▼ Derivatives

- Let f be a function.
  The derivative f' is defined as

$$\boxed{f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}}$$

provided that the limit exists.

- $f'(x)$ gives the instanteneous rate of change of $f(x)$ at x.
  e.g. $x = f(t)$ position of object moving on a line.
  $u = u(t) = f'(t) =$ velocity
  $a = a(t) = u'(t) =$ accelaration.

- It is possible to calculate $f'(x)$ by evaluating the limit. It is better however employ differentiation rules.

Differentiation Rules

1) Constant Rule

$f(x) = c \Rightarrow f'(x) = 0$

2) Power Rule

$f(x) = x^n \Rightarrow f'(x) = nx^{n-1}$

a) For $n = 0$: constant rule

b) For $n = 1/2$:

$$f(x) = \sqrt{x} \Rightarrow f'(x) = \frac{1}{2\sqrt{x}}$$

c) For $n = -1$

$f(x) = 1/x \Rightarrow f'(x) = -1/x^2$.

3) Scalar Multiple Rule

$f(x) = cg(x) \Rightarrow f'(x) = cg'(x)$

e.g: $f(x) = cx^n \Rightarrow f'(x) = ncx^{n-1}$.

examples :

$f(x) = 3x^4$

$f(x) = \dfrac{2}{3x^2}$

$f(x) = \dfrac{4}{x\sqrt{x}}$

$f(x) = \dfrac{\sqrt{x}}{\sqrt[3]{x}}$

$f(x) = (\sqrt[5]{x})^2$

4) Addition Rule

$$h(x) = f(x) + g(x) \Rightarrow h'(x) = f'(x) + g'(x)$$
$$h(x) = f(x) - g(x) \Rightarrow h'(x) = f'(x) - g'(x)$$

<u>examples</u> : $f(x) = 3x^2 + 2x$

$f(x) = x^5 + 3x^4 + 2x^3 + 1$

5) Product Rule

$$h(x) = f(x)g(x) \Rightarrow h'(x) = f'(x)g(x) + f(x)g'(x).$$

<u>examples</u> : $f(x) = (x^2+2x)(x^3+5x^2)$

$f(x) = (x^2+3x+1)(x^2-2x-3)$

6) Triple Product Rule

$$h(x) = f_1(x) f_2(x) f_3(x) \Rightarrow$$
$$\Rightarrow h'(x) = f_1'(x) f_2(x) f_3(x) + f_1(x) f_2'(x) f_3(x) + f_1(x) f_2(x) f_3'(x).$$

<u>example</u> : $f(x) = (x^2+1)(x^3+x)(2x+1)$

<u>example with radicals</u>

$f(x) = x^2 \sqrt{}$

7) Quotient Rule

$$h(x) = \frac{f(x)}{g(x)} \Rightarrow h'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$$

$$h(x) = \frac{1}{g(x)} \Rightarrow h'(x) = -\frac{g'(x)}{[g(x)]^2}$$

examples

$$f(x) = \frac{3}{x^3+3x^2+x+5}$$

$$f(x) = \frac{3x+2}{3x-2}$$

$$f(x) = \frac{x^2-x}{2x+1}$$

$$f(x) = \frac{x^3-4}{(2x+1)(3x-2)}$$

$\hookrightarrow$ Find derivative of denominator separately first

Chain Rule

- Let f, g be functions with derivatives f', g'. Then,

$$\boxed{h(x) = f(g(x)) \Rightarrow h'(x) = f'(g(x))\, g'(x)}$$

- The chain rule is a rule that generates new differentiation rules. This is done when we choose a specific function for f but leave g arbitrary.

example : Quotient rule.

Choose $f(x) = 1/x \Rightarrow f'(x) = -1/x^2$.
Let
$$h(x) = f(g(x)) = \frac{1}{g(x)}$$
By chain rule:

$$h'(x) = f'(g(x))\, g'(x) = \left[-\frac{1}{[g(x)]^2}\right] g'(x)$$
$$= -\frac{g'(x)}{[g(x)]^2} \leftarrow \text{Quotient Rule.}$$

① Generalized Power Rule

$$\boxed{h(x) = [g(x)]^n \Rightarrow h'(x) = n[g(x)]^{n-1} g'(x)}$$

Proof

Choose $f(x) = x^n \Rightarrow f'(x) = nx^{n-1}$

Let

$h(x) = f(g(x)) = [g(x)]^n \Rightarrow$

$\Rightarrow h'(x) = f'(g(x))\, g'(x) =$

$= [n (g(x))^{n-1}]\, g'(x) = n[g(x)]^{n-1} g'(x)$. □

examples

$f(x) = (3+2x)^7$

$f(x) = (x^2+3x+1)^4$

$f(x) = (2x+1)^3 (2x-1)^2$.

