Method of sign tables

The method of sign tables relies on the following facts:

1) $f(x) = ax + b$

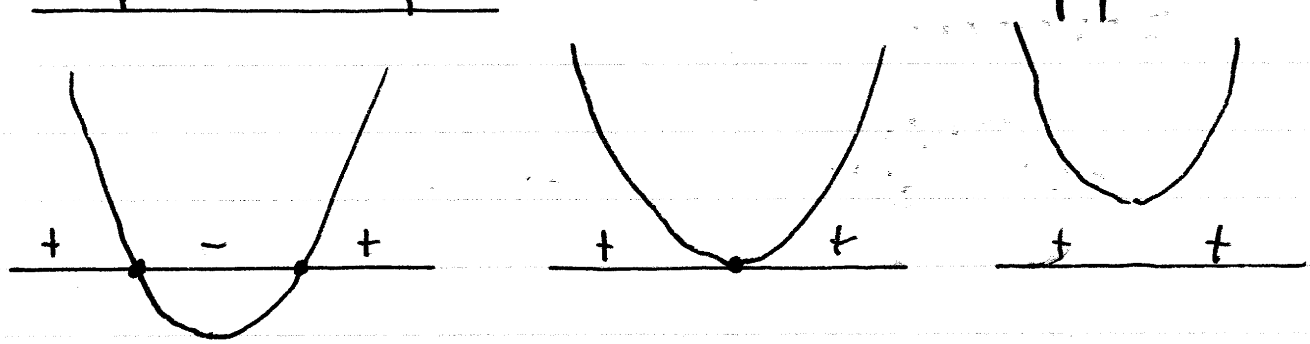
$a > 0 \Rightarrow f$ increases $\Rightarrow$ goes from negative $\rightarrow$ positive

$a < 0 \Rightarrow f$ decreases $\Rightarrow$ goes from positive $\rightarrow$ negative.

2) $f(x) = ax^2 + bx + c$

Always has the same sign as coefficient "a"
UNLESS $\Delta = b^2 - 4ac > 0$ (two zeroes) and
x is between the two zeroes (i.e. $\rho_1 \leq x \leq \rho_2$).

Graphical examples: when $a > 0$ (convex up parabola)



$\Delta > 0$ — two zeroes $\downarrow$ $\rho_{1,2} = \dfrac{-b \pm \sqrt{\Delta}}{2a}$

$\Delta = 0$ — one zero $\downarrow$ $\rho_1 = \rho_2 = \dfrac{-b}{2a}$

$\Delta < 0$ — no zeroes $\downarrow$ $\rho_{1,2} \notin \mathbb{R}$.

3) $f(x) = (ax+b)^{2k}$ or $f(x) = (ax^2+bx+c)^{2k}$

with $k \in \mathbb{N}$

are <u>ALWAYS POSITIVE</u>

4) $f(x) = (ax+b)^{2k+1}$ or $f(x) = (ax^2+bx+c)^{2k+1}$

$\boxed{\text{have the same sign as}}$

$f(x) = ax+b$ $\qquad$ $f(x) = ax^2+bx+c$

Applications of sign tables

1) Polynomial inequalities
2) Rational inequalities
3) Elimination of absolute values
4) Monotonicity and Convexity (Calculus).

Polynomial Inequalities

Form: $a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 \gtrless 0$

Method:
- 1 Move everything to left side
- 2 Factor the left side
- 3 Identify and sort the zeroes of each factor.
- 4 Make a sign table for each factor.
- 5 Multiply the signs to get signs for inequality as a whole.
- 6 Identify carefully the intervals that correspond to the desired solution.

Examples

1) ~~xdeo~~ $x^3 - 4x \geqslant -3$

Solution: $x^3 - 4x \geqslant -3 \Leftrightarrow x^3 - 4x + 3 \geqslant 0 \Leftrightarrow$

$$\Leftrightarrow x^3 - x - 3x + 3 \geqslant 0 \Leftrightarrow$$

$$\Leftrightarrow x(x^2-1) - 3(x-1) \geqslant 0$$

$$\Leftrightarrow x(x-1)(x+1) - 3(x-1) \geqslant 0 \Leftrightarrow$$

$$\Leftrightarrow (x-1)[x(x+1) - 3] \geqslant 0$$

$$\Leftrightarrow (x-1)(x^2 + x - 3) \geqslant 0$$

$x-1 \to$ zeroes : $+1$

$x^2+x-3 \to \Delta = 1^2 - 4\cdot 1\cdot(-3) = 13 > 0$

$\Rightarrow$ two zeroes

$$\rho_{1,2} = \frac{-1 \pm \sqrt{13}}{2}$$

Sort zeroes: $(-1-\sqrt{13})/2$, 1 , $(-1+\sqrt{13})/2$

To compare these two:

$$\frac{-1+\sqrt{13}}{2} \geq \frac{-1+\sqrt{9}}{2} = \frac{-1+3}{2} = 1$$

- check the nearest perfect squares of the number under the root. This is usually sufficient for a decisive comparison.

x		$(-1-\sqrt{13})/2$		1		$(-1+\sqrt{13})/2$	
$x-1$	$-$		$-$	0	$+$		$+$
x^2+x-3	$+$	0	$-$		$-$	0	$+$
$f(x)$	$-$	0	$+$	0	$-$	0	$+$

thus $x \in \left[\frac{-1-\sqrt{13}}{2}, 1\right] \cup \left[\frac{-1+\sqrt{13}}{2}, +\infty\right)$.

