Rates of Change

- Let $x = f(t)$ be the location of an object traveling on a straight line.
  At time $t + \Delta t$:

a) Displacement: $\Delta x = f(t+\Delta t) - f(t)$

b) Average velocity: $u = \dfrac{\Delta x}{\Delta t} = \dfrac{f(t+\Delta t) - f(t)}{\Delta t}$

Note that $u = u(t, \Delta t)$.

<u>example</u> : $x = t^2 + 3t \leftarrow \begin{array}{l} \Delta x \\ \Delta x / \Delta t \end{array}$

- To define <u>instanteneous velocity</u> we want Δt to be infinitely small
  This motivates the <u>Leibnitz infinitesimal</u>

↳ <u>Leibnitz infinitesimal</u>

It is a very special number ε such that it satisfies these rules:

① It respects all the laws of algebra

e.g. $\varepsilon + \varepsilon = 2\varepsilon$

$\varepsilon / \varepsilon = 1$

$\varepsilon^2 / \varepsilon = \varepsilon$, etc.

② $\boxed{\varepsilon > 0}$

③ For any numbers a, b; with $\underline{\underline{a \neq 0}}$:

$$\boxed{a + b\varepsilon = a}$$

- Note that

$a\varepsilon + b\varepsilon^2 = \varepsilon(a + b\varepsilon) = \varepsilon \cdot a = a\varepsilon$

Similarly

$a_0 + a_1\varepsilon + a_2\varepsilon^2 + \cdots + a_n\varepsilon^n = a_0$

↦ <u>Instanteneous rate of change</u>

Now we can define instanteneous velocity:

$$\boxed{u(t) = \frac{f(t+\varepsilon) - f(t)}{\varepsilon}}$$

More generally:

Average Velocity $\longrightarrow$ Average Rate of change

Instanteneous velocity $\rightarrow$ Instanteneous rate of change

example : $x = t^2 + 3t \longrightarrow u(t)$?

example : cost function

$C(x) = x^3$

marginal cost function

$$MC(x) = \frac{C(x+\varepsilon) - C(x)}{\varepsilon} = \dots = 3x^2.$$

History of infinitesimal

1) Introduced by G.W. Leibnitz (16th century)
2) British Counterattack (why?)
   a) Newton: He stole it from me
   b) Bishop Berkeley: Attacked ε
3) Cauchy-Weirstrass : (17th-18th cent)
   Calculus with limits instead of ε
4) Abraham Robinson: (20th century)
   Infinitesimal is legitimate.

Limits

- Let a, b be numbers, with $b \neq 0$

$$\boxed{\lim_{x \to a} f(x) = b \Leftrightarrow f(a+\varepsilon) = f(a-\varepsilon) = b}$$

<u>example</u> : $\lim_{x \to 3} x^2 = 9$

For $f(x) = x^2$

$$\left.\begin{array}{l} f(3+\varepsilon) = (3+\varepsilon)^2 = 3^2 = 9 \\ f(3-\varepsilon) = (3-\varepsilon)^2 = 3^2 = 9 \\ f(3) = 9 \end{array}\right\} \Rightarrow \lim_{x \to 3} x^2 = 9.$$

$\hookrightarrow$ In general, if f is a polynomial

$f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$

then

$\lim_{x \to a} f(x) = f(a).$

<u>example</u> : $f(x) = x^3 + 3x^2 - x + 1 \leftarrow \lim_{x \to 2} f(x).$

If f, g are polynomials then

$$\boxed{g(a) \neq 0 \Rightarrow \lim_{x \to a} \frac{f(x)}{g(x)} = \frac{f(a)}{g(a)}}$$

example: $f(x) = \dfrac{x^2+3x}{x+1} \leftarrow \lim_{x \to 3} f(x)$

If $\boxed{\lim_{x \to a} f(x) = b > 0 \Rightarrow \lim_{x \to a} \sqrt{f(x)} = \sqrt{b}}$

example: $f(x) = \sqrt{x^2+2x+3} \leftarrow \lim_{x \to 1} f(x).$

0/0 Limits

We use binomial quotient identities:

