- To compare a^{x_1} with a^{x_2}:

$\left.\begin{array}{l} x_1 < x_2 \\ a > 1 \end{array}\right\} \Rightarrow a^{x_1} < a^{x_2}$	$\left.\begin{array}{l} x_1 < x_2 \\ 0 < a < 1 \end{array}\right\} \Rightarrow a^{x_1} > a^{x_2}$

EXAMPLES

a) Simplify:

$$\left[\left(\sqrt{5\sqrt{5}}\right)^{-1/3}\right]^{6} = \left(\sqrt{5\sqrt{5}}\right)^{(-1/3)\cdot 6} = \left(\sqrt{5\sqrt{5}}\right)^{-2}$$

$$= \frac{1}{\left(\sqrt{5\sqrt{5}}\right)^{2}} = \frac{1}{5\sqrt{5}} = \frac{\sqrt{5}}{5\sqrt{5}\sqrt{5}} = \frac{\sqrt{5}}{5\cdot 5}$$

$$= \frac{\sqrt{5}}{25}$$

b) Simplify:

$$\left(\frac{1}{2}\right)^{-2/3}\left(\frac{1}{4}\right)^{-2/3} = \left(\frac{1}{2}\,\frac{1}{4}\right)^{-2/3} = \left(\frac{1}{8}\right)^{-2/3} =$$

$$= 8^{2/3} = \left(\sqrt[3]{8}\right)^{2} = 2^{2} = 4.$$

c) Compare $(5/3)^{-1/2}$ with 1.

<u>Solution</u>

$$\left.\begin{array}{l} 5/3 > 1 \\ -1/2 < 0 \end{array}\right\} \Rightarrow (5/3)^{-1/2} < 1^{-1/2} \Rightarrow (5/3)^{-1/2} < 1.$$

d) Compare $(1/3)^{-2/3}$ with $(1/3)^{-4/5}$

<u>Solution</u>

$$\left.\begin{array}{l} 2/3 < 4/5 \Rightarrow -2/3 > -4/5 \\ 0 < 1/3 < 1 \end{array}\right\} \Rightarrow$$

$$\Rightarrow (1/3)^{-2/3} < (1/3)^{-4/5}.$$

e) Compare $(\sqrt{7})^{\sqrt{3}}$ with $(\sqrt{5})^{\sqrt{2}}$.

<u>Solution</u>

$$\left.\begin{array}{l} 7>5 \Rightarrow \sqrt{7} > \sqrt{5} \\ \sqrt{3} > 0 \end{array}\right\} \Rightarrow (\sqrt{7})^{\sqrt{3}} > (\sqrt{5})^{\sqrt{3}} \quad (1)$$

$$\left.\begin{array}{l} 3>2 \Rightarrow \sqrt{3} > \sqrt{2} \\ \sqrt{5} > 1 \end{array}\right\} \Rightarrow (\sqrt{5})^{\sqrt{3}} > (\sqrt{5})^{\sqrt{2}} \quad (2)$$

From (1) and (2): $(\sqrt{7})^{\sqrt{3}} > (\sqrt{5})^{\sqrt{2}}$.

↳ Note that $1^x = 1$, $\forall x \in \mathbb{R}-\{0\}$ and

$$\left.\begin{array}{l} a>b \Rightarrow a^{1/2} > b^{1/2} \\ 1/2 > 0 \end{array}\right\} \Rightarrow \sqrt{a} > \sqrt{b}$$

f) Compare $(2/3)^{3/4}$ with $(3/4)^{2/3}$.

<u>Solution</u>

$$\left.\begin{array}{l} 3/4 > 2/3 \\ 0 < 2/3 < 1 \end{array}\right\} \Rightarrow (2/3)^{3/4} < (2/3)^{2/3} \quad (1)$$

$$\left.\begin{array}{l} 2/3 < 3/4 \\ 2/3 > 0 \end{array}\right\} \Rightarrow (2/3)^{2/3} < (3/4)^{2/3} \quad (2)$$

From (1) and (2): $(2/3)^{3/4} < (3/4)^{2/3}$.

The power function

Let $f(x) = a^x$ with $a > 0$

Domain: $A = \mathbb{R}$

Range: $f(A) = (0, +\infty)$

Monotonicity: $a > 1 \Rightarrow f \nearrow \mathbb{R}$

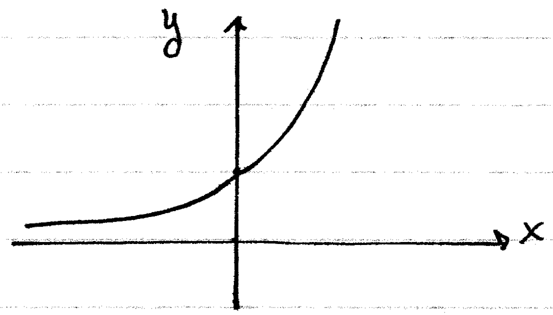
$a = 1 \Rightarrow f$ constant in $\mathbb{R}$

$0 < a < 1 \Rightarrow f \searrow \mathbb{R}$

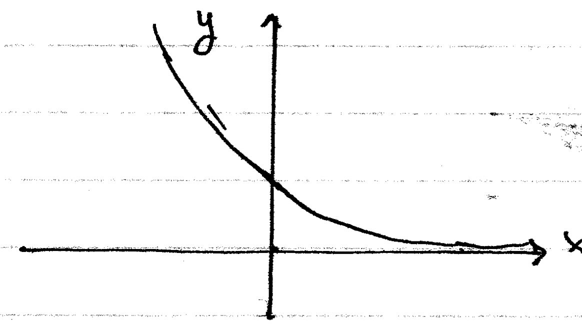
One-to-one: $a \neq 1 \Rightarrow f$ one-to-one

thus: $a^{x_1} = a^{x_2} \Leftrightarrow x_1 = x_2$ (for $a \neq 1$)

Graph: Passes through $(0,1)$.



$a > 1$



$0 < a < 1$

EXAMPLES

a) Find all $a \in \mathbb{R}$ such that the function
$f(x) = (3a+2)^x$ is decreasing in $\mathbb{R}$.

Solution

$$f \searrow \mathbb{R} \Leftrightarrow 0 < 3a+2 < 1 \Leftrightarrow \begin{cases} 3a+2 < 1 \\ 3a+2 > 0 \end{cases} \Leftrightarrow \begin{cases} 3a < -1 \\ 3a > -2 \end{cases}$$

$$\Leftrightarrow \begin{cases} a < -1/3 \\ a > -2/3 \end{cases} \Leftrightarrow -2/3 < a < -1/3 \Leftrightarrow$$

$$\Leftrightarrow a \in (-2/3, -1/3)$$

b) Find the default domain to the function
$f(x) = (x^3 - 4x)^{2x+1}$

Solution

We require that

$$x^3 - 4x > 0 \Leftrightarrow x(x^2-4) > 0 \Leftrightarrow x(x-2)(x+2) > 0 \quad (1)$$

x		-2		0		2	
x	$-$		$-$	0	$+$		$+$
$x-2$	$-$		$-$		$-$	0	$+$
$x+2$	$-$	0	$+$		$+$		$+$
ineq	$-$	0	$+$	0	$-$	0	$+$

therefore:

$(1) \Leftrightarrow x \in (-2,0) \cup (2,+\infty)$

It follows that

$$\mathrm{dom}(f) = (-2,0) \cup (2,+\infty)$$

In general, for any function $f(x) = a(x)^{b(x)}$ we have to require $\underline{a(x) > 0}$ in addition to any other requirements that may be necessary to evaluate $a(x)$ and $b(x)$.

