Asymptotes - Definitions

① (l): $y = ax + b$ asymptote of $f(x)$ at $\pm\infty$

$$\left.\begin{array}{l}(l): y = ax+b \\ \text{asymptote of } f(x) \\ \text{at } \pm\infty\end{array}\right\} \Leftrightarrow \left\{\begin{array}{l}\lim\limits_{x \to \pm\infty} \dfrac{f(x)}{x} = a \\ \lim\limits_{x \to \pm\infty} \left[f(x) - ax\right] = b.\end{array}\right.$$

__notation__ : $f(x) \sim ax + b$ with $x \to \pm\infty$

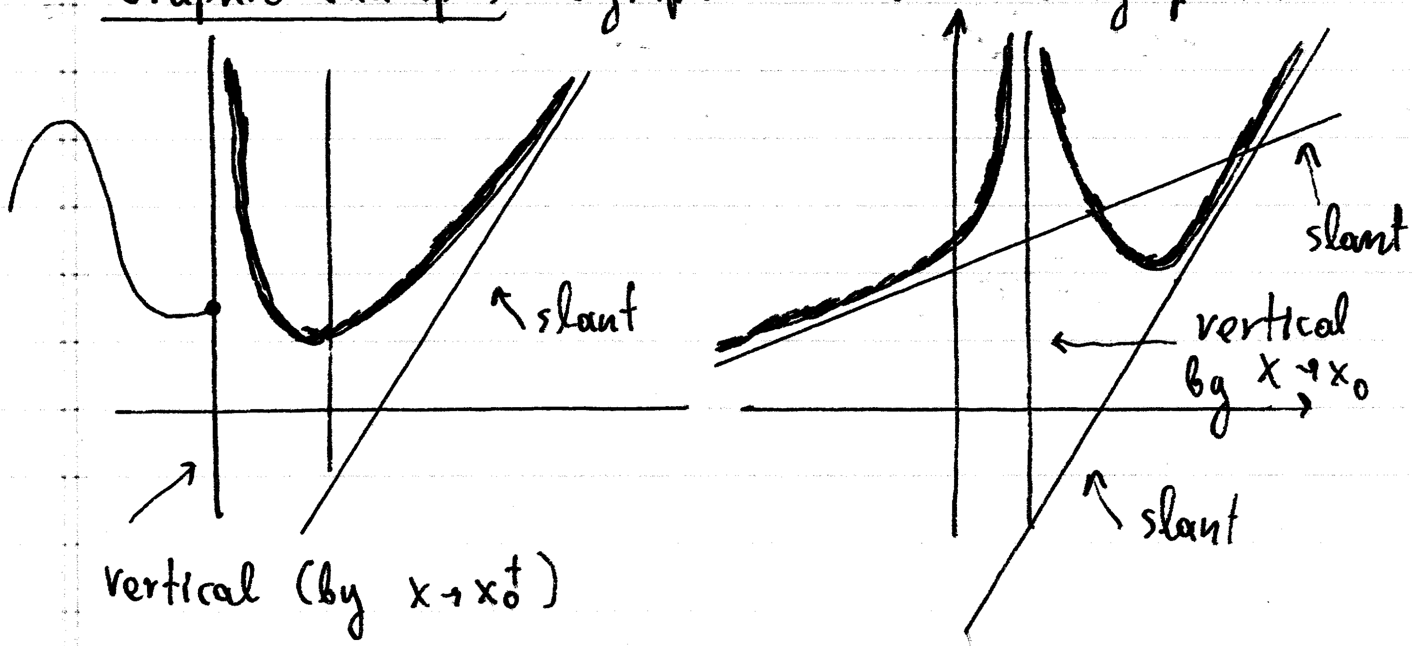
if $a \neq 0 \Rightarrow$ slant asymptotes

$a = 0 \Rightarrow$ horizontal asymptote.

② (l): $x = x_0$ vertical asymptote of $f(x)$

$$\left.\begin{array}{l}(l): x = x_0 \\ \text{vertical asymptote} \\ \text{of } f(x)\end{array}\right\} \Leftrightarrow \begin{array}{l}\lim\limits_{x \to x_0^+} f(x) = +\infty \text{ OR } -\infty \\ \qquad \text{OR} \\ \lim\limits_{x \to x_0^-} f(x) = +\infty \text{ OR } -\infty\end{array}$$

__Graphic examples__ (graph can "touch" asymptotes)



Method for slant / horizontal asymptotes

1) For Rational functions

$f(x) = P(x)/Q(x)$, P, Q polynomials

▶ If there is one it is unique at both $x \to \pm\infty$ so do the limits together:

$$\left.\begin{array}{l} \lim\limits_{x\to\pm\infty} \dfrac{f(x)}{x} = \dots = a \in \mathbb{R} \\ \lim [f(x) - ax] = \dots = b \in \mathbb{R} \end{array}\right\} \Rightarrow (\ell): y = ax + b \text{ is the slant asymptote.}$$

▶ If $\deg P(x) \leq \deg Q(x) \Rightarrow a = 0$ so skip a and investigate.

$$\lim_{x\to\pm\infty} f(x) = \dots = b \in \mathbb{R} \Rightarrow (\ell): y = b \text{ is the horizontal asymptote.}$$

2) For $f(x) = \sqrt{ax^2 + bx + c}$

do the limits $x \to \pm\infty$ separately.

