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ON NONLINEAR VIBRATION OF NONUNIFORM BEAM WITH RECTANGULAR CROSS-SECTION AND PARABOLIC THICKNESS VARIATION

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We present the exact solution of differential equation in the linear case of free bending vibrations of nonuniform beam with rectangular cross-section using the factorization method. This beam with constant width and parabolic thickness is a good approximation of the gear tooth profile. It permits a nonlinear bending vibrations study (moderately large curvatures) of the gear tooth (the cantilever beam case). The case of the beam with a sharp end is considered. We use the method of multiple scales to treat the governing partial-differential equations and boundary conditions directly. In the absence of internal resonance (weakly nonlinear systems) the nonlinear modes are taken to be perturbed versions of the linear modes. We determine the nonlinear planar mode shapes and natural frequencies of a gear tooth with a sharp end variation (the cantilever beam case).

1. Introduction

Recently, the concept of the natural mode of motion of linear systems has been generalized to nonlinear systems. A basic property of a natural mode of linear systems is invariance. In nonlinear systems invariant motions on a two-dimensional manifold can be found. These motions are known as nonlinear mode motion and have been treated in many papers.

The introduction of [4] presents the actual situation of this area. The concept of nonlinear normal modes for undamped, multi-degree-of-freedom system with n-masses interconnected by strongly nonlinear symmetric springs has been introduced by the paper [6]. The existence of similar normal modes in a symmetric, conservative system has been provided by [7].

And other authors have determined the nonlinear modes of conservative nonlinear systems. The nonlinear modes may be constructed using the linear modes in the case of weakly nonlinear systems. [8,9] have used a center-manifold type reduction to find the nonlinear modes. Nayfeh [3] has shown that the method of multiple scales can be used to obtain equivalent results with Shaw and Pierre method.

In this paper we study the nonlinear vibrations of nonuniform beam with rectangular cross-section, constant width, parabolic thickness variation and a sharp end. The nonlinear model of moderately large amplitude of vibrations is presented. The general solution of fourth-order differential equation of bending vibrations (the linear case) is found. Being a weakly nonlinear system, the nonlinear modes are taken to be perturbed versions of the linear modes.

The nonlinear mode shapes and natural frequencies are determined using the method of multiple scales directly to the governing partial-differential equation of motion and boundary conditions (the cantilever beam case).

2. The Differential Equation

For an elastic beam the elementary strain energy is considered as:

$$\mathrm{d}U = \frac{1}{2} EIk^2 \mathrm{d}x \,, \tag{1}$$

where we have used the curvature k and the flexural rigidity EI. For the case of moderately large amplitude, the curvature is considered as:

$$k \cong y''\left(1 - \frac{3}{2} y'^2\right), \tag{2}$$

where y is considered the transverse displacement. So, the strain energy will be:

$$U = \int_0^{\ell} \frac{1}{2} EIy''\left(1 - 3y'^2\right)\mathrm{d}x^* \ . \tag{3}$$

We denote by ℓ the length of the beam. Using the kinetic energy, the Lagrange partial-differential equation of motion is obtained as

$$\frac{\partial^2}{\partial x^{*2}}\left\{EI\left(x^*\right)\frac{\partial^2 y}{\partial x^{*2}}\left[1-3\left(\frac{\partial y}{\partial x^*}\right)^2\right]\right\}-\frac{\partial}{\partial x^*}\left[-3EI\left(x^*\right)\left(\frac{\partial^2 y}{\partial x^{*2}}\right)^2\frac{\partial^2 y}{\partial x^{*2}}\right]+\rho_0 A\left(x^*\right)\frac{\partial^2 y}{\partial t^{*2}}=0 \ , \tag{4}$$

where the mass density ρ_0, the cross-sectional area of beam, the Young's modulus E and the moment of inertia I are considered. We can rewrite the equation (4) as:

$$\frac{1}{A\left(x^*\right)}\frac{\partial^2}{\partial x^{*2}}\left[EI\left(x^*\right)\frac{\partial^2 y}{\partial x^{*2}}\right]+\frac{3E}{A\left(x^*\right)}\left\{\frac{\partial^2}{\partial x^{*2}}\left[I\left(x^*\right)\frac{\partial^2 y}{\partial x^{*2}}\left(\frac{\partial y}{\partial x^*}\right)^2\right]+\right.$$

$$\left.+\frac{\partial}{\partial x^*}\left[I\left(x^*\right)\left(\frac{\partial^2 y}{\partial x^{*2}}\right)^2\frac{\partial y}{\partial x^*}\right]\right\}+\rho_0\frac{\partial^2 y}{\partial t^{*2}}=0 \ . \tag{5}$$

We introduce the nondimensional quantities defined by:

$$x=\frac{x^*}{\ell} \quad , \quad t=t^*\frac{1}{\ell^2}\sqrt{\frac{EI_0}{\rho_0 A_0}} \quad , \quad w=\frac{y}{W} \,, \tag{6}$$

where W is the characteristic transverse displacement, usually taken as ℓ. So, the partial-differential equation (5) becomes

$$\frac{1}{A(x)}\frac{\partial^2}{\partial x^2}\left[I(x)\frac{\partial^2 w}{\partial x^2}\right]+\frac{\alpha^*}{A(x)}\left\{\frac{\partial^2}{\partial x^2}\left[I(x)\frac{\partial^2 w}{\partial x^2}\left(\frac{\partial w}{\partial x}\right)^2\right]+\frac{\partial}{\partial x}\left[I(x)\left(\frac{\partial^2 w}{\partial x^2}\right)^2\frac{\partial w}{\partial x}\right]\right\}+\frac{\partial^2 w}{\partial t^2}=0, \quad (7)$$

where

$$\alpha^* = \frac{3W^2}{\ell^2}. \quad (8)$$

3. Formulation of the Problem

The nonuniform beam with constant width, parabolic thickness variation and a sharp end case is considered:

$$a(x)=a_0 \quad , \quad b(x)=b_0\left(1-x^2\right) \quad , \quad x\in[x_0,1). \quad (9)$$

In this way, the cross-sectional area and the moment of inertia at $x=0$, and the dimensionless forms of them, are:

$$A_0=a_0 b_0 \quad , \quad I_0=\frac{a_0 b_0}{12} \quad , \quad A(x)=1-x^2 \quad , \quad I(x)=\left(1-x^2\right)^3. \quad (10)$$

We examine the nonlinear modes (the case of moderately large amplitude of a cantilever beam) of the problem

$$L_2[w(x,t)]+\alpha^* N[w(x,t)]+\frac{\partial^2 w}{\partial t^2}=0 \ , \quad (11)$$

$$w=\frac{\partial w}{\partial x}=0 \text{ at } x=x_0 \quad \text{and} \quad w \text{ finite at } x=1, \quad (12)$$

where the linear and the nonlinear operators are:

$$L_2[w(x,t)]=\frac{1}{A(x)}\frac{\partial^2}{\partial x^2}\left[I(x)\frac{\partial^2 w}{\partial x^2}\right] \ , \quad (13)$$

$$N[w(x,t)]=\frac{1}{A(x)}\left\{\frac{\partial^2}{\partial x^2}\left[I(x)\frac{\partial^2 w}{\partial x^2}\left(\frac{\partial w}{\partial x}\right)^2\right]+\frac{\partial}{\partial x}\left[I(x)\left(\frac{\partial^2 w}{\partial x^2}\right)^2\frac{\partial w}{\partial x}\right]\right\}. \quad (14)$$

4. The Solution of the Linear Case

The linear case of the problem (11), (12) means:

$$L_2\left[w(x,t)\right]+\frac{\partial^2 w}{\partial t^2}=0 \ , \tag{15}$$

$$w=\frac{\partial w}{\partial x}=0 \ \text{ at } x=x_0 \quad \text{and} \quad w \ \text{ finite at } x=1 . \tag{16}$$