EXERCISES

① Simplify the following arithmetic expressions, using root notation

a) $\dfrac{2\cdot 2^{-3}}{\sqrt{2}}$ b) $\dfrac{2^{1/2}\,5^{1/2}}{\sqrt{10}}$

c) $\left(\frac{1}{2}\right)^{-3}\left(\frac{1}{3}\right)^{-2}$ d) $5^{-2}\cdot 2^{-5}$

e) $4^{1/5}\cdot 8^{1/5}$ (show it equals 2)

f) $\left[\left(\sqrt{2}\right)^{-\sqrt{2}}\right]^{\sqrt{2}}$ (show it equals 1/2)

g) $\left\{\left[\left(\frac{2}{3}\right)^{3/2}\right]^{1/3}\right\}^{2}$

h) $\dfrac{\sqrt{2\sqrt{2}}}{\sqrt{2}}$ i) $\left[\left(\sqrt{3\sqrt{3}}\right)^{-1/2}\right]^{8}$

② For what values of $a\in\mathbb{R}$ are the following functions increasing in $\mathbb{R}$? decreasing in $\mathbb{R}$?

a) $f(x) = \left(\dfrac{a+1}{a-1}\right)^x$ b) $f(x) = [a(a+2)]^x$

c) $f(x) = \left(\dfrac{a^2}{a+1}\right)^x$

③ Compare the following numbers with 1

a) $(2/5)^{2/3}$ b) $(3/2)^{2/3}$ c) $(\sqrt{2})^{-3/2}$

d) $(1/3)^{-\sqrt{2}/2}$ e) $(5/4)^{-1/3}$ f) $(\sqrt{3})^{-\sqrt{2}}$

g) $(2-\sqrt{2})^{\sqrt{2}-1}$ h) $(\sqrt{2})^{1-\sqrt{3}}$

④ Compare the following numbers with each other:

a) $(3/5)^{2/3}$, $(3/5)^{3/4}$
b) $(4/3)^{1/2}$, $(4/3)^{1/3}$
c) $(2/5)^{-2/3}$, $(2/5)^{-3/4}$
d) $(\sqrt{2})^{\sqrt{2}}$, $(\sqrt{2})^{\sqrt{3}}$
e) $(1/2)^{1/3}$, $(1/3)^{1/4}$
f) $(1/3)^{1/2}$, $(1/4)^{1/3}$
g) $(\sqrt{3})^{-\sqrt{2}}$, $\dfrac{1}{\sqrt{2}}$
h) $(\sqrt{5})^{\sqrt{3}}$, $2^{\sqrt{2}}$

(e)–(h): harder.

(5) Find the default domain of the following functions:

a) $f(x) = (3x^2 - 10x + 3)^{2x+1}$

b) $f(x) = (x^3 - 2x^2 + 1)^x$

c) $f(x) = \left[\dfrac{x+1}{x-1}\right]^x$

* d) $f(x) = (1/x)^{1/x}$

* e) $f(x) = (x+1)^{1/(x+2)}$

* f) $f(x) = (1-x^2)^{1/x}$

↳ To find the domain of $f(x) = a(x)^{b(x)}$:

$$\boxed{\mathrm{dom}(f) = \mathrm{dom}(a) \cap \mathrm{dom}(b) \cap \{x \in \mathbb{R} \mid a(x) > 0\}}$$

The exponential function

- Let $a \in \mathbb{R}$ be a variable with $a \neq 0$.
  A simple <u>compounding</u> of a with rate r gives
  $$a_1 = (1+r)a$$
  Compounding n times at rate r/n gives:
  $$a_n = \left(1+\frac{r}{n}\right)^n a$$
  The sequence $a_1, a_2, a_3, \ldots$ approximates a number a_∞:
  $$a_\infty = a \cdot \lim_{n\to+\infty} \left(1+\frac{r}{n}\right)^n$$
  Thus we are motivated to define the exponential function
  $$\boxed{\exp(x) = \lim_{n\to+\infty}\left(1+\frac{x}{n}\right)^n}$$

- It can be shown that
  $$\boxed{\forall x \in \mathbb{R} : \exp(x) = e^x}$$

with $e \approx 2.718281828459$

- For $f(x) = \exp(x)$.
  Domain : $A = \mathbb{R}$
  Range : $f(A) = (0, +\infty)$
  Monotonicity : $f \nearrow \mathbb{R}$

- It can also be shown that

$$\boxed{e^x > 1 + x \quad , \forall x \in \mathbb{R}}$$

↳ <u>Method</u> : Range of exponential / power functions.

The domain of such functions is usually $A = \mathbb{R}$. Thus after we find the solvability set S for the equation $y = f(x)$, we can then claim that $f(A) = S$.
Note that we use:

$$\boxed{a^{b(y)} = c(y) \text{ has a solution } \Leftrightarrow \underline{c(y) > 0}}$$

with $b(y) = Ay + B$ a linear function.

EXAMPLE

a) $f(x) = 3^{1-2x} - 2$

The equation
$y = f(x) \Leftrightarrow y = 3^{1-2x} - 2 \Leftrightarrow 3^{1-2x} = y+2$

has a unique solution $\Leftrightarrow y+2 > 0$
$\Leftrightarrow y > -2$
Thus $S = [-2, +\infty)$
Since $A = \mathbb{R} \Rightarrow f(A) = S = [-2, +\infty)$.

→ <u>Method:</u> Monotonicity

Usually, the best approach is to work with the definition of monotonicity.

EXAMPLE

$f(x) = 3^{1-2x} - 2 \leftarrow A = \mathbb{R}$

Let $x_1, x_2 \in \mathbb{R}$. with $x_1 < x_2$.

$x_1 < x_2 \Rightarrow -2x_1 > -2x_2 \Rightarrow 1-2x_1 > 1-2x_2$

$\underset{(*)}{\Rightarrow} 3^{1-2x_1} > 3^{1-2x_2}$ (because $3 > 1$)

$\Rightarrow 3^{1-2x_1} - 2 > 3^{1-2x_2} - 2 \Rightarrow f(x_1) > f(x_2)$

Thus $f \downarrow \mathbb{R}$.

EXERCISES

(6) Find the range and monotonicity for the following functions.

a) $f(x) = 2 - 5^{3-2x}$

b) $f(x) = 3^{x-1} + 1$

c) $f(x) = \left(\frac{1}{2}\right)^{1-2x} - 3$

d) $f(x) = 2e^{1-x} - 1$

e) $f(x) = \left(\frac{1}{2e}\right)^{x-2} + e$

f) $f(x) = e^{-x^2}$

g) $f(x) = \exp(x^2 - 5x + 6)$

$\left.\begin{matrix} f) \\ g) \end{matrix}\right\}$ Monotonicity only!

h) $f(x) = 3e - e^{e-x}$

i) $f(x) = (1-e)e^{-x+1}$

j) $f(x) = 5^x + 5^{x+1}$

k) $f(x) = \left(\frac{1}{3}\right)^{1-x} - \left(\frac{1}{3}\right)^{2-x}$

$\left.\begin{matrix} j) \\ k) \end{matrix}\right\}$ Use factoring of 5^x or $(1/3)^{-x}$.

