Monotonicity of a function

- The sign of the derivative $f'(x)$ of a function $f(x)$ can be used to determine the intervals where $f(x)$ is increasing or decreasing.

- Methodology : Monotonicity.

- 1 Calculate $f'(x)$
- 2 Factor $f'(x)$
- 3 Construct a sign chart for $f'(x)$ with an additional entry for $f(x)$.
- 4 $f(x)$ is increasing when $f'(x) > 0$
  $f(x)$ is decreasing when $f'(x) < 0$
- 5 f has a local max when f' changes from + to −
  f has a local min when f' changes from − to +
  However, singular points cannot be local min or local max.

<u>example</u> : $f(x) = (x-1)^2 (x+2)^3$

•1 $A_f = \mathbb{R}$.

•2 $f'(x) = \dots = (x-1)(x+2)^2(5x+1)$

with $A_{f'} = \mathbb{R}$.

x		-2		$-1/5$		1	
$x-1$	$-$		$-$		$-$	0	$+$
$(x+2)^2$	$+$	0	$+$		$+$		$+$
$(5x+1)$	$-$		$-$	0	$+$		$+$
f'	$+$	0	$+$	0	$-$	0	$+$
f	$\nearrow$		$\nearrow$		$\searrow$		$\nearrow$
				max		min	

$f \nearrow$ at $(-\infty,-2)$, $(-2,-1/5)$, $(1,+\infty)$

$f \searrow$ at $(-1/5, 1)$

max at $x=-1/5$

min at $x=1$

▶ $x=-2$ is not a min or max!

example : $f(x) = x^2 e^{4x} + 3$

$$f'(x) = [x^2 e^{4x}]' + 0 = (x^2)' e^{4x} + x^2 (e^{4x})' =$$
$$= 2x e^{4x} + x^2 e^{4x} \cdot (4x)' =$$
$$= 2x e^{4x} + 4x^2 e^{4x} =$$
$$= 2x e^{4x} (1 + 2x). \leftarrow \text{Zeroes: } -1/2 \,,\, 0$$

x		$-1/2$		0	
$2x$	$-$		$-$	0	$+$
e^{4x}	$+$		$+$		$+$
$1+2x$	$-$	0	$+$		$+$
$f'(x)$	$+$	0	$-$	0	$+$
$f(x)$	$\nearrow$		$\searrow$		$\nearrow$
		max		min	

$f \nearrow (-\infty, -1/2)$

$f \searrow (-1/2, 0)$

$f \nearrow (0, +\infty)$

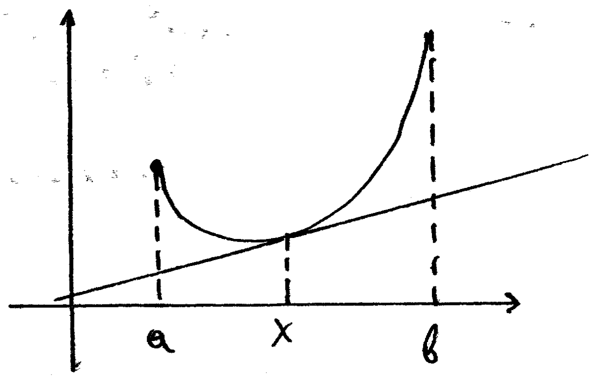
local max at $x = -1/2$

local min at $x = 0$.

Concavity

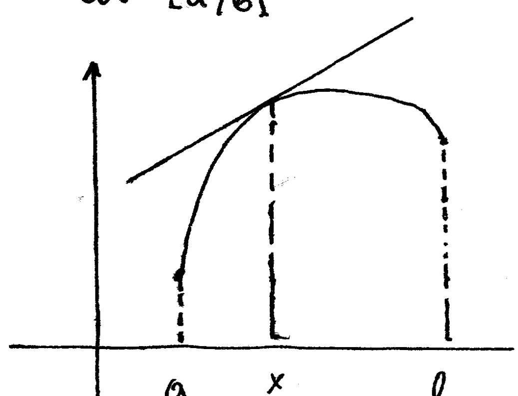
- Let f be a function. We say that f

a) f is concave up at $[a,b]$ $\Bigg\} \Leftrightarrow$ The graph of f is ABOVE every tangent line at $x \in [a,b]$