Considering the solution of partial-differential equation (15) as

$$w(x,t)=\varphi(x)\left(A_k e^{i\omega t}+\overline{A_k}e^{-i\omega t}\right) \quad , \quad A_k \ \text{complex constant,} \tag{17}$$

we obtain the following differential equation:

$$L_2\left[\varphi(x)\right]-\omega^2\varphi(x)=0 \ . \tag{18}$$

The general solution can be determined using the factorization method (see [2]). Thus, using (10), the differential equation (18) can be factored as:

$$\left[\left(1-x^2\right)\frac{\mathrm{d}^2}{\mathrm{d}x^2}-4x\frac{\mathrm{d}}{\mathrm{d}x}+\lambda_1\right]\left[\left(1-x^2\right)\frac{\mathrm{d}^2}{\mathrm{d}x^2}-4x\frac{\mathrm{d}}{\mathrm{d}x}+\lambda_2\right]\left[\varphi(x)\right]=0 \ , \tag{19}$$

where

$$\lambda_i=2+(-1)^i\sqrt{4+\omega^2} \quad , \quad i=\overline{1,2} \ . \tag{20}$$

By variable changing

$$x=1-2x' \ , \tag{21}$$

two differential Gauss equations result. Thus the constants of Gauss equations are:

$$c_i=2 \ , \ a_i+b_i=3 \ , \ a_i b_i=-2-(-1)^i\sqrt{4+\omega^2} \ , \quad i=\overline{1,2} \ . \tag{22}$$

So that the general solution of the equation (19) will be:

$$\varphi(x)=\sum_{i=1}^{2}\left[B_i\cdot {}_2F_1\left(a_i,b_i,2,\frac{1-x}{2}\right)+C_i\cdot w_{i2}\left(\frac{1-x}{2}\right)\right] \quad , \quad B_i, C_i \ , i=\overline{1,2} \ \text{are constant} \tag{23}$$

where the $w_{i2}(x)$ functions of Gauss equation theory are:

$$w_{i2}(x)={}_2F_1(a_i,b_i,2,x)\ln x+\sum\frac{(a_i)_n(b_i)_n}{n!(2)_n}x^n\left[\psi(a_i+n)-\psi(a_i)+\psi(b_i+n)-\psi(b_i)-\right.$$

$$\left.-\psi(n+2)+\psi(2)-\psi(n+1)+\psi(1)\right]-\sum_{n=1}^{1}\frac{(n-1)!(-1)_n}{(1-a_i)_n(1-b_i)_n}x^{-n} \ . \tag{24}$$

We used, the hypergeometric function ${}_2F_1(a,b,2,x)$, and ψ the logarithmic derivative of Γ function.

For the present beam, the boundary conditions (16) means, $C_i=0 \ , i=\overline{1,2}$ and the following frequency equation:

$$F(\omega)=\begin{vmatrix} {}_2F_1\left(a_1,b_1,2,\frac{1-x_0}{2}\right) & {}_2F_1\left(a_2,b_2,2,\frac{1-x_0}{2}\right) \\ \frac{a_1b_1}{3}\cdot{}_2F_1\left(a_1+1,b_1+1,3,\frac{1-x_0}{2}\right) & \frac{a_2b_2}{3}\cdot{}_2F_1\left(a_2+1,b_2+1,3,\frac{1-x_0}{2}\right) \end{vmatrix}=0\ . \quad (25)$$

So, the mode shapes are:

$$\varphi_k(x)={}_2F_1\left(a_{1k},b_{1k},2,\frac{1-x}{2}\right)+D_k\cdot{}_2F_1\left(a_{2k},b_{2k},2,\frac{1-x}{2}\right), \quad (26)$$

where

$$D_k=-\frac{{}_2F_1\left(a_{1k},b_{1k},2,\frac{1-x_0}{2}\right)}{{}_2F_1\left(a_{2k},b_{2k},2,\frac{1-x_0}{2}\right)}\ , \quad (27)$$

and $a_{ik}=a_i(\omega_k)$, $b_{ik}=b_i(\omega_k)$, $i=\overline{1,2}$.