V Logarithmic function

- Consider the function $f(x)=a^x$ with $a\in(0,+\infty)-\{1\}$. We know that f is then one-to-one, consequently the inverse f^{-1} is also a function with the same monotonicity as f.
  We call f^{-1} the logarithmic function with base a:

$$\log_a = f^{-1} \quad \text{with} \quad \boxed{\log_a : (0,+\infty) \longrightarrow \mathbb{R}}$$

Thus: $$\boxed{y=\log_a x \Leftrightarrow a^y = x}$$

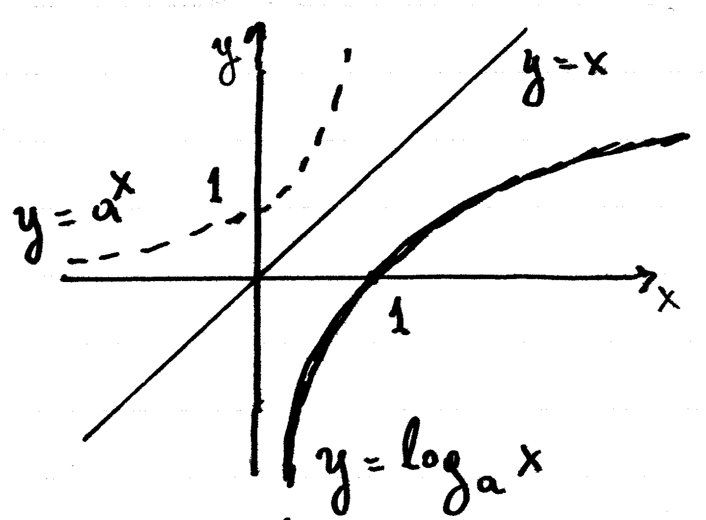
- Immediate consequences of the definition:

$\log_a 1 = 0$	$\log_a a = 1$
$a^{\log_a x} = x$	$\log_a a^x = x$

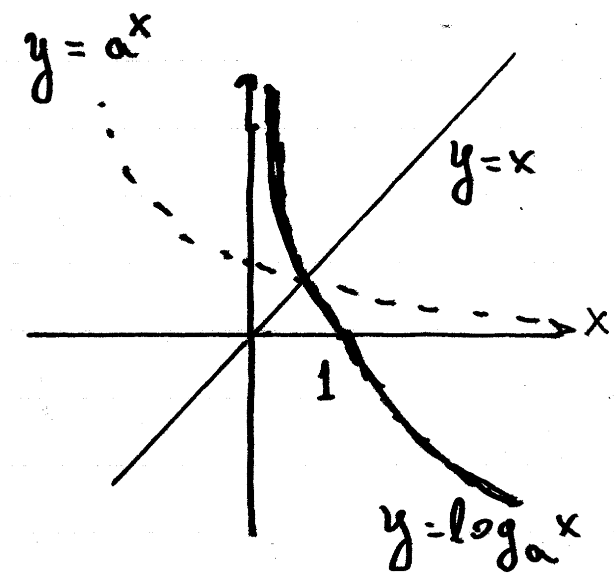
Properties of logarithmic function

For $f(x) = \log_a(x)$

- Domain : $A = (0, +\infty)$
- Range : $f(A) = \mathbb{R}$
- Monotonicity: $a > 1 \Leftrightarrow f \nearrow (0, +\infty)$
  $0 < a < 1 \Leftrightarrow f \searrow (0, +\infty)$
- Graph : Because $\log_a$ is the inverse of $g(x) = x^a$, its graph is the mirror image of the graph of g accross the line $(\ell) : y = x$



$\boxed{a > 1}$



$\boxed{a < 1}$

EXAMPLES

a) Find the default domain of the function

$$f(x) = \log_{x^2-4}(2x-1)$$

Solution

Require: $\begin{cases} 2x-1>0 & (1) \\ x^2-4>0 & (2) \\ x^2-4\neq 1 & (3) \end{cases}$

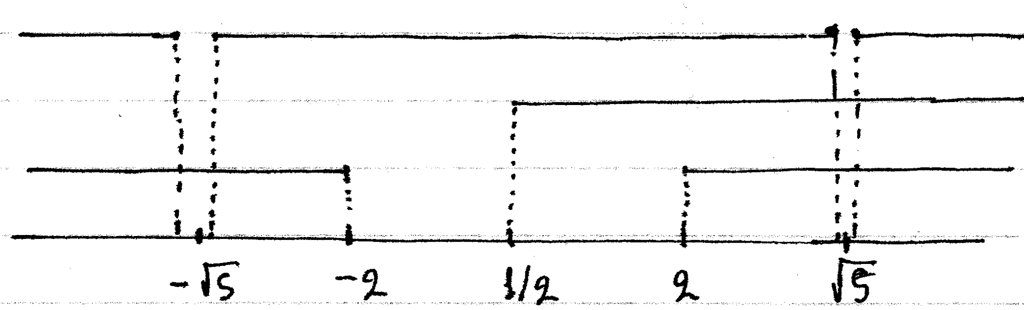
(1) $\Leftrightarrow 2x>1 \Leftrightarrow x>1/2 \Leftrightarrow \underline{x\in(1/2,+\infty)}$

(2) $\Leftrightarrow x^2-4>0 \Leftrightarrow (x-2)(x+2)>0 \Leftrightarrow \underline{x\in(-\infty,-2)\cup(2,+\infty)}$

x		-2		2	
$x-2$	$-$		$-$	0	$+$
$x+2$	$-$	0	$+$		$+$
	$+$	0	$-$	0	$+$

(3) $\Leftrightarrow x^2-4\neq 1 \Leftrightarrow x^2-5\neq 0 \Leftrightarrow (x+\sqrt{5})(x-\sqrt{5})\neq 0$

$\Leftrightarrow x\neq +\sqrt{5} \wedge x\neq -\sqrt{5} \Leftrightarrow \underline{x\in\mathbb{R}-\{-\sqrt{5},+\sqrt{5}\}}.$



Thus: $A=(1/2,+\infty)\cap[(-\infty,-2)\cup(2,+\infty)]\cap[\mathbb{R}-\{-\sqrt{5},\sqrt{5}\}]$

$= (2,\sqrt{5})\cup(\sqrt{5},+\infty).$

6) Determine the domain and monotonicity of the function f defined by
$f(x) = \sqrt{5} - 3\log_{1/3}(2 - 5e^{-3x})$.

Solution

• Domain

Require: $2 - 5e^{-3x} > 0 \Leftrightarrow -5e^{-3x} > -2 \Leftrightarrow 5e^{-3x} < 2$

$$\Leftrightarrow e^{-3x} < \frac{2}{5} \underset{*}{\Leftrightarrow} \ln(e^{-3x}) < \ln\left(\frac{2}{5}\right) \Leftrightarrow$$

$$\underset{**}{\Leftrightarrow} -3x < \ln(2/5) \Leftrightarrow x > \frac{\ln(2/5)}{3} = \frac{\ln 2 - \ln 5}{3}$$

thus: $A = \left(\frac{\ln(2/5)}{3}, +\infty\right)$

* Here we use $\ln \uparrow (0, +\infty)$ which gives:
$0 < x_1 < x_2 \Leftrightarrow \ln x_1 < \ln x_2$

** Note that $\ln(\exp(x)) = x$, $\forall x \in \mathbb{R}$.

• Monotonicity.