$a^2 - b^2 = (a-b)(a+b)$

$a^3 - b^3 = (a-b)(a^2+ab+b^2)$

etc. , or factorization

to try to cancel the infinitesimal.

examples: $\lim_{x \to 2} \dfrac{x^2-4}{x-2}$; $\lim_{x \to -3} \dfrac{x^2+6x+9}{x^2+5x+6}$

$\updownarrow \to$ Form $0/0$ with $\sqrt{f(x)} - \sqrt{g(x)}$

Use the identity

$$a - b = \frac{a^2 - b^2}{a + b}$$

to eliminate the radicals and cancel the infinitesimal.

examples : $f(x) = \dfrac{\sqrt{x-1} - 2}{x-5} \leftarrow \lim\limits_{x\to 5} f(x)$

$f(x) = \dfrac{x^2 - \sqrt{x}}{\sqrt{x} - 1} \leftarrow \lim\limits_{x \to 1} f(x)$.

Side limits and infinity

- Let a, b be numbers. ~~Then~~ with $b \neq 0$.

$$\boxed{\begin{aligned} &\lim_{x \to a^+} f(x) = b \iff f(a+\varepsilon) = b \\ &\lim_{x \to a^-} f(x) = b \iff f(a-\varepsilon) = b \end{aligned}}$$

Recall that

$$\boxed{\lim_{x \to a} f(x) = b \iff f(a+\varepsilon) = f(a-\varepsilon) = b}$$

It follows that

$$\boxed{\lim_{x \to a} f(x) = b \iff \lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x) = b}$$

- Note that if $\lim\limits_{x \to a^+} f(x) \neq \lim\limits_{x \to a^-} f(x)$,

we say that $\lim\limits_{x \to a} f(x)$ <u>does not exist</u>

example : $\cancel{\lim}\, f(x) = \dfrac{|x|}{x} \longleftarrow \lim\limits_{x \to 0^+} f(x) = 1$

$$\lim_{x \to 0^-} f(x) = -1$$

$\lim\limits_{x \to 0} f(x)$ does not exist

example

$$f(x) = \frac{x^2 + 2|x|}{x^2 - 2|x|} \longleftarrow \lim_{x \to 0^+} f(x) = -1$$

$$\lim_{x \to 0^-} f(x) = -1$$

$$\lim_{x \to 0} f(x) = -1$$

example

The case $b = 0$

For ~~every~~ $k > 0$, the number $a\varepsilon^k$ is infinitely close to 0. So we give the following definitions:

$$f(A+\varepsilon) = a\varepsilon^k \Rightarrow \lim_{x \to A^+} f(x) = 0 \quad , \text{ with } k > 0$$

$$f(A-\varepsilon) = a\varepsilon^k \Rightarrow \lim_{x \to A^-} f(x) = 0 \quad , \text{ with } k > 0$$

$$\lim_{x \to A^+} f(x) = \lim_{x \to A^-} f(x) = 0 \Rightarrow \lim_{x \to A} f(x) = 0$$

example: $\lim_{x \to 1^+} \left[(x-1)^2 + 3(x-1)(x-2) \right]$

The concept of infinity $\pm\infty$

- The symbol $\pm\infty$ does not represent a number or even an unambiguous expression involving infinitesimals.

- Note that $1/\varepsilon$, $1/\varepsilon^2$, $1/\varepsilon^3$ etc are all infinitely large. This motivates the following definitions:

$$\lim_{x\to a^+} f(x) = \begin{cases} +\infty \iff f(a+\varepsilon) \geq A/\varepsilon^k \text{ with } A>0, k>0 \\ -\infty \iff f(a+\varepsilon) \leq -A/\varepsilon^k \text{ with } A>0, k>0 \end{cases}$$

$$\lim_{x\to a^-} f(x) = \begin{cases} +\infty \iff f(a-\varepsilon) \geq A/\varepsilon^k \text{ with } A>0, k>0 \\ -\infty \iff f(a+\varepsilon) \leq -A/\varepsilon^k \text{ with } A>0, k>0 \end{cases}$$