5. Direct Approach of the Nonlinear Problem

We apply the method of multiple scales directly to the governing partial-differential system (11), (12). Introducing a small dimensionless parameter ε as a bookkeeping device we obtain the problem:

$$L_2[w(x,t)]+\varepsilon\cdot\alpha^*N[w(x,t)]+\frac{\partial^2 w}{\partial t^2}=0\,, \quad (28)$$

$$w=\frac{\partial w}{\partial x}=0 \text{ at } x=x_0 \quad \text{and} \quad w \text{ finite at } x=1\,. \quad (29)$$

A first-order uniform expansion is considered as

$$w(x,t,\varepsilon)=w_0(x,T_0,T_1)+\varepsilon\cdot w_1(x,T_0,T_1)\ , \quad (30)$$

where $T_0=t$ is a fast scale and $T_1=\varepsilon\cdot t$ is a slow scale, characterizing the nonlinearity influence on the natural frequencies. The time derivative becomes:

$$\frac{\mathrm{d}}{\mathrm{d}t}=D_0+\varepsilon\cdot D_1 \quad \text{where} \quad D_n=\frac{\partial}{\partial T_n}\ . \quad (31)$$

Using the equation (30), and equating coefficients of like powers of ε, from (28), we obtain:

Order ε^0

$$D_0^2 w_0+L_2[w_0]=0\ , \quad (32)$$

$$w_0 = \frac{\partial w_0}{\partial x} = 0 \text{ at } x = x_0 \quad \text{and} \quad w_0 \text{ finite at } x = 1 . \tag{33}$$

Order ε

$$D_0^2 w_1 + L_2[w_1] = -2D_0 D_1 w_0 - \alpha^* N(w_0) , \tag{34}$$

$$w_1 = \frac{\partial w_1}{\partial x} = 0 \text{ at } x = x_0 \quad \text{and} \quad w_1 \text{ finite at } x = 1 . \tag{35}$$

To construct the nonlinear mode, we write the solution of equation (32)

$$w_{0k}(x, T_0, T_1) = \varphi_k(x)\left[A_k(T_1)e^{i\omega T_0} + \overline{A_k}(T_1)e^{-i\omega T_0}\right], \tag{36}$$

where A_k is undetermined at this moment of approximation. Substituting (36) into (34) we obtain

$$D_0^2 w_1 + L_2[w_1] = -2i\omega_k \varphi_k(x)\left[A_k' e^{i\omega_k T_0} - \overline{A_k'} e^{-i\omega_k T_0}\right] -$$

$$-\alpha^* \cdot \left[\omega_k^2 N_1(\varphi_k) + N_2(\varphi_k)\right]\left(A_k e^{i\omega_k T_0} + \overline{A_k} e^{-i\omega_k T_0}\right)^3 . \tag{37}$$

Using (10) and (14), we have denoted the nonlinear operators

$$N_1(\varphi_k) = \varphi_k (\varphi_k')^2 , \tag{38}$$

$$N_2(\varphi_k) = (1 - x^2)^2 \left[8\varphi_k' \varphi_k'' \varphi_k''' + 3(\varphi_k'')^3\right] - 30x(1 - x^2)\varphi_k' (\varphi_k'')^2 . \tag{39}$$

This inhomogeneous equation (37) and the condition (35), have a solution only if a solvability condition is satisfied.