Let $x_1, x_2 \in A$ be given, with $x_1 < x_2$. Then:

$$x_1 < x_2 \Rightarrow -3x_1 > -3x_2 \Rightarrow e^{-3x_1} > e^{-3x_2} \Rightarrow$$

$$\Rightarrow -5e^{-3x_1} < -5e^{-3x_2} \Rightarrow$$

$$\Rightarrow 2 - 5e^{-3x_1} < 2 - 5e^{-3x_2} \Rightarrow$$

$$\Rightarrow \log_{1/3}(2 - 5e^{-3x_1}) > \log_{1/3}(2 - 5e^{-3x_2})$$

$$\Rightarrow -3\log_{1/3}(2 - 5e^{-3x_1}) < -3\log_{1/3}(2 - 5e^{-3x_2})$$

$$\Rightarrow \sqrt{5} - 3\log_{1/3}(2-5e^{-3x_1}) < \sqrt{5} - 3\log_{1/3}(2-5e^{-3x_2})$$

$$\Rightarrow \underline{f(x_1) < f(x_2)}.$$

It follows that $f \uparrow \left(\frac{\ln(2/5)}{3}, +\infty\right)$

EXERCISES

(7) Find the default domain of the following functions:

a) $f(x) = \log_3 (x^2+3x+2)$

b) $f(x) = \log_5 (2-|x-1|)$

c) $f(x) = \log_x (x-1)$

d) $f(x) = \log_{x^2-1} (x+1)$

e) $f(x) = \log_{x+2} (5-x)$

$\hookrightarrow$ For the domain of $f(x) = \log_{a(x)} (b(x))$ we require:

$$\begin{cases} b(x) > 0 \\ a(x) > 0 \\ a(x) \neq 1 \end{cases}$$

(8) Determine the domain and monotonicity of the following functions:

a) $f(x) = \log_3(2x-1)$

b) $f(x) = 3 - \log_{1/2}(2-5x)$

c) $f(x) = \frac{1}{2}\log_2(3x-1)$

d) $f(x) = 1 + \log_{1/e}(e-x)$

e) $f(x) = \log_2(x) + \log_3(1+x)$

f) $f(x) = \log_{1/2}(e^x+1)$

g) $f(x) = 2 - \log_{1/5}(3e^{-x}+1)$

↳ For monotonicity, we use the same method as in exercise 6

Manipulation of Logarithms

Properties

1) $\log_a(x_1 x_2) = \log_a(x_1) + \log_a(x_2)$

2) $\log_a\left(\frac{x_1}{x_2}\right) = \log_a(x_1) - \log_a(x_2)$

3) $\log_a x^k = k \log_a, \ \forall k \in \mathbb{R}$

$\hookrightarrow \log_a \sqrt[n]{x} = \frac{1}{n} \log_a x$

$\log_a \sqrt{x} = \frac{1}{2} \log_a x$

4) For $a > 1$: $\log_a x > 0 \Leftrightarrow x > 1$
$\log_a x < 0 \Leftrightarrow x < 1$

For $0 < a < 1$: $\log_a x > 0 \Leftrightarrow x < 1$
$\log_a x < 0 \Leftrightarrow x > 1$

For both cases:
$\log_a x = 0 \Leftrightarrow x = 1.$

▶ Decimal and Natural Logarithms

- We define $\boxed{\log x = \log_{10} x}$ (decimal logarithm)

  Thus:

  $\log 1 = 0$, $\log 10 = 1$, $\log 100 = 2$, $\log 1000 = 3$, etc.

- We also define: $\boxed{\ln(x) = \log_e(x)}$

  (natural logarithm).

Note that $\ln \nearrow (0, +\infty)$ since $e > 1$

We can also show that

$$\boxed{a^x = \exp(x \ln a)}$$

$$\boxed{\log_a(x) = \frac{\ln x}{\ln a}}$$

▶ Change of Base

$$\boxed{\log_b(x) = \frac{\log_a(x)}{\log_a(b)}}$$

EXAMPLES

a) Compare $\log_{1/2} 17$ and $\log_{1/2} 21$

Solution

$$\left.\begin{array}{l} 0<1/2<1 \Rightarrow \log_{1/2} \searrow (0,+\infty) \\ 17<21 \end{array}\right\} \Rightarrow \log_{1/2} 17 > \log_{1/2} 21.$$

b) Compare $\log 9$ with $\log 12$

Solution

$$\left.\begin{array}{l} 9<12 \\ \log \nearrow (0,+\infty) \end{array}\right\} \Rightarrow \log 9 < \log 12.$$

c) Compare $\ln 3$ with $\ln 5$

Solution

$$\left.\begin{array}{l} 3<5 \\ \ln \nearrow (0,+\infty) \end{array}\right\} \Rightarrow \ln 3 < \ln 5.$$

d) Show that $\log_2 25 \cdot \log_5 8 = 6$

Solution

$$A = \log_2 25 \log_5 8 = \frac{\ln 25}{\ln 2} \frac{\ln 8}{\ln 5} =$$

$$= \frac{\ln 5^2}{\ln 2} \frac{\ln 2^3}{\ln 5} = \frac{2\ln 5}{\ln 2} \frac{3\ln 2}{\ln 5} =$$

$$= 2 \cdot 3 = 6 = B$$

e) Show that:

e) $\log 2 + \log(2+\sqrt{2}) + \log(2+\sqrt{2+\sqrt{2}}) + \log(2-\sqrt{2+\sqrt{2}}) = 2\log 2$

Solution

$$A = \log 2 + \log(2+\sqrt{2}) + \log\left(2+\sqrt{2+\sqrt{2}}\right) + \log\left(2-\sqrt{2+\sqrt{2}}\right) =$$
$$= \log\left[2(2+\sqrt{2})\left(2+\sqrt{2+\sqrt{2}}\right)\left(2-\sqrt{2+\sqrt{2}}\right)\right] =$$
$$= \log\left[2(2+\sqrt{2})\left(2^2 - \left(\sqrt{2+\sqrt{2}}\right)^2\right)\right] =$$
$$= \log\left[2(2+\sqrt{2})(4-(2+\sqrt{2}))\right] =$$
$$= \log\left[2(2+\sqrt{2})(2-\sqrt{2})\right] =$$
$$= \log\left[2(2^2-(\sqrt{2})^2)\right] = \log[2(4-2)] =$$
$$= \log(2 \cdot 2) = \log 2 + \log 2 = 2\log 2 = B.$$

f) Show that:

$$\log_a(b^2\sqrt{b}) \log_{\sqrt{b}}\left(\frac{a^3}{\sqrt{a}}\right) = \frac{25}{2}$$

Solution

$$A = \log_a(b^2\sqrt{b}) \log_{\sqrt{b}}\left(\frac{a^3}{\sqrt{a}}\right) =$$

$$= \log_a(b^{2+1/2}) \log_{b^{1/2}}(a^{3-1/2}) =$$

$$= \frac{\ln b^{5/2}}{\ln a} \frac{\ln a^{5/2}}{\ln b^{1/2}} = \frac{(5/2)\ln b}{\ln a} \frac{(5/2)\ln a}{(1/2)\ln b} =$$

$$= \frac{(5/2)(5/2)}{1/2} = \frac{25}{2}$$

g) Write the following expression in terms of $\ln a$, $\ln b$, and $\ln c$:

$$A = \log_9 \left[\frac{9a^3\sqrt{a}}{b^2\sqrt{bc}} \right]$$

Solution

$$A = \log_9 \left[\frac{9a^3\sqrt{a}}{b^2\sqrt{bc}} \right] =$$

$$= \log_9 9 + \log_9 a^3 + \log_9 \sqrt{a} - \log_9 b^2 - \log_9 \sqrt{bc} =$$

$$= 1 + 3\log_9 a + (1/2)\log_9 a - 2\log_9 b - (1/2)[\log_9 b + \log_9 c]$$

$$= 1 + (3 + 1/2)\log_9 a + (-2 - 1/2)\log_9 b + (-1/2)\log_9 c =$$

$$= 1 + (7/2)\log_9 a - (5/2)\log_9 b - (1/2)\log_9 c =$$

$$= 1 + (7/2)\frac{\ln a}{\ln 9} - (5/2)\frac{\ln b}{\ln 9} - (1/2)\frac{\ln c}{\ln 9} =$$