<u>example</u> : $f(x) = \dfrac{1+4x}{(x-3)^3} \leftarrow \lim\limits_{x\to 3^-} f(x)$

$$f(3-\varepsilon) = \frac{1+4(3-\varepsilon)}{(3-\varepsilon-3)^3} = \frac{1+12-4\varepsilon}{(-\varepsilon)^3} = \frac{13-4\varepsilon}{-\varepsilon^3}$$

$$\simeq \frac{13}{-\varepsilon^3} = -\frac{13}{\varepsilon^3} \Rightarrow \lim_{x\to 3^-} f(x) = -\infty.$$

- We may extend definition:

$$\lim_{x\to a} f(x) = +\infty \Leftrightarrow \lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x) = +\infty$$

$$\lim_{x\to a} f(x) = -\infty \Leftrightarrow \lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x) = -\infty$$

<u>example</u> : $f(x) = \dfrac{x+1}{(x-2)^2} \leftarrow \lim_{x\to 2} f(x)$

$$f(2\pm\varepsilon) = \frac{2\pm\varepsilon+1}{(2\pm\varepsilon-2)^2} = \frac{3\pm\varepsilon}{(\pm\varepsilon)^2} = \frac{3}{(\pm\varepsilon)^2} = \frac{3}{\varepsilon^2}$$

$\Rightarrow \lim_{x\to 2} f(x) = +\infty$.

<u>example</u> : $f(x) = \dfrac{2x-1}{(x-1)^3} \leftarrow \lim_{x\to 1} f(x)$

$$f(1+\varepsilon) = \frac{2(1+\varepsilon)-1}{(1+\varepsilon-1)^3} = \frac{2+2\varepsilon-1}{\varepsilon^3} = \frac{1}{\varepsilon^3} \Rightarrow \lim_{x\to 1^+} f(x) = +\infty \qquad (1)$$

$$f(1-\varepsilon) = \frac{2(1-\varepsilon)-1}{(1-\varepsilon-1)^3} = \frac{2-2\varepsilon-1}{(-\varepsilon)^3} = \frac{1}{-\varepsilon^3} \Rightarrow$$

$\Rightarrow \lim_{x\to 1^-} f(x) = -\infty$. (2)

From (1) and (2): $\lim_{x\to 1} f(x)$ does not exist.

▸ Algebra with $\pm\infty$

- Let a be a number
  $p > 0$ be a positive number
  $n < 0$ be a negative number.

Then

$(+\infty) + (+\infty) = +\infty$	$a + (+\infty) = +\infty$	$\dfrac{a}{+\infty} = 0$
$(-\infty) + (-\infty) = -\infty$	$a + (-\infty) = -\infty$	
$-(-\infty) = +\infty$	$p(+\infty) = +\infty$	$\dfrac{a}{-\infty} = 0$
$(+\infty)(-\infty) = -\infty$	$p(-\infty) = -\infty$	
$(+\infty)(+\infty) = +\infty$	$n(+\infty) = \cancel{+\infty} -\infty$	
$(-\infty)(-\infty) = -\infty$	$n(-\infty) = +\infty$	

- Because $\pm\infty$ is an ambiguous symbol, the following expressions, when encountered in a limit evaluation are <u>indeterminate</u>

$$(+\infty) - (+\infty) \qquad 0 \cdot (+\infty) \qquad \frac{\pm\infty}{\pm\infty}$$
$$(-\infty) - (-\infty) \qquad 0 \cdot (-\infty)$$

This means that the limit could exist but we don't know yet what it equals to.