It means that, the right hand sides of (37) be orthogonal to every solution of the homogeneous problem

$$-2i\omega_k g_{1kk}\left(A_k' e^{i\omega_k T_0} - \overline{A_k'} e^{-i\omega_k T_0}\right) - \alpha^*\left(\omega_k^2 g_{2kk} + g_{3kk}\right)\left(A_k e^{i\omega_k T_0} + \overline{A_k} e^{-i\omega_k T_0}\right)^3 = 0 , \tag{40}$$

where

$$g_{1kk} = \langle \varphi_k , \varphi_k \rangle \quad , \quad g_{2kk} = \langle \varphi_k , N_1(\varphi_k) \rangle \quad , \quad g_{3kk} = \langle \varphi_k , N_2(\varphi_k) \rangle , \tag{41}$$

and the inner product between $\varphi_m(x)$ and $\varphi_n(x)$ is defined by:

$$\langle \varphi_m(x), \varphi_n(x) \rangle = \int_{x_0}^{1} (1 - x^2)\varphi_m(x)\varphi_n(x)\mathrm{d}x . \tag{42}$$

Because the operator L_2 is self-adjoint with given boundary conditions defined, the eigenfunctions $\varphi_m(x)$ corresponding to different eigenvalues ω_m, are orthogonal.

So, for the case of no internal resonance the solvability condition is:

$$-2i\omega_k g_{1kk} A_k' - 3\alpha^*\left(\omega_k^2 g_{2kk} + g_{3kk}\right)A_k^2 \overline{A_k} = 0 . \tag{43}$$

This equation governs the amplitude and phase evolution. Using (43), the equation (37) becomes

$$D_0^2 w_1 + L_2[w_1] = 3\alpha^*\left[\varphi_k\left(\omega_k^2\frac{g_{2kk}}{g_{1kk}}+\frac{g_{3kk}}{g_{1kk}}\right)-N(\varphi_k)\right]A_k^2\overline{A_k}e^{i\omega_k T_0}-\alpha^* N(\varphi_k)A_k^3 e^{3i\omega_k T_0}+cc. \quad (44)$$

The operator N is (see (34) and (37)):

$$N(\varphi_k)=\omega_k^2 N_1(\varphi_k)+N_2(\varphi_k) \ . \quad (45)$$

In this way the equation (44) can be rewrite:

$$D_0^2 w_1 + L_2[w_1] = -3\alpha^* f_{2k}(\varphi_k)A_k^2\overline{A_k}e^{i\omega_k T_0} - \alpha^* f_{1k}(\varphi_k)A_k^3 e^{3i\omega_k T_0} + cc \quad (46)$$

where

$$f_{1k}(\varphi_k)=\omega_k^2 N_1(\varphi_k)+N_2(\varphi_k), \quad (47)$$

$$f_{2k}(\varphi_k)=\omega_k^2 N_1(\varphi_k)+N_2(\varphi_k)-\varphi_k\left(\omega_k^2\frac{g_{2kk}}{g_{1kk}}+\frac{g_{3kk}}{g_{1kk}}\right). \quad (48)$$

Then, the solution w_1 can be expressed as:

$$w_1(x)=h_1(x)A_k^3 e^{3i\omega_k T_0}+h_2(x)A_k^2\overline{A_k}e^{i\omega_k T_0}+cc \ . \quad (49)$$

Replacing the express of w_1 function in equation (46), two two-point boundary-value problems result:

$$L_2[h_{1k}]-9\omega_k^2 h_{1k} = -\alpha^* f_{1k}(x) \ , \quad (50)$$

$$h_{1k} = h'_{1k} = 0 \ \text{ at } x = x_0 \quad \text{and} \quad h_{1k} \ \text{ finite at } x=1 \ , \quad (51)$$

and

$$L_2[h_{2k}]-\omega_k^2 h_{2k} = -3\alpha^* f_{2k}(x) \ , \quad (52)$$

$$h_{2k} = h'_{2k} = 0 \ \text{ at } x = x_0 \quad \text{and} \quad h_{2k} \ \text{ finite at } x=1 \,. \quad (53)$$

To solve the boundary-value problem (50), (51) we consider:

$$f_{1km}(x)=\sum_{m=1} A_{1km}\varphi_m(x) \quad , \quad A_{1km}=\frac{\langle f_{1km}(x),\varphi_m(x)\rangle}{\langle\varphi_m(x),\varphi_m(x)\rangle} \ . \quad (54)$$