You will find <u>two</u> slant asymptotes.

3) For $f(x) = ax + b$, it is its own slant asymptote!

4) For $f(x) = P(x)$ polynomial with $\deg P \geq 2$ then show that

$$\lim_{x\to\pm\infty} (f(x)/x) = +\infty \text{ or } -\infty \Rightarrow \text{no slant/horizontal asymptote.}$$

- **Method for vertical asymptotes**

1) $\boxed{\begin{array}{ll} f \text{ continuous} \Rightarrow & (\ell): x = x_0 \text{ is NOT} \\ \text{at } x_0 \in A_f & \text{a vertical asymptote} \end{array}}$

So $A_f = \mathbb{R} \Rightarrow$ no vertical asymptotes
$A_f = (-\infty, 1] \cup [3, +\infty) \Rightarrow$ no vertical asymptotes

IF you also know that
f continuous at A_f.

CAUTION: If $A_f = (-\infty, 1) \cup [3, +\infty)$
you have to check $x \to 1^-$.

2) For rational functions $f(x) = P(x)/Q(x)$
- 1 Simplify the fraction to eliminate $0/0$ situations
- 2 Expect vertical asymptotes at points where you get $A/0$
(proof needed only if requested, see examples)

3) For polynomial functions $f(x) = P(x)$

- 1 f continuous at $\mathbb{R} \Rightarrow$ no vertical asymptotes

3) $f(x) = \sqrt{P(x)}$, P polynomial

- $_1$ Find A_f.
- $_2$ Since f continuous at A_f use that to argue that you don't have vertical asymptotes.

<u>examples:</u>

1) $f(x) = \sqrt{1-x^2}$
2) $f(x) = \sqrt{\dfrac{x}{x-2}}$.

Examples on Asymptotes

1) $f(x) = \dfrac{x^2(x+1)}{x^2+3x+2}$ ← Find ALL asymptotes.

- Slant / Horizontal Asymptotes.

$$\lim_{x\to\pm\infty} \frac{f(x)}{x} = \lim_{x\to\pm\infty} \frac{x^2(x+1)}{x(x^2+3x+2)} =$$

$$= \lim_{x\to\pm\infty} \frac{x^3}{x^3} = 1 = a$$

$$\lim_{x\to\pm\infty} [f(x)-x] = \lim_{x\to\pm\infty}\left[\frac{x^2(x+1)}{x^2+3x+2} - x\right] =$$

$$= \lim_{x\to\pm\infty} \frac{x^2(x+1)-(x^2+3x+2)x}{x^2+3x+2} =$$

$$= \lim_{x\to\pm\infty} \frac{\cancel{x^3}+x^2-\cancel{x^3}-3x^2-2x}{x^2+3x+2} = \lim_{x\to\pm\infty} \frac{-2x^2}{x^2} = -2 = b$$

The line $(l): y = x-2$ is slant asymptote at $\pm\infty$.

- Vertical Asymptotes
- FIRST you must simplify $f(x)$. (eliminate 0/0 situations)

$$f(x) = \frac{x^2(x+1)}{(x+1)(x+2)} = \frac{x^2}{x+2}.$$

So we have vertical asymptote at $(\ell): x=-2$.

In general vertical asymptotes occur when you have the form $A/0$.
To PROVE it you must do the limit:

$$\left.\begin{array}{l}\lim\limits_{x\to -2^+} \dfrac{1}{x+2} = +\infty \\ \lim\limits_{x\to -2^+} x^2 = 4>0\end{array}\right\} \Rightarrow \lim_{x\to -2^+} \frac{x^2}{x+2} = +\infty .$$

$\Rightarrow (\ell): x=-2$ vertical asymptote

- A proof is needed UPON REQUEST.
  If the vertical asymptote is obvious, you can state it without proof.

2) $f(x) = x^3+5x^2+3x+1$. (Polynomial)

f continuous in $\mathbb{R} \Rightarrow$ no vertical asymptotes

$$\lim_{x\to\pm\infty} \frac{f(x)}{x} = \lim_{x\to\pm\infty} \frac{x^3+5x^2+3x+1}{x} =$$

$$= \lim_{x\to\pm\infty} \frac{x^3}{x} = \lim_{x\to\pm\infty} x^2 = +\infty$$

$\Rightarrow$ no slant or horizontal asymptotes.

3) $f(x) = \sin x + \cos x$ (trig. functions)

f continuous at $\mathbb{R} \Rightarrow$ no vertical asymptotes.