Applications

a) $$\lim_{x \to a^+} \frac{1}{x-a} = +\infty$$

b) $$\lim_{x \to a^-} \frac{1}{x-a} = -\infty$$

c) $$\lim_{x \to a} \frac{1}{(x-a)^{2k}} = +\infty \, , \forall k \in \mathbb{N}-\{0\}.$$

examples

1) $$f(x) = \frac{1-3x}{(x-2)^2} \leftarrow \lim_{x \to 2} f(x)$$

2) $$f(x) = \frac{2x+1}{2x-1} \leftarrow \lim_{x \to 1/2^+} f(x)$$

3) $$f(x) = \frac{3-4x}{(x-3)^3} \leftarrow \lim_{x \to 3} f(x)$$

4) $$f(x) = \frac{x^2+3x+2}{x^2+4x+4} \leftarrow \lim_{x \to -2^+} f(x).$$

Examples with Side Limits

1) $f(x) = \dfrac{2x-1}{2-3x} \leftarrow \lim\limits_{x \to 2/3^+} f(x).$

Solution:

$$f(x) = \frac{2x-1}{2-3x} = (2x-1)\frac{1}{-3} \cdot \frac{1}{x-2/3} =$$

$$= -\frac{2x-1}{3} \cdot \frac{1}{x-2/3}.$$

$$\lim_{x \to 2/3^+}\left[-\frac{2x-1}{3}\right] = -\frac{2\cdot(2/3)-1}{3} = -\frac{1/3}{3} < 0 \quad (1)$$

$$\lim_{x \to 2/3^+} \frac{1}{x-2/3} = +\infty \quad (2)$$

Multiply (1) and (2): $\lim\limits_{x \to 2/3^+} f(x) = -\infty.$

2) $f(x) = \dfrac{5-2x}{(x-5)^2} \leftarrow \lim\limits_{x \to 5} f(x).$

Solution:

$$f(x) = (5-2x) \cdot \frac{1}{(x-5)^2}.$$

$$\left.\begin{array}{l} \lim\limits_{x \to 5} (5-2x) = 5-10 < 0 \\ \lim\limits_{x \to 5} \dfrac{1}{(x-5)^2} = +\infty \end{array}\right\} \Rightarrow \lim_{x \to 5} f(x) = -\infty.$$

3) $f(x) = \dfrac{x+1}{(x-1)^3} \longleftarrow \lim\limits_{x \to 1} f(x)$

<u>Solution</u>

$$f(x) = (x+1)\frac{1}{(x-1)^3}$$

$$\lim_{x \to 1} (x+1) = 1+1 = 2 \qquad (1)$$

$$\lim_{x \to 1^-} \frac{1}{(x-1)^3} = (-\infty)(-\infty)(-\infty) = -\infty \qquad (2)$$

$$\lim_{x \to 1^+} \frac{1}{(x-1)^3} = (+\infty)^3 = +\infty \qquad (3)$$

From (1) and (2): $\lim\limits_{x \to 1^-} f(x) = -\infty \qquad (4)$

From (1) and (3): $\lim\limits_{x \to 1^+} f(x) = +\infty \qquad (5)$

From (4) and (5): $\lim\limits_{x \to 1} f(x)$ does not exist.

Limits at infinity $x \to \pm\infty$

- Let f be a function.
  Then we define:

  a) $\lim\limits_{x \to +\infty} f(x) = a \neq 0 \iff f(1/\varepsilon) = a$

  b) $\lim\limits_{x \to +\infty} f(x) = 0 \iff f(1/\varepsilon) = A\,\varepsilon^{+K}$ with $K > 0$

  c) $\lim\limits_{x \to +\infty} f(x) = +\infty \iff f(1/\varepsilon) \geq A\,\varepsilon^{-k}$ with $k > 0$ and $A > 0$

  d) $\lim\limits_{x \to +\infty} f(x) = -\infty \iff f(1/\varepsilon) \leq A\,\varepsilon^{-k}$ with $k > 0$ and $A < 0$.