Replacing (54) into (50) and using for h_1 the expression

$$h_{1k}(x)=\sum_{m=1}\Gamma_{1km}\varphi_m(x), \quad (55)$$

it results:

$$\Gamma_{1km}=\frac{\omega_k^2\dfrac{g_{2km}}{g_{1mm}}+\dfrac{g_{3km}}{g_{1mm}}}{\omega_m^2-9\omega_k^2}\quad\text{where}\quad g_{2km}=\left\langle N_1(\varphi_k),\varphi_m\right\rangle,\ g_{3km}=\left\langle N_2(\varphi_k),\varphi_m\right\rangle\ . \tag{56}$$

Analogously the boundary-value problem (52), (53) has the solution

$$h_{2k}(x)=\sum_{m\neq k}\Gamma_{2km}\varphi_m(x), \tag{57}$$

where

$$\Gamma_{2km}=\frac{\omega_k^2\dfrac{g_{2km}}{g_{1mm}}+\dfrac{g_{3km}}{g_{1mm}}}{\omega_m^2-\omega_k^2}\ . \tag{58}$$

Expressing A_k in the polar form

$$A_k=\frac{1}{2}a_k e^{i\beta_k}\ , \tag{59}$$

where a_k and β_k are real and separating equation (43) into real and imaginary parts, we obtain for the amplitude and for the phase:

$$a_k'=0\quad,\quad\omega_k a_k\beta_k'=\frac{3\alpha^*}{8}\left(\omega_k^2\frac{g_{2kk}}{g_{1kk}}+\frac{g_{3kk}}{g_{1kk}}\right)a_k^3\ . \tag{60}$$

Hence

$$a_k=\text{constant}\quad,\quad\beta_k=\frac{3\alpha^*}{8\omega_k}\left(\omega_k^2\frac{g_{2kk}}{g_{1kk}}+\frac{g_{3kk}}{g_{1kk}}\right)a_k^2\varepsilon t+\beta_{k0}\ ,\quad\beta_{k0}=\text{constant}\ . \tag{61}$$

So, using (30), (36), (49), (59) and (61), and considering $\varepsilon=1$, the displacement in terms of real variables will be

$$w(x,t)=a_k\varphi_k(x)\cos(\omega_{Nk}t+\beta_{k0})+\frac{a_k^3}{4}\left[h_1(x)\cos 3(\omega_{Nk}t+\beta_{k0})+h_2(x)\cos(\omega_{Nk}t+\beta_{k0})\right], \tag{62}$$

where a_k , β_{k0} are constants which represent, respectively, first approximations to the amplitude and phase of the motion. The nonlinear natural frequency of the *k*th mode is:

$$\omega_{Nk}=\omega_k+\frac{3\alpha^*}{8\omega_k}\left(\omega_k^2\frac{g_{2kk}}{g_{1kk}}+\frac{g_{3kk}}{g_{1kk}}\right)a_k^2\ . \tag{63}$$

We express the displacement given by equation (62) in terms of position and velocity coordinates by letting

$$q_k(t)=a_k\cos(\omega_{Nk}t+\beta_{k0})\quad,\quad\dot{q}_k(t)=-\omega_{Nk}a_k\sin(\omega_{Nk}t+\beta_{k0})\ . \tag{64}$$

Thus, considering

$$a_k^3 \cos(\omega_{Nk} t + \beta_{k0}) = q_k^3(t) + \frac{1}{\omega_k^2} q_k(t) \dot{q}_k^2(t) \ , \tag{65}$$

$$a_k^3 \cos 3(\omega_{Nk} t + \beta_{k0}) = q_k^3(t) - 3 \frac{1}{\omega_k^2} q_k(t) \dot{q}_k^2(t) \ , \tag{66}$$

the displacement will be

$$w(x,t) = \varphi_k(x) q_k(t) + \frac{1}{4} \left\{ [h_1(x) + h_2(x)] q_k^3(t) + \frac{1}{\omega_k^2} [h_2(x) - 3h_1(x)] q_k(t) \dot{q}_k^2(t) \right\} \ . \tag{67}$$