$$= 1 + \frac{7\ln a - 5\ln b - \ln c}{\ln 9}$$

EXERCISES

(9) Compare the numbers

a) $\log_2 5$, $\log_2 3$ c) $\log 15$, $\log 2$

b) $\log_{1/3} 11$, $\log_{1/3} 12$ d) $\ln 2$, $\ln 3$

(10) Show that

a) $\log 3 + 2\log 4 - \log 12 = 2\log 2$

b) $\frac{1}{2}\log 25 + \frac{1}{3}\log 8 + \frac{1}{5}\log 32 = 1 + \log 2$

c) $3\log 2 + \log 5 - \log 4 = 1$

d) $\log_2 3 \log_3 4 = 2$

e) $\log_a b \cdot \log_b c \cdot \log_c a = 1$, $\forall a, b, c \in (0,1) \cup (1, +\infty)$

* f) $a^{\log b} = b^{\log a}$, $\forall a, b \in (0, +\infty)$

(11) If $a, b, c \in (0, +\infty)$ and $a \neq b \neq c \neq 0$, and

$$\frac{\log a}{b-c} = \frac{\log b}{c-a} = \frac{\log c}{a-b}$$

show that $a^a b^b c^c = 1$.

(12) Let $x, y \in (0, +\infty)$ with $x^2 + y^2 = 23xy$.
Show that

$$\log_a \left[\frac{x+y}{5}\right] = \frac{1}{2}(\log_a x + \log_a y)$$

(Hint: $x^2 + y^2 = (x+y)^2 - 2xy$)

(13) Write the following in terms of $\ln a$, $\ln b$ and $\ln c$:

a) $\log_3 \left[\dfrac{3a^2}{5b\sqrt{c}}\right]$

b) $\log \left[\dfrac{3a^3 \sqrt[4]{b^2 c}}{5b^2 \sqrt[3]{a^2 b c^2}}\right]$

(14) If $a, b \in (0,1) \cup (1, +\infty)$, show that

a) $\log_a \left(\frac{1}{b^5}\right) \log_b a^2 = -10$

b) $\log_b (a^2) \log_a (b\sqrt{b}) = 3$

(15) If $a, b, c \in (0,1) \cup (1, +\infty)$ show that

a) $\log_a (bc) = \dfrac{1}{\log_b a} + \dfrac{1}{\log_c a}$

b) $\log_{ab} (c) = \dfrac{\log_b (c)}{1 + \log_b (a)}$, c) $\log_a b = -\log_{1/a} (b)$

Logarithmic equations

These are equations that contain a logarithm of the unknown or a logarithm of a function of the unknown.

- 1 Find the domain of the equation using the initial form of the equation.
  Remember that each term $\log_{a(x)} b(x)$ contributes the conditions

$$\begin{cases} a(x) > 0 \\ a(x) \neq 1 \\ b(x) > 0 \end{cases}$$

- 2 Use the properties of logarithms to reduce the initial equation to one of the following forms:

1) $\log_x a = b \Leftrightarrow a = x^b \Leftrightarrow \dots$

2) $\log_a f(x) = b \Leftrightarrow f(x) = a^b \Leftrightarrow \dots$

3) $\log_a f(x) = \log_a g(x) \Leftrightarrow f(x) = g(x) \Leftrightarrow \dots$

- 3 Accept or reject the solutions based on whether they belong to the domain, found in step 1.

EXAMPLES

a) Solve: $\log_x 64 = 4$

Solution

Domain: Require $\begin{cases} x > 0 \\ x \neq 1 \end{cases} \Leftrightarrow x \in (0,1) \cup (1,+\infty)$

thus $A = (0,1) \cup (1,+\infty)$.

$\log_x 64 = 4 \Leftrightarrow x^4 = 64 \Leftrightarrow x^2 = 8 \vee x^2 = -8 \Leftrightarrow$

$\Leftrightarrow x^2 = 8 \Leftrightarrow x = 2\sqrt{2} \vee x = -2\sqrt{2}$

$\Leftrightarrow x = 2\sqrt{2}$ (Reject $x = -2\sqrt{2}$).

Thus $S = \{2\sqrt{2}\}$.

b) Solve $\log x = -2$

Solution

Domain: Require $x > 0$, thus $A = (0,+\infty)$.

$\log x = -2 \Leftrightarrow x = 10^{-2} \Leftrightarrow x = 0.01 \longleftarrow$ accepted

thus $S = \{0.01\}$.

c) Solve: $\log_{1/2}(x^2 - 4x) = -2$

Solution

Domain: Require $x^2 - 4x > 0 \Leftrightarrow x(x-4) > 0 \Leftrightarrow$

$\Leftrightarrow x \in (-\infty, 0) \cup (4, +\infty)$.

x		0		4	
x	$-$	0	$+$		$+$
$x-4$	$-$		$-$	0	$+$
	$+$	0	$-$	0	$+$

thus $A = (-\infty, 0) \cup (4, +\infty)$.

$\log_{1/2}(x^2-4x) = -2 \Leftrightarrow x^2-4x = (1/2)^{-2}$

$\Leftrightarrow x^2-4x = 4 \Leftrightarrow x^2-4x-4=0 \quad (1).$

$\Delta = b^2-4ac = (-4)^2-4\cdot 1\cdot(-4) = 16+16 = 32 \Rightarrow (4\sqrt{2})^2$

$$\Rightarrow x_{1,2} = \frac{-(-4) \pm 4\sqrt{2}}{2\cdot 1} = +2 \pm 2\sqrt{2}$$

$$\left.\begin{array}{l} 2-2\sqrt{2} < 0 \Rightarrow 2-2\sqrt{2} \in A \\ 2+2\sqrt{2} > 2+2 = 4 \Rightarrow 2+2\sqrt{2} \in A \end{array}\right\} \Rightarrow$$

$\Rightarrow S = \{2-2\sqrt{2},\ 2+2\sqrt{2}\}$.

↳ We use the equation's domain to accept or reject solutions.

d) Solve: $\log_x 81 = (\log_x 3)^2 + 4$

Solution

Domain: Require $\begin{cases} x > 0 \\ x \neq 1 \end{cases}$, thus $A = (0,1) \cup (1, +\infty)$.

Define $y = \log_x 3$ and note that

$\log_x 81 = \log_x 3^4 = 4\log_x 3 = 4y$.