↳ combine with product rule.

② Generalized Root Law

$$h(x) = \sqrt{g(x)} \Rightarrow h'(x) = \frac{g'(x)}{2\sqrt{g(x)}}$$

(Proof : Choose $f(x) = \sqrt{x}$ ...)

<u>examples</u>

$f(x) = \sqrt{3x+2}$

$f(x) = \sqrt{x^3+3x^2+1}$

$f(x) = 2x^2\sqrt{x+1}$

Marginal Analysis

- Let $p =$ price of product
  $x =$ amount of product produced
- We assume that x, p are related by a demand function:
  $$p = f(x).$$

- Total revenue: $R(x) = xp = xf(x)$
- Marginal revenue:
  $$MR(x) = R'(x) = [xf(x)]' = f(x) + xf'(x)$$

- Cost Function
  $C(x) =$ cost of producing x amount

- Marginal cost function
  $MC(x) = C'(x)$.

- Profit Function
  $P(x) = R(x) - C(x) = xf(x) - C(x)$

- Marginal Profit Function
  $$MP(x) = P'(x) = R'(x) - C'(x)$$
  $$= f(x) + xf'(x) - C'(x).$$

<u>example</u> : Linear model:

Demand: $p = a - bx$, $a > 0,\ b > 0$

Cost : $C(x) = c_0 + c_1 x$, $c_0 > 0,\ c_1 > 0$

c_0 = overhead cost

c_1 = cost per product item

Revenue:

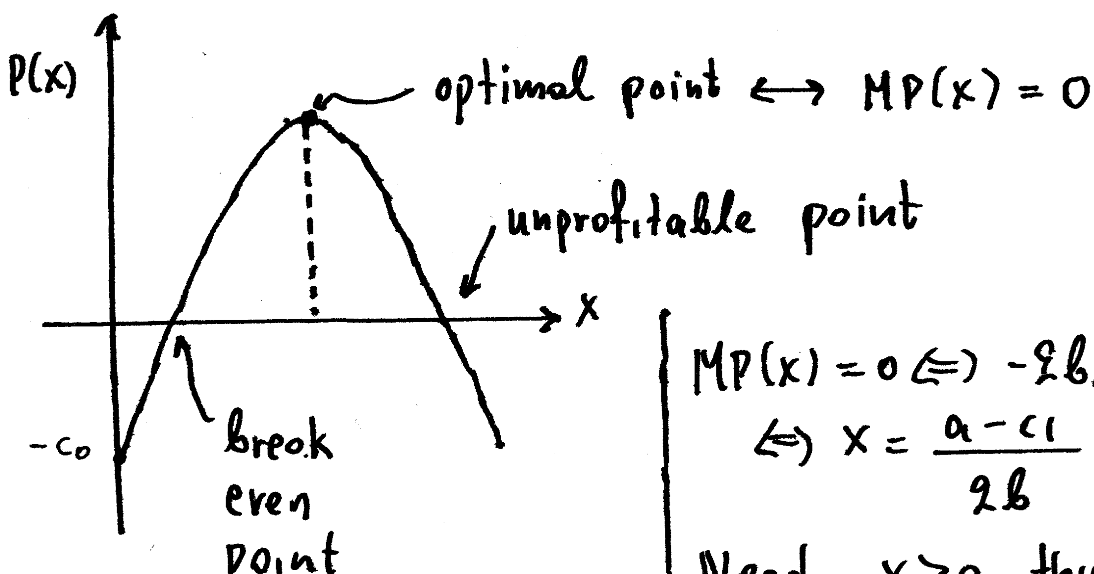
$$R(x) = px = (a - bx)x = ax - bx^2$$

Profit:

$$P(x) = R(x) - C(x) = [ax - bx^2] - [c_0 + c_1 x]$$
$$= ax - bx^2 - c_0 - c_1 x =$$
$$= -bx^2 + (a - c_1)x - c_0$$

Marginal Profit

$$MP(x) = P'(x) = [-bx^2 + (a - c_1)x - c_0]$$
$$= -2bx + (a - c_1)$$



$MP(x) = 0 \Leftrightarrow -2bx + (a - c_1) = 0$

$\Leftrightarrow x = \dfrac{a - c_1}{2b}$

Need $x > 0$ thus $a > c_1$

Average cost

- The average cost per item is given by

$$\bar{C}(x) = \frac{C(x)}{x}$$

and it is dependent on x.