- We show that $\lim\limits_{x \to \pm\infty} \frac{f(x)}{x} = 0$

$$\left|\frac{f(x)}{x}\right| = \left|\frac{\sin x + \cos x}{x}\right| \leq \frac{|\sin x| + |\cos x|}{|x|}$$

$$\left.\begin{array}{l} \leq \frac{1+1}{|x|} = \frac{2}{|x|}, \ \forall x \in \mathbb{R} \\ \lim\limits_{x \to \pm\infty} \frac{2}{|x|} = 0 \end{array}\right\} \Rightarrow$$

$\Rightarrow \lim\limits_{x \to \pm\infty} \frac{f(x)}{x} = 0 \Rightarrow$ no slant asymptotes.

- We show that $\lim\limits_{x \to \pm\infty} f(x)$ does not exist.

For $a_n = 2n\pi \Rightarrow f(a_n) = \sin(2n\pi) + \cos(2n\pi)$
$= 0 + 1 = 1$

$b_n = 2n\pi + \pi/3 \Rightarrow f(b_n) = \sin(\pi/3) + \cos(\pi/3)$
$= \sqrt{3}/2 + 1/2 \neq 1$

$$\text{Since} \left.\begin{array}{l} \lim a_n = +\infty \\ \lim b_n = +\infty \\ \lim f(a_n) = 1 \\ \lim f(b_n) = \sqrt{3}/2 + 1/2 \end{array}\right\} \Rightarrow \lim\limits_{x \to +\infty} f(x) \text{ does not exist}$$

similar argument that $\lim_{x \to -\infty} f(x)$ does not exist.

Thus f has no horizontal asymptotes

4) $f(x) = \sqrt{3x^2 - x}$ (radicals)

Slant Asymptotes

$$\lim_{x \to \pm} \frac{f(x)}{x} = \frac{\sqrt{3x^2 - x}}{x} = \frac{|x|\sqrt{3 - 1/x}}{x} =$$

$$= \frac{|x|}{x} \sqrt{3 - 1/x} \Rightarrow$$

$$\Rightarrow \lim_{x \to \pm\infty} \frac{f(x)}{x} = \lim_{x \to \pm\infty} \frac{|x|}{x} \cdot \lim_{x \to \pm\infty} \sqrt{3 - 1/x} =$$

$$= \sqrt{3} \lim_{x \to \pm\infty} \frac{|x|}{x}$$

$$= \begin{cases} +\sqrt{3}, & \text{when } x \to +\infty \\ -\sqrt{3}, & \text{when } x \to -\infty \end{cases}$$

So $a_+ = \sqrt{3}$ and $a_- = -\sqrt{3}$.

$$b_+ = \lim_{x \to +\infty} [f(x) - a_+ x] =$$

$$= \lim_{x \to +\infty} \left[\sqrt{3x^2 - x} - \sqrt{3}\, x\right] =$$

$$= \lim_{x \to +\infty} \left[\frac{(3x^2 - x) - 3x^2}{\sqrt{3x^2 - x} + x\sqrt{3}}\right] =$$

$$= \lim_{x \to +\infty} \frac{-x}{\sqrt{3x^2 - x} + x\sqrt{3}} =$$

$$= \lim_{x \to +\infty} \frac{-x}{x\left[\sqrt{3 - 1/x} + \sqrt{3}\right]} =$$

$$= \lim_{x \to +\infty} \frac{-1}{\sqrt{3 - 1/x} + \sqrt{3}} = \frac{-1}{2\sqrt{3}}$$

$$b_- = \lim_{x \to -\infty} \left[f(x) - a_- x \right] =$$

$$= \lim_{x \to -\infty} \left[\sqrt{3x^2 - x} + \sqrt{3}\, x \right]$$

$$= \lim_{x \to -\infty} \frac{(3x^2 - x) - (-\sqrt{3}\, x)^2}{\sqrt{3x^2 - x} - \sqrt{3}\, x} =$$

$$= \lim_{x \to -\infty} \frac{3x^2 - x - 3x^2}{-x\sqrt{3 - 1/x} - \sqrt{3}\, x}$$

CAUTION : $|x| = -x$ when $x \to -\infty$!!

$$= \lim_{x \to -\infty} \frac{-x}{-x\left[\sqrt{3 - 1/x} + \sqrt{3}\right]} =$$

$$= \lim_{x \to -\infty} \frac{1}{\sqrt{3 - 1/x} + \sqrt{3}} = \frac{1}{2\sqrt{3}} .$$

To summarize, we found

$a_+ = +\sqrt{3}$

$a_- = -\sqrt{3}$

$b_+ = -1/2\sqrt{3}$

$b_- = +1/2\sqrt{3}$

thus we have Two slant asymptotes

$$(\ell_1) : y = \sqrt{3}\,x - \frac{1}{2\sqrt{3}} \quad \text{when} \quad x \to +\infty$$

$$(\ell_2) : y = -\sqrt{3}\,x + \frac{1}{2\sqrt{3}} \quad \text{when} \quad x \to -\infty.$$

- Vertical Asymptotes.

Domain of f:

$3x^2 - x \geq 0 \iff x(3x-1) \geq 0$

$\iff x \in (-\infty, 1/3] \cup [1/3, +\infty)$

thus $A_f = (-\infty, 1/3] \cup [1/3, +\infty)$

f continuous at A_f including the points $1/3$ and $-1/3$, so no vertical asymptotes.