It follows that:

$\log_x 81 = (\log_x 3)^2 + 4 \Leftrightarrow 4y = y^2 + 4 \Leftrightarrow$

$\Leftrightarrow y^2 - 4y + 4 = 0 \Leftrightarrow (y-2)^2 = 0 \Leftrightarrow y - 2 = 0 \Leftrightarrow y = 2$

$\Leftrightarrow \log_x 3 = 2 \Leftrightarrow 3 = x^2 \Leftrightarrow x = \sqrt{3} \vee x = -\sqrt{3}$

Since:

$\sqrt{3} \in A$ and $-\sqrt{3} \notin A$

it follows that $S = \{\sqrt{3}\}$.

e) $\ln(x+2) + \ln(x+1) = \ln 6$

<u>Solution</u>

Require: $\begin{cases} x+2>0 \\ x+1>0 \end{cases} \Leftrightarrow \begin{cases} x>-2 \\ x>-1 \end{cases} \Leftrightarrow x > -1$

thus, domain is $A = (-1, +\infty)$. It follows that

$\ln(x+2) + \ln(x+1) = \ln 6 \Leftrightarrow \ln[(x+2)(x+1)] = \ln 6 \Leftrightarrow$

$\Leftrightarrow (x+2)(x+1) = 6 \Leftrightarrow x^2 + 3x + 2 = 6 \Leftrightarrow x^2 + 3x + 2 - 6 = 0$

$\Leftrightarrow x^2 + 3x - 4 = 0 \Leftrightarrow (x+4)(x-1) = 0 \Leftrightarrow$

$\Leftrightarrow x + 4 = 0 \vee x - 1 = 0 \Leftrightarrow x = -4 \vee x = 1$

Since $-4 \notin A$ and $1 \in A$, then $S = \{1\}$.

f) $\ln(\ln(x^2 + x)) = 0$

<u>Solution</u>

Require $\begin{cases} x^2+x>0 \\ \ln(x^2+x)>0 \end{cases} \Leftrightarrow \begin{cases} x(x+1)>0 \\ \ln(x^2+x)>\ln 1 \end{cases} \Leftrightarrow$

$\Leftrightarrow \begin{cases} x(x+1)>0 \\ x^2+x>1 \end{cases} \Leftrightarrow \begin{cases} x(x+1)>0 & (1) \\ x^2+x-1>0 & (2) \end{cases}$

We note that for (1):

$x(x+1)>0 \Leftrightarrow \underline{x \in (-\infty, 0) \cup (1, +\infty)}$

x		0		1	
x	$-$	0	$+$		$+$
$x+1$	$-$		$-$	0	$+$
	$+$	0	$-$	0	$+$

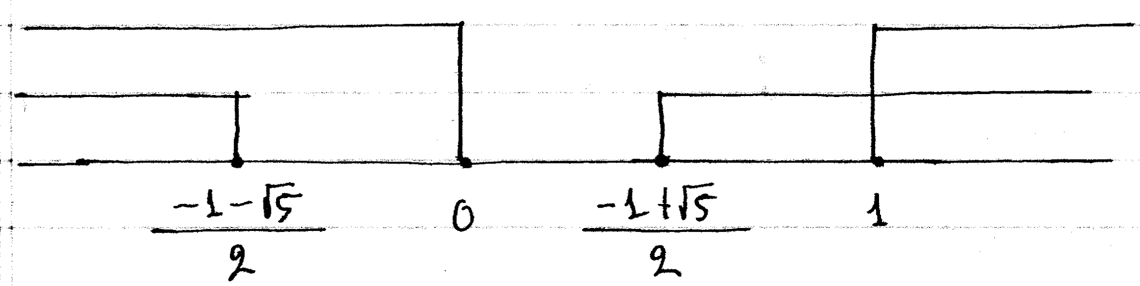
and for (2): $\Delta = b^2-4ac = 1^2-4\cdot1\cdot(-1) = 1+4 = 5 \Rightarrow$

$$\Rightarrow x_{1,2} = \frac{-b\pm\sqrt{\Delta}}{2a} = \frac{-1\pm\sqrt{5}}{2\cdot1}$$

x		$(-1-\sqrt{5})/2$		$(-1+\sqrt{5})/2$	
x^2+x-1	$+$	0	$-$	0	$+$

thus:

$x^2+x-1>0 \Leftrightarrow \underline{x \in \left(-\infty, \frac{-1-\sqrt{5}}{2}\right) \cup \left(\frac{-1+\sqrt{5}}{2}, +\infty\right)}.$



It follows that the domain is:

$$A = \left[(-\infty,0)\cup(1,+\infty)\right]\cap\left[\left(-\infty,\frac{-1-\sqrt{5}}{2}\right)\cup\left(\frac{-1+\sqrt{5}}{2},+\infty\right)\right]$$

$$= \left(-\infty,\frac{-1-\sqrt{5}}{2}\right)\cup(1,+\infty).$$

Solving the equation gives:

$$\ln(\ln(x^2+x)) = 0 \Leftrightarrow \ln(\ln(x^2+x)) = \ln 1 \Leftrightarrow$$

$$\Leftrightarrow \ln(x^2+x) = 1 \Leftrightarrow \ln(x^2+x) = \ln e \Leftrightarrow$$

$$\Leftrightarrow x^2+x = e \Leftrightarrow x^2+x-e = 0$$

$$\Delta = b^2-4ac = 1^2-4\cdot 1\cdot(-e) = 1+4e \Rightarrow$$

$$\Rightarrow x_{1,2} = \frac{-b\pm\sqrt{\Delta}}{2a} = \frac{-1\pm\sqrt{1+4e}}{2}$$

For $x_1 = \dfrac{-1-\sqrt{1+4e}}{2} < \dfrac{-1-\sqrt{5}}{2} \Rightarrow \dfrac{-1-\sqrt{1+4e}}{2} \in A.$

For $x_2 = \dfrac{-1+\sqrt{1+4e}}{2} > \dfrac{-1+\sqrt{1+4\cdot 2}}{2} = \dfrac{-1+\sqrt{1+8}}{2} =$

$$= \frac{-1+\sqrt{9}}{2} = \frac{3-1}{2} = 1 \Rightarrow \frac{-1+\sqrt{1+4e}}{2} \in A$$

Since we accept both solutions:

$$S = \left\{\frac{-1-\sqrt{1+4e}}{2}, \frac{-1+\sqrt{1+4e}}{2}\right\}.$$

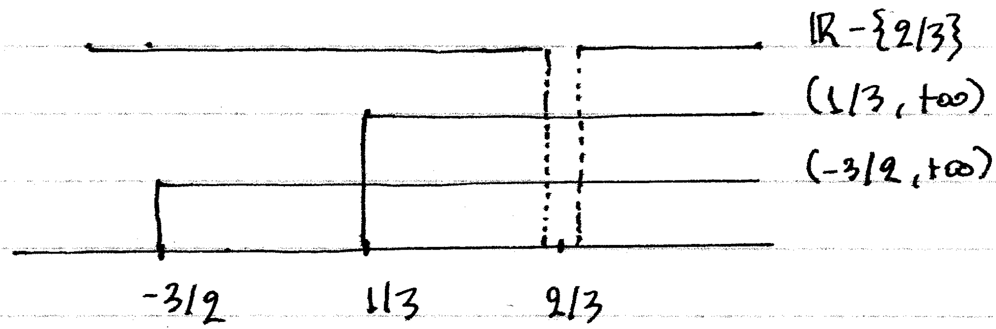
g) $\log_9(2x+3)\log_{3x-1}3 = 1$

Solution

Require $\begin{cases} 2x+3>0 \\ 3x-1>0 \\ 3x-1\neq 1 \end{cases} \Leftrightarrow \begin{cases} 2x>-3 \\ 3x>1 \\ 3x\neq 1+1 \end{cases} \Leftrightarrow \begin{cases} x>-3/2 \\ x>1/3 \\ x\neq 2/3 \end{cases}$

thus domain of equation is:

$$A = (-3/2, +\infty)\cap(1/3,+\infty)\cap(\mathbb{R}-\{2/3\}) =$$
$$= (1/3, 2/3)\cup(2/3,+\infty).$$