2) $(2x^4 - x^2)(x^2-3)^2(2-x)^3 < 0$ (1)

▸ Note that now we do not allow equality with 0.

Solution:

$(1) \Leftrightarrow x^2(2x^2-1)(x^2-3)^2(2-x)^3 < 0.$

Zeroes: $0, \pm 1/\sqrt{2} = \pm \frac{\sqrt{2}}{2}, \pm\sqrt{3}, 2$

Sorted: $-\sqrt{3}, -\sqrt{2}/2, 0, \sqrt{2}/2, \sqrt{3}, 2.$

x		$-\sqrt{3}$		$-\sqrt{2}/2$		0		$\sqrt{2}/2$		$\sqrt{3}$		2	
x^2	+		+		+	0	+		+		+		+
$2x^2-1$	+		+	0	−		−	0	+		+		+
$(x^2-3)^2$	+	0	+		+		+		+	0	+		+
$(2-x)^3$	+		+		+		+		+		+	0	−
$f(x)$	+	0	+	0	−	0	−	0	+	0	+	0	−

Thus $(1) \Leftrightarrow x \in (-\sqrt{2}/2, 0) \cup (0, \sqrt{2}/2) \cup (2, +\infty).$

Rational Inequalities

Form : $\dfrac{P(x)}{Q(x)} \gtrless 0$

with P, Q polynomials.

Method : The method entails the same steps as with polynomial inequalities. However, the zeroes of numerator factors must be distinguished from the zeroes of denominator factors.

- Denominator zeroes are shown with the $\ddagger$ symbol instead of ϕ in the last entry of your sign table because at these zeroes, the expression is undefined.
- Denominator zeroes are to be excluded from the solution set.

examples

1) $\dfrac{x-5}{x-3} \geq \dfrac{x-2}{x-1} \qquad (1)$

Solution:

$$(1) \Leftrightarrow \frac{x-5}{x-3} - \frac{x-2}{x-1} \geq 0 \Leftrightarrow \frac{(x-5)(x-1)-(x-2)(x-3)}{(x-3)(x-1)} \geq 0$$

$$\Leftrightarrow \frac{(x^2-6x+5)-(x^2-5x+6)}{(x-3)(x-1)} \geqslant 0$$

$$\Leftrightarrow \frac{(-6+5)x+(5-6)}{(x-3)(x-1)} \geqslant 0$$

$$\Leftrightarrow \frac{-x-1}{(x-3)(x-1)} \geqslant 0. \quad (2)$$

Zeroes: $-1, 3, 1$

x		-1		1		3	
$-x-1$	$+$	0	$-$		$-$		$-$
$x-3$	$-$		$-$		$-$	0	$+$
$x-1$	$-$		$-$	0	$+$		$+$
$f(x)$	$+$	0	$-$	‖	$+$	‖	$-$

$(2) \Leftrightarrow x \in (-\infty, -1] \cup (1, 3)$

$\hookrightarrow$ Note that -1 is a zero of $f(x)$ but 1 and 3 are not, so they are not included in the solution.

→ <u>CAUTION</u> : If the fraction has cancellations then you must find the domain of the inequality before solving it:

<u>example</u> : $\dfrac{(x+1)(x^2+4x+4)}{(x^2+5x+6)} \geqslant 0. \quad (1)$

$$(1) \Leftrightarrow \frac{(x+1)(x+2)^2}{(x+2)(x+3)} \geqslant 0 \Leftrightarrow \frac{(x+1)(x+2)}{x+3} \geqslant 0$$

▶ Domain:

$x^2+5x+6 \neq 0 \Leftrightarrow \underline{x \in \mathbb{R} - \{-2, -3\} = A}$

x		-3		-2		-1	
$x+1$	$-$		$-$		$-$	0	$+$
$x+2$	$-$		$-$	0	$+$		$+$
$x+3$	$-$	0	$+$		$+$		$+$
$f(x)$	$-$	$\|$	$+$	$\|$	$-$	0	$+$

thus

$(1) \Leftrightarrow \underline{x \in (-3, -2) \cup [-1, +\infty)}$

-2 looks like a numerator zero but it cannot solve the original inequality because the domain

$$A = \mathbb{R} - \{-2, -3\}$$

of the inequality EXCLUDES -2 !!

Elimination of absolute values

Method : Sign tables can be used in conjuction with expressions that involve absolute values.

Example : Simplify
$$f(x) = |x^2+3x+2| + |x+5|.$$

Solution:

Zeroes : $-2, -1, -5$

x		-5		-2		-1	
x^2+3x+2	$+$		$+$	0	$-$	0	$+$
$x+5$	$-$	0	$+$		$+$		$+$

Distinguish 4 cases

a) $x \in (-\infty, -5)$
$$f(x) = (x^2+3x+2) - (x+5) = x^2+2x-3$$
b) $x \in [-5, -2)$
$$f(x) = (x^2+3x+2) + (x+5) = x^2+4x+7$$
c) $x \in [-2, -1)$
$$f(x) = -(x^2+3x+2) + (x+5) = -x^2-2x+3$$
d) $x \in [-1, +\infty)$
$$f(x) = \cdots = x^2+4x+7.$$

thus

$$f(x) = \begin{cases} x^2+2x-3, & x \in (-\infty,-5) \\ x^2+4x+7, & x \in [-5,-2) \cup [-1,+\infty) \\ -x^2-2x+3, & x \in [-2,-1). \end{cases}$$

$\hookrightarrow$ Similar applications are possible to equations / inequalities with absolute values where a <u>case distinction</u> is necessary in setting up your solution