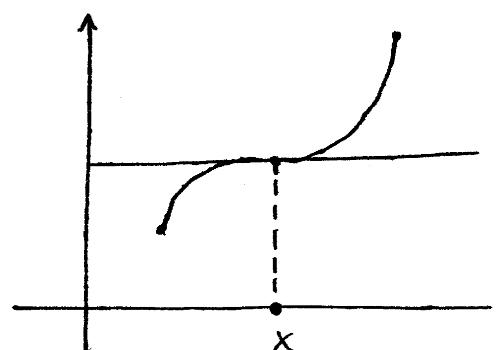


"concave up"

b) f is concave down at $[a,b]$ $\Bigg\} \Leftrightarrow$ The graph of f is BELLOW every tangent line at $x \in [a,b]$



"concave down"



"inflection point"

c) An inflection point x is a point where the function's concavity changes.

- The concavity of $f(x)$ depends on the sign of the second derivative $f''(x)$ which is defined as

$$f''(x) = [f'(x)]'$$

- Methodology : To determine concavity

- 1 Calculate and factor $f'(x)$ and then $f''(x)$
- 2 Make a sign chart for $f''(x)$ with an additional entry for $f(x)$.
- 3 f is concave up when $f''(x) > 0$
  f is concave down when $f''(x) < 0$
- 4 Inflection points are located at the zeroes of $f''(x)$ where the sign changes.

- Methodology: Curve Analysis

- 1 First make a monotonicity chart
- 2 Then make a concavity chart.
- 3 Merge the two charts into a curve analysis (variation) chart consisting of
  a) The zeroes of both f', f'' charts
  b) The entries f', f'', f
  c) Label f as: , , ,

example : $f(x) = (x^2+1)e^{-x}$

$$f'(x) = \dots = -(x-1)^2 e^{-x}$$
$$f''(x) = \dots = (x-3)(x-1)e^{-x}$$

Monotonicity:

x		1	
$-(x-1)^2$	$-$	0	$-$
e^{-x}	$+$		$+$
$f'(x)$	$-$	0	$-$
$f(x)$	↘		↘

Concavity:

x		1		3	
$x-3$	$-$		$-$	0	$+$
$x-1$	$-$	0	$+$		$+$
e^{-x}	$+$		$+$		$+$
$f''(x)$	$+$	0	$-$	0	$+$
$f(x)$	$\cup$		$\cap$		$\cup$
		infl.		infl.	

Curve Analysis

x		1		3	
$f'(x)$	$-$	0	$-$		$-$
$f''(x)$	$+$	0	$-$	0	$+$
$f(x)$	↘		↘		↘

No local min or max
Inflection points at
$x=1$ and $x=3$.

example : $f(x) = \dfrac{x^3}{x^2-1} \longleftarrow \begin{cases} \text{Monotonicity} \\ \text{Convexity} \\ \text{Variation} \end{cases}$

$$f'(x) = \dots = \frac{x^2(x^2-3)}{(x-1)^2(x+1)^2}$$

$$f''(x) = \dots = \frac{2x\,(x^2+3)}{(x-1)^3(x+1)^3}$$

- Monotonicity

x		$-\sqrt{3}$		-1		0		1		$+\sqrt{3}$	
x^2	+		+		+	0	+		+		+
x^2-3	+	0	−		−		−		−	0	+
$(x-1)^2$	+		+		+		+	0	+		+
$(x+1)^2$	+		+	0	+		+		+		+
$f'(x)$	+	0	−	$\Vert$	−	0	−	$\Vert$	−	0	+
$f(x)$	$\nearrow$		$\searrow$		$\searrow$		$\searrow$		$\searrow$		$\nearrow$

- Convexity

x		-1		0		1	
$2x$	−		−	0	+		+
x^2+3	+		+		+		+
$(x-1)^3$	−		−		−	0	+
$(x+1)^3$	−	0	+		+		+
f''	−	$\Vert$	+	0	−	$\Vert$	+
f	$\cap$		$\cup$		$\cap$		$\cup$

- Variation Table

x		$-\sqrt{3}$		-1		0		1		$\sqrt{3}$	
f'	$+$	0	$-$	$\parallel$	$-$	0	$-$	$\parallel$	$-$	0	$+$
f''	$-$		$-$	$\parallel$	$+$	0	$-$	$\parallel$	$+$		$+$
f	$\nearrow$		$\searrow$	$\parallel$	$\searrow$		$\searrow$	$\parallel$	$\searrow$		$\nearrow$
		max		vertical asymptote		infl.		vertical asymptote		min	