Solving the equation:

$$\log_9(2x+3)\log_{3x-1}3 = 1 \Leftrightarrow \frac{\ln(2x+3)}{\ln 9}\,\frac{\ln 3}{\ln(3x-1)} = 1$$

$$\Leftrightarrow \frac{\ln(2x+3)}{\ln(3x-1)}\,\frac{\ln 3}{2\ln 3} = 1 \Leftrightarrow \frac{\ln(2x+3)}{2\ln(3x-1)} = 1 \Leftrightarrow$$

$$\Leftrightarrow \ln(2x+3) = 2\ln(3x-1) \Leftrightarrow \ln(2x+3) = \ln[(3x-1)^2]$$

$$\Leftrightarrow 2x+3 = (3x-1)^2 \Leftrightarrow 2x+3 = 9x^2-6x+1 \Leftrightarrow$$

$$\Leftrightarrow 9x^2+(-6-2)x+1-3=0 \Leftrightarrow 9x^2-8x-2=0$$

$$\Delta = b^2 - 4ac = (-8)^2 - 4\cdot 9 \cdot (-2) = 64 + 72 = 136 = 2^3 \cdot 17 \Rightarrow$$

$$\Rightarrow x_{1,2} = \frac{-(-8) \pm 2\sqrt{34}}{2\cdot 9} =$$

$$= \frac{4 \pm \sqrt{34}}{9}$$

$$\begin{array}{r|l} 136 & 2 \\ 68 & 2 \\ 34 & 2 \\ 17 & 17 \\ 1 & \end{array}$$

To accept/reject solutions, we note that

$$\left.\begin{array}{l} \text{for } x_1 = \dfrac{4+\sqrt{34}}{9} > \dfrac{4}{9} > \dfrac{3}{9} = \dfrac{1}{3} \\ \text{and } \dfrac{4+\sqrt{34}}{9} \neq \dfrac{2}{3} \end{array}\right\} \Rightarrow$$

$$\Rightarrow \frac{4+\sqrt{34}}{9} \in A.$$

for $x_2 = \dfrac{4-\sqrt{34}}{9} = \dfrac{\sqrt{16}-\sqrt{34}}{9} < 0 \Rightarrow$

$$\Rightarrow \frac{4-\sqrt{34}}{9} \notin A$$

It follows that $S = \left\{ \dfrac{4+\sqrt{34}}{9} \right\}$.

EXERCISES

(16) Solve the equations (1st form)

1) $\log_x\left(\frac{81}{16}\right) = 4$ 2) $\log_x \sqrt{8} = \frac{3}{4}$

3) $\log_x 25 = 8$ 4) $\log_x 16 = \frac{2}{3}$ 5) $\log_x 5 = \frac{1}{3}$

6) $\log_x 16 = -2$ 7) $\log_x \frac{1}{81} = -4$ 8) $\log_x 64 = -2$

(17) Solve the equations (2nd form)

1) $\log_4 x = 3$ 2) $\log x = -3$ 3) $\ln x = 2$

4) $\log_8 x = -\frac{7}{3}$ 5) $\log_8 x = -\frac{7}{3}$ 6) $\log_{2\sqrt[3]{5}} x = -6$

7) $\log_3 (x^2 - x + 3) = 2$ 8) $\log (x^2 - 5x + 16) = 1$

9) $\log_{1/2} (x^2 - 3x) = -1$

(18) Solve the equations (use substitution)

1) $2(\log_x 8)^2 + \log_x 64 + \log_x 8 = 9$

2) $\log_x 256 = (\log_x 4)^2 + 3$

(19) Solve the equations (3rd form)

1) $\log(4x-1) = 2\log 2 + \log(x^2-1)$

2) $\frac{1}{2}\log(x+2) + \log\sqrt{x+3} = 1 + \log\sqrt{3}$

3) $2\log x - \log(x+1) = \log 4 - \log 3$

4) $\log_4(x+2) - \log_4(x-3) = 3$

5) $\log_3 x \cdot \log_9 x = 2$

6) $\log_x 2 + \log_2 x = \frac{5}{2}$

7) $\log[\log(2x^2+x-2)] = 0$

8) $\log[\log(2x^2+x-11)] = 0$

(20) Solve the equations

1) $\log_4(x+12)\log_x 2 = 1$

2) $\log_x(5x^2)[\log_5 x]^2 = 1$

3) $\log_4(\log_3(\log_2 x)) = 0$

Equations with exponentials

① <u>Form</u> : $\boxed{a^{f(x)} = b} \Leftrightarrow \ln a^{f(x)} = \ln b$

$\Leftrightarrow f(x)\ln a = \ln b$

$\Leftrightarrow \ldots$

EXAMPLE

a) Solve : $5^{x^2+x} = 2$.

<u>Solution</u>

$$5^{x^2+x} = 2 \Leftrightarrow \ln 5^{x^2+x} = \ln 2 \Leftrightarrow (x^2+x)\ln 5 = \ln 2$$

$$\Leftrightarrow (\ln 5)x^2 + (\ln 5)x - \ln 2 = 0.$$

$$\Delta = b^2 - 4ac = (\ln 5)^2 - 4(\ln 5)(-\ln 2)$$

$$= (\ln 5 + 4\ln 2)\ln 5 \Rightarrow$$

$$\Rightarrow x_{1,2} = \frac{-b \pm \sqrt{\Delta}}{2a} = \frac{-\ln 5 \pm \sqrt{(\ln 5 + 4\ln 2)\ln 5}}{2\ln 5}$$

thus

$$S = \left\{ \frac{-\ln 5 - \sqrt{(\ln 5 + 4\ln 2)\ln 5}}{2\ln 5}, \frac{-\ln 5 + \sqrt{(\ln 5 + 4\ln 2)\ln 5}}{2\ln 5} \right\}$$

② <u>Form</u> : $\boxed{a^{f(x)} = b^{g(x)}} \Leftrightarrow \ln a^{f(x)} = \ln b^{g(x)} \Leftrightarrow$

$\Leftrightarrow f(x)\ln a = g(x)\ln b$

$\Leftrightarrow \ldots$

EXAMPLE

Solve: $3^{2x+1} = 7^{3x-2}$.

Solution

$$3^{2x+1} = 7^{3x-2} \Leftrightarrow \ln 3^{2x+1} = \ln 7^{3x-2} \Leftrightarrow$$
$$\Leftrightarrow (2x+1)\ln 3 = (3x-2)\ln 7 \Leftrightarrow$$
$$\Leftrightarrow (2\ln 3)x + \ln 3 = (3\ln 7)x - 2\ln 7 \Leftrightarrow$$
$$\Leftrightarrow (2\ln 3 - 3\ln 7)x = -\ln 3 - 2\ln 7 \Leftrightarrow$$
$$\Leftrightarrow x = \frac{-\ln 3 - 2\ln 7}{2\ln 3 - 3\ln 7} = \frac{2\ln 7 + \ln 3}{3\ln 7 - 2\ln 3}$$

③ Form $\boxed{f(a^x) = g(a^x)}$ → Let $y = a^x$ and solve $f(y) = g(y)$ first.

EXAMPLE

Solve $e^x - e^{-x} = 2$.