example

$$f(x) = \frac{x^2 + 3x}{2x^2 - 5} \leftarrow \lim_{x \to +\infty} f(x)$$

$$f(x) = \frac{-x^3 + 5x}{3x^2 + 2} \leftarrow \lim_{x \to +\infty} f(x)$$

▸ Solution using infinitesimals

$$f(x) = \frac{x^2+3x}{2x^2-5}$$

$$f(1/\varepsilon) = \frac{(1/\varepsilon)^2+3(1/\varepsilon)}{2(1/\varepsilon)^2-5} =$$

$$= \frac{\varepsilon^2[(1/\varepsilon)^2+3(1/\varepsilon)]}{\varepsilon^2[2(1/\varepsilon)^2-5]}$$

$$\simeq \frac{1+3\varepsilon}{2-5\varepsilon^2} \simeq \frac{1}{2} \Rightarrow$$

$$\Rightarrow \lim_{x\to+\infty} f(x) = \frac{1}{2}.$$

$$f(x) = \frac{-x^3+5x}{3x^2+2}$$

$$f(1/\varepsilon) = \frac{-(1/\varepsilon)^3+5(1/\varepsilon)}{3(1/\varepsilon)^2+2} = \frac{\varepsilon^3[-(1/\varepsilon)^3+5(1/\varepsilon)]}{\varepsilon^3[3(1/\varepsilon)^2+2]}$$

$$= \frac{-1+5\varepsilon^2}{3\varepsilon+2\varepsilon^3} \simeq \frac{-1}{3\varepsilon} \Rightarrow \lim_{x\to+\infty} f(x) = -\infty.$$

- A 2nd more efficient method follows later.

- We give a similar definition for $x \to -\infty$.

$$\lim_{x \to -\infty} f(x) = a \neq 0 \iff f(-1/\varepsilon) = a$$

$$\lim_{x \to -\infty} f(x) = 0 \iff f(-1/\varepsilon) = A\varepsilon^{+k} \text{ with } k > 0$$

$$\lim_{x \to -\infty} f(x) = +\infty \iff f(-1/\varepsilon) \geq A\varepsilon^{-k} \text{ with } k > 0 \text{ and } A > 0$$

$$\lim_{x \to -\infty} f(x) = -\infty \iff f(-1/\varepsilon) \leq A\varepsilon^{-k} \text{ with } k > 0 \text{ and } A < 0$$

▸ Case 1: Monomials

- You should be able to write answer immediately. Can be justified with infinitesimals.

examples: $\lim_{x \to -\infty} (3x^3)$ $\lim_{x \to -\infty} (-2x^6)$

$\lim_{x \to +\infty} \frac{-5}{2x^2}$ $\lim_{x \to +\infty} \frac{10}{x\sqrt{x}}$

▸ Case 2: Polynomials

• If $f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$
then

$$\boxed{\lim_{x \to \pm\infty} f(x) = \lim_{x \to \pm\infty} a_n x^n}$$

examples

$f(x) = 2x^3 + 3x + 1 \leftarrow \lim\limits_{x \to -\infty} f(x)$

$f(x) = 3x + x^2 - x^4 + 1 \leftarrow \lim\limits_{x \to +\infty} f(x)$

$f(x) = (x^2 + 3x)(5x - 1) \leftarrow \lim\limits_{x \to -\infty} f(x)$

▸ Case 3: Rational functions

• If $f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$
$\quad\;\; g(x) = b_m x^m + b_{n-1} x^{n-1} + \cdots + b_1 x + b_0$
then

$$\boxed{\lim_{x \to \pm\infty} \frac{f(x)}{g(x)} = \lim_{x \to \pm\infty} \frac{a_n x^n}{b_m x^m}}$$

examples

1) $f(x) = \dfrac{x + x^3 + 1}{2x - x^2} \leftarrow \lim\limits_{x \to -\infty} f(x)$

2) $f(x) = \dfrac{x^2 + 3x + 1}{3x^2 - 2} \leftarrow \lim\limits_{x \to +\infty} f(x)$

3) $f(x) = \dfrac{2x^2 + 1}{x^4 - x} \leftarrow \lim\limits_{x \to -\infty} f(x)$