The spatial correction to the linear mode shape is given by $h_1(x)$ and $h_2(x)$. Using (55) and (57), the displacement (67) becomes:

$$w(x,t) = \varphi_k(x) q_k(t) + \sum_{m \neq k} \varphi_m(x) \left[\Gamma_{1km}^* q_k^3(t) + \Gamma_{2km}^* \dot{q}_k^2(t) q_k(t) \right], \tag{68}$$

where

$$\Gamma_{1km}^* = \frac{1}{2\omega_k^2} \left(\omega_k^2 \frac{g_{2km}}{g_{1mm}} + \frac{g_{3km}}{g_{1mm}} \right) \frac{\omega_m^2 - 5\omega_k^2}{\Delta} \ , \tag{69}$$

$$\Gamma_{2km}^* = \frac{1}{2} \left(\omega_k^2 \frac{g_{2km}}{g_{1mm}} + \frac{g_{3km}}{g_{1mm}} \right) \frac{\omega_m^2 + 3\omega_k^2}{\Delta} \ , \tag{70}$$

$$\Delta = (\omega_m^2 - \omega_k^2)(\omega_m^2 - 9\omega_k^2) \ . \tag{71}$$

When $\Delta = 0$ this construction of nonlinear mode breaks down, it means the case of one-to-one internal resonance ($\omega_m \approx \omega_k$) and the case of three-to-one internal resonance ($\omega_m \approx 3\omega_k$). We assume ω_m is away from ω_k or $3\omega_k$.

6. Conclusion

In this paper, we used the method of multiple scales to determine the expressions of the nonlinear mode shapes and natural frequencies of a cantilever beam with a constant width, parabolic thickness variation and a sharp end, in the case of large amplitude.

This method is applied directly to the nonlinear partial differential equation and boundary conditions. The linear mode shapes are determined using the factorization method. This beam is a good approximation of a gear tooth with a sharp end.

7. References

1. Caruntu, D. (1996) On bending vibrations of some kinds of beams of variable cross-section using orthogonal polynomials, *Rev.Roum.Sci.Techn.-Méc.Appl.* **41**, 265-272.
2. Caruntu, D. (1996) Relied studies on factorization of the differential operator in the case of bending vibration of a class of beams with variable cross-section, *Rev.Roum.Sci.Techn.-Méc.Appl.* **41**, 389-397.
3. Nayfeh, A. H. and Nayfeh, S. A. (1994) On Nonlinear Modes of Continuous Systems, *Journal of Vibration*

and Acoustics **116**, 129-136.
4. Nayfeh, A. H. , Chin, C, and Nayfeh, S. A. (1995) Nonlinear Normal Modes of Cantilever Beam, *Journal of Vibration and Acoustics* **117**, 477-481.
5. Nayfeh, A. H. and Nayfeh, S. A. (1995) Nonlinear Normal Modes of a Continuous System with Quadratic Nonlinearities, *Journal of Vibration and Acoustics* **117**, 199-205.
6. Rosenberg, R. M. (1962) The Normal Modes of Nonlinear N-Degree-of Freedom Systems, *ASME Journal of Applied Mechanics* **30**, 7-14.
7. Rosenberg, R. M. (1966) On nonlinear Vibrations of Systems with Many Degrees of Freedom, *Advances in Applied Mechanics* **9**, 155-242.
8. Shaw, S. W. and Pierre, C. (1994) Normal Modes of Non-Linear Vibratory Systems, *Journal of Sound and Vibration* **164**, 85-124.
9. Shaw, S. W. and Pierre, C, (1994) Normal Modes of Vibration for Non-Linear Continuous Systems, *Journal of Sound and Vibration* **169**, 319-347.