Solution

Let $y = e^x$. Then $e^{-x} = \frac{1}{e^x} = \frac{1}{y}$, and it follows that:

$$e^x - e^{-x} = 2 \Leftrightarrow y - \frac{1}{y} = 2 \Leftrightarrow y^2 - 1 = 2y \Leftrightarrow$$

$$\Leftrightarrow y^2 - 2y - 1 = 0 \qquad (1)$$

$\Delta = b^2 - 4ac = (-2)^2 - 4 \cdot 1 \cdot (-1) = 4 + 4 = 8 \Rightarrow$

$$\Rightarrow \quad y_{1,2} = \frac{-b \pm \sqrt{\Delta}}{2a} = \frac{-(-2) \pm 2\sqrt{2}}{2 \cdot 1} = -1 \pm \sqrt{2}$$

It follows that

$y = -1 - \sqrt{2} \ \vee \ y = -1 + \sqrt{2} \Leftrightarrow$

$\Leftrightarrow e^x = -1 - \sqrt{2} \ \vee \ e^x = -1 + \sqrt{2} \Leftrightarrow$

$\Leftrightarrow e^x = -1 + \sqrt{2} \Leftrightarrow \ln e^x = \ln(\sqrt{2} - 1) \Leftrightarrow$

$\Leftrightarrow x = \ln(\sqrt{2} - 1)$

and therefore $S = \{\ln(\sqrt{2} - 1)\}$

↳ Note that the equation $e^x = -1 - \sqrt{2}$ is <u>inconsistent</u> because $-1 - \sqrt{2} < 0$. and $e^x > 0, \forall x \in \mathbb{R}$.

④ <u>Form</u> : $\boxed{A \cdot a^x = B b^x} \Leftrightarrow \ln(A a^x) = \ln(B b^x) \Leftrightarrow$

$\Leftrightarrow \ln A + x \ln a = \ln B + x \ln b \Leftrightarrow$

$\Leftrightarrow \ldots$

EXAMPLE

Solve: $2^{x+4} - 5^{x+2} = 2^{x+2} - 5^x$

<u>Solution</u>

$2^{x+4} - 5^{x+2} = 2^{x+2} - 5^x \Leftrightarrow 2^{x+4} - 2^{x+2} = 5^{x+2} - 5^x \Leftrightarrow$

$\Leftrightarrow 2^{x+2}(2^2 - 1) = 5^x(5^2 - 1) \Leftrightarrow 3 \cdot 2^{x+2} = 24 \cdot 5^x$

$\Leftrightarrow 2^{x+2} = 8 \cdot 5^x \Leftrightarrow \ln(2^{x+2}) = \ln(8 \cdot 5^x) \Leftrightarrow$

$\Leftrightarrow (x+2)\ln 2 = \ln 8 + x\ln 5 \Leftrightarrow$

$\Leftrightarrow (\ln 2)x + 2\ln 2 = 3\ln 2 + x\ln 5 \Leftrightarrow$

$\Leftrightarrow (\ln 2 - \ln 5)x = 3\ln 2 - 2\ln 2 \Leftrightarrow$

$\Leftrightarrow (\ln 2 - \ln 5)x = \ln 2 \Leftrightarrow$

$$\Leftrightarrow x = \frac{\ln 2}{\ln 2 - \ln 5}$$

thus $S = \left\{ \dfrac{\ln 2}{\ln 2 - \ln 5} \right\}$

5 Form : $\boxed{A a^{2x} + B a^x b^x + C b^{2x} = 0} \Leftrightarrow$

$$\Leftrightarrow A\frac{a^{2x}}{b^{2x}} + B\frac{a^x b^x}{b^{2x}} + C\frac{b^{2x}}{b^{2x}} = 0 \Leftrightarrow$$

$$\Leftrightarrow A\left(\frac{a}{b}\right)^{2x} + B\left(\frac{a}{b}\right)^x + C = 0$$

Let $y = \left(\frac{a}{b}\right)^x$, and the equation yields:

$Ay^2 + By + C = 0 \Leftrightarrow \dots \Leftrightarrow y = y_1 \vee y = y_2 \Leftrightarrow$

$$\Leftrightarrow \left(\frac{a}{b}\right)^x = y_1 \vee \left(\frac{a}{b}\right)^x = y_2 \Leftrightarrow \dots$$

EXAMPLE

Solve: $2^{2x+1} + 5\cdot 10^x - 5^{2x} = 0$

Solution

$$2^{2x+1} + 5\cdot 10^x - 5^{2x} = 0 \Leftrightarrow 2\cdot 2^{2x} + 5\cdot 2^x\cdot 5^x - 5^{2x} = 0$$

$$\Leftrightarrow 2\left(\frac{2}{5}\right)^{2x} + 5\left(\frac{2}{5}\right)^{x} - 1 = 0 \qquad (1)$$

Let $y = (2/5)^x$. It follows that

$$(1) \Leftrightarrow 2y^2 + 5y - 1 = 0 \qquad (2)$$

$$\Delta^2 = 5^2 - 4\cdot 2\cdot(-1) = 25 + 8 = 33 \Rightarrow$$

$$\Rightarrow y_{1,2} = \frac{-b \pm \sqrt{\Delta}}{2a} = \frac{-5 \pm \sqrt{33}}{2\cdot 2} = \frac{-5 \pm \sqrt{33}}{4}$$

and therefore:

$$(2/5)^x = \frac{-5-\sqrt{33}}{4} \lor (2/5)^x = \frac{-5+\sqrt{33}}{4} \Leftrightarrow$$

$$\Leftrightarrow (2/5)^x = \frac{\sqrt{33}-5}{4} \Leftrightarrow \ln(2/5)^x = \ln\left(\frac{\sqrt{33}-5}{4}\right) \Leftrightarrow$$

$$\Leftrightarrow x\ln(2/5) = \ln(\sqrt{33}-5) - \ln 4 \Leftrightarrow$$

$$\Leftrightarrow x(\ln 2 - \ln 5) = \ln(\sqrt{33}-5) - \ln 4 \Leftrightarrow$$

$$\Leftrightarrow x = \frac{\ln(\sqrt{33}-5) - 2\ln 2}{\ln 2 - \ln 5}$$

Thus: $S = \left\{ \dfrac{\ln(\sqrt{33}-5) - 2\ln 2}{\ln 2 - \ln 5} \right\}$

EXERCISES

(21) Solve the equations

a) $3^{x^2-5x+11} = 243$

b) $7^{2-|3x|} = 1$

c) $5^{\sqrt{x}} = 625$

d) $4^{x^3-5x^2+6x+3} = 64$

e) $5^{x^4-10x^2+9} = 1$

f) $5^{3x-2} = 7$

g) $2^{2x} = 3^{x+1}$

h) $e^{2x} - 3e^{x} + 2 = 0$

i) $2^{x} + \dfrac{6}{2^{x}} = 5$

(22) Solve the equations

a) $2 \cdot 9^{x} - 7 \cdot 3^{x} + 3 = 0$

b) $4^{x} - 7 \cdot 2^{x} - 8 = 0$

c) $9^{x} - 3^{x+1} - 3^{x} + 3 = 0$

d) $5^{2x-1} + 3 \cdot 5^{x+1} = 80$

e) $2^{2x+1} + 1 = 3 \cdot 2^{x}$

f) $3 \cdot \left(\frac{3}{2}\right)^{x} + 2\left(\frac{2}{3}\right)^{x} = 5$

g) $3^{x+1} - 2^{x} = 3^{x-1} + 2^{x+3}$

h) $3^{2x+1} - 5 \cdot 6^{x} + 2 \cdot 4^{x} = 0$

i) $5 \cdot 3^{2x} + 3 \cdot 25^{x} = 8 \cdot 15^{x}$

j) $5^{x-2} - 3 \cdot 2^{x-3} = 7 \cdot 5^{x-3} - 2^{x}$

k) $3^{x+2} + 9^{x-1} = 1458$