- The marginal average cost per item is:

$$\overline{MC}(x) = [\bar{C}(x)]' = \left[\frac{C(x)}{x}\right]' =$$

$$= \frac{C'(x)\,x - C(x)(x)'}{x^2} =$$

$$= \frac{MC(x)\cdot x - C(x)}{x^2}$$

thus we showed that

$$\boxed{\overline{MC}(x) = \frac{x\,MC(x) - C(x)}{x^2}}$$

<u>example</u> : $C(x) = \dfrac{2x+1}{x+3}$

Want $\bar{C}(x)$, $\overline{MC}(x)$.

Emploee model

- Let $x =$ amount of product produced
  $p =$ sale price per unit
  $N =$ number of employees.

  Want profit as a function of N.

- Assume demand curve:
  $p = f(x)$
- Assume production curve
  $x = g(N)$

- Revenue Model:

$$R(N) = px = f(x)\,x = f(g(N))\,g(N)$$

$$MR(N) = R'(N) = [f(g(N))\,g(N)]' =$$

$$= [f(g(N))]'\,g(N) + f(g(N))\,[g(N)]'$$

$$= f'(g(N))\,g'(N)\,g(N) + f(g(N))\,g'(N) =$$

$$= [f'(g(N))\,g(N) + f(g(N))]\,g'(N).$$

- Cost model

$$C(N) = c_0 + c_1 x + c_2 N = $$
$$= c_0 + c_1 g(N) + c_2 N$$

with c_0 = overhead cost
c_1 = cost of raw materials
c_2 = cost of employees

$$MC(N) = C'(N) = c_1 g'(N) + c_2$$

<u>example</u> : $\begin{cases} p = 100/x \\ x = 5\sqrt{n} \end{cases}$ $c_0 = 2$, $c_1 = 3$, $c_2 = 5$

Derivative of exponential function

- The main result is

$$\boxed{f(x) = \exp(x) = e^x \Rightarrow f'(x) = \exp(x) = e^x}$$

<u>examples</u> : $f(x) = (x^2+3x)\,e^x$

$f(x) = \dfrac{e^x+1}{e^x-1}$

$f(x) = (e^x+x^2)^3$

- From the chain rule we get the more general result:

$$\boxed{f(x) = \exp(g(x)) \Rightarrow f'(x) = g'(x)\exp(g(x))}$$

<u>examples</u> : $f(x) = e^{-x^2}$

$f(x) = \exp(x-2x^3)$

$f(x) = (x+1)\exp(-x^3)$

- For more general power function:

$$\boxed{f(x) = a^x \Rightarrow f'(x) = a^x \ln a}$$

<u>examples</u> : $f(x) = (x^3+x)\,3^x$

$f(x) = \dfrac{2^x}{3^x+1}$

- From the chain rule:

$$\boxed{f(x) = a^{g(x)} \Rightarrow f'(x) = g'(x)\, a^{g(x)} \ln a}$$

<u>examples</u> : $f(x) = x^2\, 5^{x-x^3}$

Derivative of Logarithmic Function

- The main result is

$$\boxed{\begin{array}{ll} f(x) = \ln x \Rightarrow f'(x) = \dfrac{1}{x} & \text{for } x>0 \\ f(x) = \ln|x| \Rightarrow f'(x) = \dfrac{1}{x} & \text{for } x\neq 0 \end{array}}$$

<u>examples</u> : $f(x) = (x^3+5x^2)\ln x$

$f(x) = \ln(4x^6)$

$f(x) = \dfrac{\ln|x|}{1+\ln|x|}$ $\qquad f(x) = e^x \ln x$

- From the chain rule we get the generalized rule that

$$\boxed{f(x) = \ln(g(x)) \Rightarrow f'(x) = \frac{g'(x)}{g(x)}}$$

<u>example</u> : $f(x) = \ln(x^3+3x^2+x)$ $\quad f(x) = \ln(\ln x)$

$f(x) = \ln\left[\dfrac{x^2+1}{x^2-1}\right]$

$f(x) = \ln\sqrt{x^3+4x}$, $f(x) = \ln(e^x+1)$

- For the decimal logarithm:

$$f(x) = \log x \Rightarrow f'(x) = \frac{1}{x \ln 10}, \quad \text{if } x > 0$$

$$f(x) = \log|x| \Rightarrow f'(x) = \frac{1}{x \ln 10}, \quad \text{if } x \neq 0$$

$$f(x) = \log[g(x)] \Rightarrow f'(x) = \frac{g'(x)}{g(x) \ln 10}$$

<u>examples</u>: $f(x) = (x^2 + x) \log x$

$f(x) = \log(3x^4)$

$f(x) = \log(x^3 + 3x + 1)$